Lie Algebra Math 7970 Key

Exercises on Section 2

1. We need to show that $[\operatorname{Der} L, \operatorname{ad} L] \subset \operatorname{ad} L$. From L is an $\mathbb{F}$-algebra in which the product is the bracket. So $\delta \in \operatorname{Der} L$ means $\delta[x, y] = [x, \delta(y)] + [\delta(x), y]$ (see 1.3). Then for all $x, y \in L$,

$$\begin{aligned}[\delta, \operatorname{ad} x]y &= (\delta(\operatorname{ad} x) - (\operatorname{ad} x)\delta)(y) = \delta[x, y] - \operatorname{ad} x(\delta(y)) \\ &= [x, \delta(y)] + [\delta(x), y] - [x, \delta(y)] = \operatorname{ad}(\delta(x))(y),\end{aligned}$$

i.e., $[\delta, \operatorname{ad} x] = \operatorname{ad}(\delta(x))$.

2. We are going to show that $[\mathfrak{gl}(n, \mathbb{F}), \mathfrak{gl}(n, \mathbb{F})] = \mathfrak{sl}(n, \mathbb{F})$. Notice that when $i \neq j$, $[e_{ij}, e_{jj}] = e_{ij}$ and $e_{ii} - e_{jj} = [e_{ij}, e_{ji}]$. So $\mathfrak{sl}(n, \mathbb{F}) \subset [\mathfrak{gl}(n, \mathbb{F}), \mathfrak{gl}(n, \mathbb{F})]$. It suffices to show that $[\mathfrak{gl}(n, \mathbb{F}), \mathfrak{gl}(n, \mathbb{F})] \subset \mathfrak{sl}(n, \mathbb{F})$. Use $\operatorname{Tr} xy = \operatorname{Tr} yx$ for all $x, y \in \mathfrak{gl}(n, \mathbb{F})$.

3. Clearly the center of $\mathfrak{gl}(n, \mathbb{F})$ contains $\mathfrak{s}(n, \mathbb{F})$ since scalar matrices commutes with all elements in $\mathfrak{gl}(n, \mathbb{F})$. Now $[e_{ij}, x] = 0$ implies $x_{jj} = x_{ii}$ and $x_{jk} = 0$ if $k \neq j$ and $x_{ki} = 0$ if $k \neq i$. So x is a scalar matrix. By Exercise 1.7, unless $\operatorname{char} \mathbb{F}$ divides n, $\mathfrak{gl}(n, \mathbb{F}) = \mathfrak{sl}(n, \mathbb{F}) \oplus \mathfrak{s}(n, \mathbb{F})$. So the center of $\mathfrak{sl}(n, \mathbb{F})$ is trivial. If $\operatorname{char} \mathbb{F}$ divides n, then $\mathfrak{s}(n, \mathbb{F}) \subset \mathfrak{sl}(n, \mathbb{F})$ so that the center of $\mathfrak{sl}(n, \mathbb{F})$ is $\mathfrak{s}(n, \mathbb{F})$.

4. Suppose that $L^1 = [L, L] \subset Z(L)$ has dimension 1. Let $L^1 = \mathbb{F}x$ and we write $L = \operatorname{span}\{x, y, z\}$ where (x, y, z) is a basis for L. Since $L^1 \subset Z(L)$, i.e., $[x, y] = [x, z] = 0$ so that $L^1 = \mathbb{F}[y, z]$. Hence we may assume $[y, z] = x$. Thus

$$[y, z] = x, \quad [x, y] = 0, \quad [x, z] = 0.$$

It is easy to see that the Jacobi identity is satisfied. See p.11-14 of Jacobson's Lie Algebra for the classification of 3-dimensional Lie algebras.

5. Method 1: Recall that L is simple if (i) it has no ideals except itself and 0 and (ii) it is not abelian. Suppose $\dim L = 3$ and $L = [L, L]$. Clearly L is not abelian since $[L, L] = L \neq 0$. If L were not simple, there would exist a proper ideal $I \neq 0$ of L. We would have the following two cases.

1. $\dim I = 1$. Then $I = \mathbb{F}x$. Extend x to a basis (x, y, z) of L. So $[x, y]$ and $[x, z]$ are multiples of x. The elements of L are linear combinations of x, y, z so that the elements of $[L, L]$ are linear combinations of $[x, y], [x, z], [y, z]$. So $[L, L]$ would be at most 2-dimensional, a contradiction.
2. $\dim I = 2$. Then $I = \text{span}\,\{x, y\}$ for some linearly independent $x, y \in L$. So $[x, y]$, $[x, z]$, $[y, z] \in I$. So $[L, L]$ would be at most 2-dimensional, a contradiction.

Method 2: $[L, L] = L$ implies that $[\phi(L), \phi(L)] = \phi(L)$ for any homomorphism of L. Notice that L is not abelian (because $[L, L] = L$ and $\dim L = 3$). If $I \neq 0$ is a proper idea of L, then there is an homomorphism $\phi : L \to L'$ such that $\operatorname{Ker}\phi = I$ (p.7). Moreover $L/\operatorname{Ker}\phi \cong \operatorname{Im}\phi$ has dimension either 1 or 2. The dimension cannot be 1, otherwise $\operatorname{Im}\phi$ is abelian, i.e., $0 = [\phi(L), \phi(L)] = \phi(L)$, i.e., $I = \ker\phi = L$. The dimension cannot be 2 either, $\operatorname{Im}\phi$ is isomorphic to the 2-dimensional algebra in 1.4 on p.5 defined by $[x, y] = x$; then $[L, L] = \mathbb{F}x \neq L$.

Application: From Exercise 1.3 we know that $[L, L] = L$ when $L := \mathfrak{sl}(2, \mathbb{F})$ whose dimension is 3. So it is simple by the above argument.

6. Suppose char $\neq 3$. Clearly $L := \mathfrak{sl}(3, \mathbb{F})$ is not abelian. Let $h_1 := e_{11} - e_{22}$, $h_2 := e_{22} - e_{33}$, e_{ij} $(1 \leq i \neq j \leq 3)$ be the standard basis of L. If x is a diagonal matrix, then

$$\operatorname{ad} x(e_{ij}) = [x, e_{ij}] = (x_i - x_j)e_{ij}, \quad 1 \leq i \neq j \leq 3.$$

Since h_1, h_2 are diagonal matrices,

$$\operatorname{ad} h_k(e_{ij}) = [h_k, e_{ij}] = ((h_k)_i - (h_k)_j)e_{ij}.$$

Thus $\operatorname{ad} h_1, \operatorname{ad} h_2 \in \operatorname{End} L$ have the same $8 = \dim \mathfrak{sl}(3, \mathbb{F})$ linearly independent eigenvectors respectively, namely h_1, h_2, e_{ij} $(1 \leq i \neq j \leq 3)$ (but corresponding to possibly different eigenvalues). In other words, $\operatorname{ad} h_k$ is diagonalizable, $k = 1, 2$.

1. $\operatorname{ad} h_1$ has eigenvalues 0 (eigenvectors h_1, h_2), 1 (eigenvectors e_{13}, e_{32}), 2 (eigenvector e_{12}), -1 (eigenvectors e_{31}, e_{23}), -2 (eigenvector e_{21})
2. $\operatorname{ad} h_2$ has eigenvalues 0 (eigenvectors h_1, h_2), 1 (eigenvectors e_{21}, e_{13}), 2 (eigenvector e_{23}), -1 (eigenvectors e_{12}, e_{31}), -2 (eigenvector e_{32})

The eigenvalues $0, \pm 1, \pm 2$ are distinct since char $\mathbb{F} \neq 3$.

If $I \neq 0$ is an ideal of L, then clearly $\operatorname{ad} h_k(I) = [h_k, I] \subset I$, i.e., I is an invariant subspace of $\operatorname{ad} h_k \in \operatorname{End} L$, $k = 1, 2$. From matrix theory, I must be a direct sum of eigenspaces for $\operatorname{ad} h_1$ or $\operatorname{ad} h_2$. Case by case study shows that none of the combinations is a nonzero ideal, except L. For example,

1. if $\dim I = 1$, then $I = \mathbb{F}x$ where x is one of the standard basis vectors. But I is not an ideal.
2. if $\dim I = 2$, then I is spanned by x, y where x, y are among the standard basis vectors. But I is not an ideal....

7. Clearly $K \subset N_L(K) := \{x \in L : [x, K] \subset K\}$.

$\mathfrak{d}(n, \mathbb{F})$: If $x \in \mathfrak{gl}(n, \mathbb{F})$, with $x_{ij} \neq 0$ and $i \neq j$, then $[e_{ii}, x_{ij}] = x_{ij}$, i.e., x is not in the normalizer of $\mathfrak{d}(n, \mathbb{F})$. So $\mathfrak{d}(n, \mathbb{F})$ is self-normalizing.

$\mathfrak{t}(n, \mathbb{F})$: If $x \in \mathfrak{gl}(n, \mathbb{F})$, with $x_{ij} \neq 0$ and $i < j$, then $[e_{ii}, x_{ij}] = x_{ij}$, i.e., x is not in the normalizer of $\mathfrak{t}(n, \mathbb{F})$. So $\mathfrak{t}(n, \mathbb{F})$ is self-normalizing.

$\mathfrak{n}(n, \mathbb{F})$: By the definition of matrix multiplication $\mathfrak{n}(n, \mathbb{F})\mathfrak{t}(n, \mathbb{F}) \subset \mathfrak{n}(n, \mathbb{F})$ and $\mathfrak{t}(n, \mathbb{F})\mathfrak{n}(n, \mathbb{F}) \subset \mathfrak{n}(n, \mathbb{F})$. So $[\mathfrak{n}(n, \mathbb{F}), \mathfrak{t}(n, \mathbb{F})] \subset \mathfrak{n}(n, \mathbb{F})$. Hence $\mathfrak{t}(n, \mathbb{F})$ is a subset of the normalizer of $\mathfrak{n}(n, \mathbb{F})$ in $\mathfrak{gl}(n, \mathbb{F})$. Suppose $x \notin \mathfrak{t}(n, \mathbb{F})$, i.e., $x_{ij} \neq 0$ for some $i > j$. Then $[e_{ji}, x] = e_{ji}x - xe_{ji} \neq 0$ since its (j, j) entry is x_{ij}, i.e., x is not in the normalizer of $\mathfrak{n}(n, \mathbb{F})$.

8. $A_{\ell-1}$ Let $\mathfrak{d}_A(\ell, \mathbb{F}) := \mathfrak{d}(\ell, \mathbb{F}) \cap \mathfrak{sl}(\ell, \mathbb{F}) \subset \mathfrak{sl}(\ell, \mathbb{F})$, i.e., the set of diagonal matrices in $\mathfrak{sl}(\ell, \mathbb{F})$. Since char $F = 0$, $\mathfrak{gl}(\ell, \mathbb{F}) = \mathfrak{sl}(\ell, \mathbb{F}) \oplus \mathfrak{s}(\ell, \mathbb{F})$ by Exercises 1.3. So if x is in the normalizer of $\mathfrak{d}_A$ in $\mathfrak{sl}(\ell, \mathbb{F})$, x is also in the normalizer of $\mathfrak{d}_A$ in $\mathfrak{gl}(\ell, \mathbb{F})$. By Exercise 7, x must be diagonal and thus in $\mathfrak{d}_A$.

B_ℓ Let $\mathfrak{d}_B(2\ell, \mathbb{F}) := \mathfrak{d}(2\ell, \mathbb{F}) \cap \mathfrak{sp}(2\ell, \mathbb{F}) = \{\operatorname{diag}(d, -d) : d \in \mathbb{F}^\ell\}$. Using $e_{ii} + e_{\ell+i,\ell+i}$ to conclude that each x of the normalizer of $\mathfrak{d}_D$ in $\mathfrak{sp}(2\ell, \mathbb{F})$ is of the form.

D_ℓ Similar.

C_ℓ Similar.

9. (a) It is well-known that for a linear map $\phi : L \to L'$,

$$\dim L/\mathrm{Ker}\,\phi = \dim L - \dim \mathrm{Ker}\,\phi = \dim \mathrm{Im}\,\phi.$$

So $L/\mathrm{Ker}\,\phi \cong \mathrm{Im}\,\phi$. One can use the natural map $\hat{\phi} : L/\mathrm{Ker}\,\phi \to \mathrm{Im}\,\phi$ (induced by ϕ) as the isomorphism such that $x + \mathrm{Ker}\,\phi \mapsto \phi(x)$. The map $\hat{\phi}$ is clearly linear and is one-to-one since $\hat{\phi}(x + \mathrm{Ker}\,\phi) = 0$ if and only if $\phi(x) = 0$, i.e., $x \in \mathrm{Ker}\,\phi$. In other words, $x + \mathrm{Ker}\,\phi = 0 + \mathrm{Ker}\,\phi$. Thus use the dimension theorem to conclude that $\hat{\phi}$ is a vector space isomorphism. Finally

$$\begin{aligned}
\hat{\phi}([x + \mathrm{Ker}\,\phi, y + \mathrm{Ker}\,\phi]) &= \phi([x, y] + \mathrm{Ker}\,\phi) \\
&= [\phi(x), \phi(y)] \\
&= [\hat{\phi}(x + \mathrm{Ker}\,\phi), \hat{\phi}(y + \mathrm{Ker}\,\phi)]
\end{aligned}$$

i.e., $\hat{\phi}$ respects the bracket.

If I is an ideal of L included in $\mathrm{Ker}\,\phi$, then clearly I is an ideal of $\mathrm{Ker}\,\phi$ and thus L/I is a Lie algebra. We now claim that there exists a unique homomorphism $\psi : L/I \to L'$ such that $\pi \circ \psi = \phi$ where $\pi : L \to L/I$ is the canonical map.

Existence: Define $\psi : L/I \to L'$ by $x + I \mapsto \phi(x)$ so that $\pi \circ \psi = \phi$. It is well defined since if $x + K = y + K$, i.e., $x - y \in K$. Hence $\phi(x - y) = 0$ as $\mathrm{Ker}\,\phi \subset I$ so that $\phi(x) = \phi(y)$. It is a homomorphism because ϕ is a homomorphism.

Uniqueness is clear.

(b) Clearly if $I \subset J$ are ideals of L, then I is an ideal of J so that we have Lie algebra J/I as well as L/I. We now show that J/I is an ideal of L/I. For $x + I \in J/I$ and $y + I \in L/I$, i.e., $x \in J$ and $y \in L$, then

$$[x + I, y + I] = [x, y] + I \in J/I$$

since J is an ideal of L. So $(L/I)/(J/I)$ is a Lie algebra. Now we give a natural isomorphism

$$\xi : (L/I)/(J/I) \to L/J, \qquad (x + I) + J/I \mapsto x + J, \quad x \in L.$$

It is quite clear that ξ is linear and respects the bracket since

$$\xi([(x+I)+J/I,(y+I)+J/I] = \xi([x+I,y+I]+J/I) = \xi(([x,y]+I)+J/I) = \xi([x,y]).$$

It is a monomorphism (one-to-one) since $\xi((x+I)+J/I) = 0$ means that $x+J=0$, i.e., $x \in J$ or equivalently $x+I = J/I$; in other words, $(x+I)+J/I = J/I$.
By the dimension theorem

$$\dim(L/I)/(J/I) = \dim(L/I) - \dim(J/I) = \dim L - \dim I - (\dim J - \dim I) = \dim L - \dim J,$$

the natural map ξ is an isomorphism.

(c) It is well-known from linear algebra that

$$\dim I + \dim J - \dim(I \cap J) = \dim(I+J)$$

if I, J are subspaces of L.
Notice that if I and J are ideals of L, $J \subset I+J$ is an ideal of $I+J$ and $I \cap J \subset I$ is an ideal of I. So $(I+J)/J$ and $I/(I \cap J)$ are Lie algebras.
Applying the dimension theorem to our situation,

$$\dim(I+J)/J = \dim(I+J) - \dim J = \dim I + \dim J - \dim(I \cap J) - \dim J = \dim I/(I \cap J).$$

So the two vector spaces are isomorphic. The natural isomorphism is

$$\xi : (I+J)/J \to I/(I \cap J), \quad (x+y)+J = x+J \mapsto x+(I \cap J), \quad x \in I, y \in J.$$

The map is well defined because if $x'+y' = x+y$ where $x, x' \in I$ and $y, y' \in J$, then $x - x' = y - y' \in I \cap J$ so that $x + (I \cap J) = x' + (I \cap J)$. Clearly it is linear. Moreover it respects bracket since for $x, x' \in I$ and $y, y' \in J$,

$$\begin{aligned} \xi([(x+y)+J,(x'+y')+J]) &= \xi([x+J, x'+J]) = \xi([x,x']+J) \\ &= [x,x'] = [\xi((x+y)+J), \xi((x'+y')+J)]. \end{aligned}$$

10. Recall $\sigma = \exp \operatorname{ad} x \cdot \exp \operatorname{ad}(-y) \cdot \exp \operatorname{ad} x \in \mathrm{Int} L$ where $L = \mathfrak{sl}(2, \mathbb{F})$ and (x, h, y) are given in 2.1. By Exercise 3 on p.5, $\operatorname{ad} x, \operatorname{ad} y \in \operatorname{End}(L)$ are nilpotent with

$k = 3$, the index of nilpotence and $[x, y] = h$, $[h, x] = 2x$, $[h, y] = -2y$,

$$\begin{aligned} \sigma(x) &= \exp \operatorname{ad} x \cdot \exp \operatorname{ad}(-y) \cdot \exp \operatorname{ad} x(x) \\ &= \exp \operatorname{ad} x \cdot (1 + \operatorname{ad}(-y) + (\operatorname{ad}(-y))^2/2!)x \\ &= \exp \operatorname{ad} x \cdot (x + h - y) \\ &= (1 + \operatorname{ad} x + (\operatorname{ad} x)^2/2!)(x + h - y) \\ &= x + (h - 2x) - (y + h - x) = -y \end{aligned}$$

Similar computation for $\sigma(y) = -x$ and $\sigma(h) = -h$.

11. Clearly the map $\phi : L \to L$ given by $x \mapsto -gx^t g^{-1}$ is linear, invertible and respects the bracket:

$$\phi([x, y]) = -g[x, y]^t g^{-1} = -g[y^t, x^t]g^{-1} = -[gy^t g^{-1}, gx^t g^{-1}] = [gx^t g^{-1}, gy^t g^{-1}] = [\phi(x), \phi(y)].$$

So $\phi \in \operatorname{Aut} L$.

If $n = 2$ and g is the identity, then $\phi(x) = -x^t$. Then

$$\phi(x) = -y, \quad \phi(h) = -h, \quad \phi(y) = -x$$

with the standard basis (x, h, y) for $\mathfrak{sl}(2, \mathbb{F})$. In other words, $\phi = \sigma$ where σ is given on p.9. which is an inner automorphism.

12. The map $\phi : L \to L$ given by $x \mapsto gxg^{-1}$ is well-defined, i.e., $gxg^{-1} \in L$. It is because $g^t sg = s$ and $sg = (g^t)^{-1}s$. Then, if $x \in L$, i.e., $sx = -x^t s$, we have

$$sgxg^{-1} = (g^t)^{-1}sxg^{-1} = -(g^t)^{-1}x^t sg^{-1} = -(g^t)^{-1}x^t g^t s = -(gxg^{-1})^t s.$$

In other words, $gxg^{-1} \in L$. The map is clearly linear and invertible and respects the bracket.

Remark: About the name “orthogonal”: Usually when we say orthogonal matrix in matrix theory, we mean g such that $g^t g = I$ where I is the identity matrix, or we write $g^t I g = I$. Notice that I defines a symmetric bilinear form, i.e., $f(x, y) = x^t y$. So orthogonal here is with respect to s, i.e., we may call g s-orthogonal.