Lie Algebra Math 7970 Solutions and Remarks

For historical background of Lie algebra and Lie group see the book review by Brian Parshall on "Essays in the history of Lie groups and algebraic groups", by Armand Borel. Also early history. See a biography of Marius Sophus Lie.

1.1

Example: Verify that the cross product on $L = \mathbb{R}^3$ makes L a Lie algebra, i.e.,

$$x \times y = (x_2y_3 - x_3y_2, -x_1y_3 + x_3y_1, x_1y_2 - x_2y_1),$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. See Exercises 1.

Questions:

1. Show that (L2') $[x, y] = -[y, x]$ if and only if (L2) $[x, x] = 0$ for all $x, y \in L$, provided that char $\mathbb{F} \neq 2$ (why char $\mathbb{F}$ causes trouble?).
2. Show that $[x, 0] = 0$ for all $x \in L$.
3. Show that if $[x, y] \neq 0$, then $x, y \in L$ are linearly independent.

1.2

Questions:

1. Verify that the bracket $[x, y] = xy - yx$ makes $\operatorname{End} V$ a Lie algebra over $\mathbb{F}$.
2. What is the difference between $\mathfrak{gl}(V)$ and $\operatorname{End}(V)$?
3. Verify $e_{ij}e_{kl} = \delta_{jk}e_{il}$ and equation (*) $[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}$.
4. Verify $\mathfrak{n}(n, \mathbb{F}), \mathfrak{d}(n, \mathbb{F}), \mathfrak{t}(n, \mathbb{F})$ are Lie subalgebras of $\mathfrak{gl}(n, \mathbb{F})$.
5. Is it true that $[\mathfrak{n}(n, \mathbb{F}), \mathfrak{n}(n, \mathbb{F})] = \mathfrak{n}(n, \mathbb{F})$? Explain.
6. Show that $[L, L]$ is a subalgebra of L. Is it true that $[L, L] = L$ for general L? Give an example if your answer is not.

$\mathfrak{o}(n, \mathbb{F})$: Let char $\mathbb{F} \neq 2$. Let f be a nondegenerate symmetric bilinear form on the vector space V over $\mathbb{F}$. A symmetric bilinear form $f : V \times V \to \mathbb{F}$ is bilinear and

$$f(v, w) = f(w, v), \qquad \text{for all } v, w \in V.$$

Nondegenerate means that $f(v, w) = 0$ for all $w \in V$ implies $v = 0$.

Let $(e_1, \ldots, e_n)$ be a basis of V. Then the matrix $s = (f(e_i, e_j))$ (nonsingular symmetric matrix) completely characterizes f, i.e., if $v = \sum_i v_ie_i$, $w = \sum_i w_ie_i$, and if we identify v and w as $(v_1, \ldots, v_n)^t$ and $(w_1, \ldots, w_n)^t$ respectively, then $f(v, w) = v^tsw$.

Now

$$L := \mathfrak{o}(V) = \{x \in \mathfrak{gl}(V) : f(x(v), w) + f(v, x(w)) = 0 \text{ for all } v, w \in V\}.$$

It is a subalgebra of $\mathfrak{gl}(V)$.

Proof. Since we use the usual bracket of $\mathfrak{gl}(V)$, (L1), (L2) and (L3) are clearly satisfied. We only need to show that $\mathfrak{o}(V)$ is closed under bracket. Let $x, y \in \mathfrak{o}(V)$. Then for all $v, w \in V$

$$\begin{aligned}
& f([x,y](v),w) + f(v,[x,y](w)) \\
= \;& f((xy-yx)(v),w) + f(v,(xy-yx)(w)) \\
= \;& f(xy(v),w) - f(yx(v),w) + f(v,xy(w)) - f(v,yx(w)) \quad \text{bilinearity of } f \\
= \;& f(x(y(v)),w) - f(y(x(v)),w) + f(v,x(y(w))) - f(v,y(x(w))) \\
= \;& -f(y(v),x(w)) + f(x(v)),y(w)) - f(x(v),y(w)) + f(y(v),x(w)) \quad \text{since } x,y \in \mathfrak{o}(V) \\
= \;& 0
\end{aligned}$$

□

Remark: (1) You can also verify it in terms of matrices (see below).

(2) In the above proof we did not use the symmetric property of f. Indeed the proof works for nondegenerate skew symmetric bilinear form f, i.e. $f(v,w) = -f(w,v)$ for all $v, w \in V$, i.e., a proof for $\mathfrak{sp}(V)$ being a subalgebra of $\mathfrak{gl}(V)$. But it turns out skew symmetric nondegenerate form f on V implies that V is even dimensional.

Remark: We can describe $\mathfrak{o}(V)$ in terms of matrices, i.e., $\mathfrak{o}(n,\mathbb{F})$ as we see on p.3. On p.3, the form f on V is defined by the matrix s as we just saw, i.e., $f(v,w) = v^t s w$ for all $v, w \in V$, where

$$s = \begin{cases} \begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix} & \text{if } n = 2\ell \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{pmatrix} & \text{if } n = 2\ell+1. \end{cases}$$

Notice that s is nonsingular symmetric. So

$$\begin{aligned}
\mathfrak{o}(2\ell+1,\mathbb{F}) &= \left\{ x = \begin{pmatrix} 0 & b_1 & b_2 \\ -b_2^t & m & n \\ -b_1^t & p & -m^t \end{pmatrix} \in \mathfrak{sl}(2\ell+1,\mathbb{F}) : n^t = -n,\ p^t = -p, m \in \mathbb{F}_{\ell\times\ell} \right\} \\
\mathfrak{o}(2\ell,\mathbb{F}) &= \left\{ x = \begin{pmatrix} m & n \\ p & -m^t \end{pmatrix} \in \mathfrak{sl}(2\ell,\mathbb{F}) : n^t = -n,\ p^t = -p, m \in \mathbb{F}_{\ell\times\ell} \right\}.
\end{aligned}$$

Suppose $\mathbb{F} = \mathbb{C}$. Let r be an (complex) orthogonal matrix such that $r^t r = s$, namely

$$r := \frac{e^{-i\pi/4}}{\sqrt{2}} \begin{pmatrix} I & iI \\ iI & I \end{pmatrix}$$

Then the map $x \mapsto rxr^{-1}$ sends $x \in \mathfrak{o}(2\ell,\mathbb{C})$ to a skew symmetric matrix since

$$sx = -x^t s \Leftrightarrow r^t r x = -x^t r^t r \Leftrightarrow (rxr^{-1})^t + rxr^{-1} = 0$$

and it is clearly an (algebra) isomorphism. Similar for $\mathfrak{o}(2\ell+1,\mathbb{C})$.

Remark: Indeed the above argument works for algebraically closed field $\mathbb{F}$ and char $\mathbb{F} \neq 2$ since every symmetric bilinear form on the vector space V over $\mathbb{F}$ has an orthogonal basis if char $\mathbb{F} \neq 2$ (by induction on the dimension of V, e.g., Serge Lang, Algebra, p.575). In other word, the symmetric matrix s over $\mathbb{F}$ can be written as $s = u^t d u$ where d is a diagonal matrix over $\mathbb{F}$, then set $r = \sqrt{d} u$ since $\mathbb{F}$ is algebraically closed, i.e., $s = r^t r$.

1.3

Remark:

1. An $\mathbb{F}$-algebra $\mathfrak{A}$ is a vector space over $\mathbb{F}$ equipped with an additional binary operation (multiplication) from $\mathfrak{A} \times \mathfrak{A}$ to $\mathfrak{A}$, $(x, y) \mapsto xy$ denoted by juxtaposition. The binary operation is bilinear which means that the following identities hold for any three elements $x, y, z \in \mathfrak{A}$, and all scalars $a, b \in \mathbb{F}$,
   (a) $(x + y)z = xz + yz$.
   (b) $x(y + z) = xy + xz$.
   (c) $(ax)(by) = (ab)(xy)$.

   Multiplication is not necessarily associative.
2. $\operatorname{End} V$ is an $\mathbb{F}$-algebra (associative) with the usual composition as the multiplication. It is associative but not commutative (when $\dim V \geq 2$).
3. A Lie algebra is an $\mathbb{F}$-algebra in which the bracket is the multiplication, i.e., $xy = [x, y]$.
4. By (L1) $\operatorname{ad} x : L \to L$ is a linear map for each $x \in L$ and the adjoint representation $\operatorname{ad} : L \to \mathfrak{gl}(L)$ is a linear map. By the Jacobi identity, $\operatorname{ad} x \in \operatorname{Der} L$ for each $x \in L$. Check!

Questions:

1. Is the term “derivation” justified for a linear map $\delta : \mathfrak{A} \to \mathfrak{A}$ such that $\delta(ab) = a\delta(b) + \delta(a)b$?
2. Do Exercise 11 on p.6. Hence $\operatorname{Der} \mathfrak{A}$ is a subalgebra of $\mathfrak{gl}(\mathfrak{A})$ and thus $\operatorname{Der} L$ is a subalgebra of $\mathfrak{gl}(L)$.
3. Verify that $\operatorname{ad} x \in \operatorname{Der} L$, i.e., $\operatorname{ad} x[y, z] = [\operatorname{ad} x(y), z] + [y, \operatorname{ad} x(z)]$. So $\operatorname{ad} : L \to \mathfrak{gl}(L)$ can be considered as $\operatorname{ad} : L \to \operatorname{Der} L$.
4. Is $[H, K] = \{[x, y] : x \in H, y \in K\}$?
5. Is $[H, K]$ always a subalgebra of L if $H, K \subset L$ are subalgebras of L? Explain.

1.4

Examples: The following is a realization of the Lie algebra $\mathbb{R}^3$ with cross product in terms of linear Lie algebra (Ado’s theorem). Let $\phi : L \to L'$, $(x_1, x_2, x_3) \mapsto \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$, where L' is the algebra of 3×3 real matrices. It is a vector space isomorphism: Check that it is one-to-one since L and L' have the same dimension and that

$$\begin{aligned}
& \phi(x \times y) \\
= {} & \phi(x_2y_3 - x_3y_2, -x_1y_3 + x_3y_1, x_1y_2 - x_2y_1) \\
= {} & \begin{pmatrix} 0 & -(x_1y_2 - x_2y_1) & -x_1y_3 + x_3y_1 \\ x_1y_2 - x_2y_1 & 0 & -(x_2y_3 - x_3y_2) \\ -(-x_1y_3 + x_3y_1) & x_2y_3 - x_3y_2 & 0 \end{pmatrix} \\
= {} & \left[\begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix} \right] \\
= {} & [\phi(x), \phi(y)].
\end{aligned}$$

Remark: "A moment of thought on p.5": With $[x_i, x_j] = \sum_k a_{ij}^k x_k$ (warning a_{ij}^k does not mean the power $(a_{ij})^k$; k is just an index)

$$[x_i, x_i] = 0 = [x_i, x_j] + [x_j, x_i] \quad \text{for all } i, j \Leftrightarrow a_{ii}^k = 0 = a_{ij}^k + a_{ji}^k, \quad \text{for all } i, j, k \tag{1}$$

is implied by (L2). Conversely (L2) follows from (1) since if $x = \sum_i c_i x_i$, then

$$[x, x] = [\sum_i c_i x_i, \sum_j c_j x_j] = \sum_{i,j} c_i c_j [x_i, x_j] = \sum_i c_i^2 [x_i, x_i] + \sum_{i \neq j} c_i c_j [x_i, x_j] = 0.$$

Notice that

$$[[x_i, x_j], x_\ell] = [\sum_k a_{ij}^k x_k, x_\ell] = \sum_k a_{ij}^k [x_k, x_\ell] = \sum_k a_{ij}^k \sum_m a_{k\ell}^m x_m = \sum_m (\sum_k a_{ij}^k a_{k\ell}^m) x_m.$$

So the Jacobi identity implies

$$[[x_i, x_j], x_\ell] + [[x_j, x_\ell], x_i] + [[x_\ell, x_i], x_j] = 0 \Leftrightarrow \sum_k (a_{ij}^k a_{k\ell}^m + a_{j\ell}^k a_{ki}^m + a_{\ell i}^k a_{kj}^m) = 0, \tag{2}$$

for all i, j, ℓ, m. Set

$$f(x_i, x_j, x_\ell) := [[x_i, x_j], x_\ell] + [[x_j, x_\ell], x_i] + [[x_\ell, x_i], x_j].$$

Then $f(x_i, x_j, x_\ell) = 0$ implies that $f(x_{\sigma(i)}, x_{\sigma(j)}, x_{\sigma(\ell)}) = 0$ for all permutations $\sigma : \{i, j, \ell\} \to \{i, j, \ell\}$, i.e. checking (2) is sufficient for the ordered triple (i, j, ℓ). The bilinearity of the bracket and (2) imply the Jacobi identity for x, y, z: Let $x = \sum_k a_k x_k$, $y = \sum_k b_k x_k$, $z = \sum_k c_k x_k$. Then

$$\begin{aligned}
& [[x, y], z] + [[y, z], x] + [[z, x], y] \\
= & \sum_k a_k [x_k, y], z] + \sum_k a_k [[y, z], x_k] + \sum_k a_k [[z, x_k], y] \\
= & \sum_k a_k ([x_k, y], z] + [[y, z], x_k] + [[z, x_k], y]) \\
= & \sum_k a_k \sum_i a_i ([x_k, x_i], z] + [[x_i, z], x_k] + [[z, x_k], x_i]) \\
= & \sum_k \sum_i \sum_j a_k a_i a_j ([x_k, x_i], x_j] + [[x_i, x_j], x_k] + [[x_j, x_k], x_i]) \\
= & \; 0
\end{aligned}$$

Questions:

1. Check: $[x, y] = x$ actually defines a Lie algebra. Notice that $a_{12}^1 = 1, a_{12}^2 = 0$.
2. Summarize the 1-dimensional and 2-dimensional Lie algebras.

Exercises on Section 1

1. (L1) We know that the cross product on $\mathbb{R}^3$ is bilinear, i.e.,

$$[a_1x_1 + a_2x_2, y] = (a_1x_1 + a_2x_2) \times y = a_1x_1 \times y + a_2x_2 \times y = a_1[x_1, y] + a_2[x_2, y]$$

for all $x_1, x_2, y \in \mathbb{R}^3$ and $a_1, a_2 \in \mathbb{R}$ and similar for the second coordinate.

(L2) $[x, y] = x \times y = -y \times x = -[y, x]$ is also well-known.

(L3) By vector triple product $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$ we have

$$\begin{aligned} & [x,[y,z]] + [y,[z,x]] + [z,[x,y]] \\ = \; & x \times (y \times z) + y \times (z \times x) + z \times (x \times y) \\ = \; & y(x \cdot z) - z(x \cdot y) + z(y \cdot x) - x(y \cdot z) + x(z \cdot y) - y(z \cdot x) = 0. \end{aligned}$$

Since $[e_1, e_2] = e_3$, $[e_1, e_3] = -e_2$, $[e_2, e_3] = e_1$, the structure constants are

$$\begin{aligned} & a_{12}^1 = 0, a_{12}^2 = 0, a_{12}^3 = 1, \\ & a_{13}^1 = 0, a_{13}^2 = -1, a_{13}^3 = 0, \\ & a_{23}^1 = 1, a_{23}^2 = 0, a_{23}^3 = 0. \end{aligned}$$

2. Given a basis (x, y, z) of L, $[x, y] = z$, $[x, z] = y$, $[y, z] = 0$, the structure constants are

$$\begin{aligned} & a_{12}^1 = a_{12}^2 = 0, a_{12}^3 = 1, \\ & a_{13}^1 = a_{13}^3 = 0, a_{13}^2 = 1, \\ & a_{23}^1 = a_{23}^2 = a_{23}^3 = 0. \end{aligned}$$

They satisfy the structure equations (see p.5)

$$\begin{aligned} & a_{ii}^k = 0 = a_{ij}^k + a_{ji}^k, \quad \text{(the first equality is implied by (L2))} \\ & \sum_k (a_{ij}^k a_{k\ell}^m + a_{j\ell}^k a_{ki}^m + a_{\ell i}^k a_{kj}^m) = 0, \end{aligned}$$

or for the last equality we just observe (see the note on p.5)

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = [x,0] + [z,z] + [y,-y] = 0.$$

Remark: Such checking is not practical when $\dim L$ is getting large.

3. Let

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{F}).$$

Now

$$\begin{aligned} \operatorname{ad} x(x) &= 0 = 0 \cdot x + 0 \cdot h + 0 \cdot y \\ \operatorname{ad} x(h) &= [x, h] = xh - hx = -2x = -2x + 0 \cdot h + 0 \cdot y \\ \operatorname{ad} x(y) &= [x, y] = xy - yx = h = 0 \cdot x + h + 0 \cdot y. \end{aligned}$$

So the matrix of $\operatorname{ad} x$ with respect to the basis (x, h, y) is $\operatorname{ad} x = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Similarly

$$\begin{aligned}
\operatorname{ad} h(x) &= -\operatorname{ad} x(h) = 2x \\
\operatorname{ad} h(h) &= 0 \\
\operatorname{ad} h(y) &= -2y.
\end{aligned}$$

Hence $\operatorname{ad} h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$.

$$\begin{aligned}
\operatorname{ad} y(x) &= -\operatorname{ad} x(y) = -h \\
\operatorname{ad} y(h) &= -\operatorname{ad} h(y) = 2y \\
\operatorname{ad} y(y) &= 0.
\end{aligned}$$

Hence $\operatorname{ad} y = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$.

4. Given $[x, y] = x$. Now $\operatorname{ad}(x)y = x$ and $\operatorname{ad} y(x) = -\operatorname{ad} x(y) = -x$. So

$$\operatorname{ad} x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \operatorname{ad} y = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $\operatorname{ad} : L \to \operatorname{End}(L)$ is injective. Thus $\operatorname{ad} L \subset \operatorname{End}(L)$ is a Lie algebra isomorphic to L. Via the adjoint representation ad, we have a matrix model for the 2-dimensional L (predicted by Ado's theorem p.4). Clearly

$$\operatorname{ad} x \operatorname{ad} y - \operatorname{ad} y \operatorname{ad} x = [\operatorname{ad} x, \operatorname{ad} y] = \operatorname{ad}[x, y] = \operatorname{ad} x.$$

See Example 10 on p.4 of Notes on Lie Algebras to see how it arises as the affine Lie algebra of the line consisting of all real 2×2 matrices with second row 0.

5. It is easy to see that $\mathfrak{t}(n, \mathbb{F})$, $\mathfrak{n}(n, \mathbb{F})$ and $\mathfrak{d}(n, \mathbb{F})$ are invariant (or stable) under the (matrix) bracket. So they are subalgebras of $\mathfrak{gl}(n, \mathbb{F})$. We have

$$\mathfrak{t}(n, \mathbb{F}) = \mathfrak{n}(n, \mathbb{F}) \oplus \mathfrak{d}(n, \mathbb{F})$$

since each $A \in \mathfrak{t}(n, \mathbb{F})$ is the sum of its diagonal part, $\operatorname{diag} A \in \mathfrak{d}(n, \mathbb{F})$, and its strictly upper triangular part, and clearly the intersection of $\mathfrak{d}(n, \mathbb{F})$ and $\mathfrak{n}(n, \mathbb{F})$ is trivial, i.e., the zero space. Notice that $\mathfrak{d}(n, \mathbb{F})\mathfrak{n}(n, \mathbb{F}), \mathfrak{n}(n, \mathbb{F})\mathfrak{d}(n, \mathbb{F}) \subset \mathfrak{n}(n, \mathbb{F})$ so that

$$[\mathfrak{d}(n, \mathbb{F}), \mathfrak{n}(n, \mathbb{F})] \subset \mathfrak{n}(n, \mathbb{F}).$$

(here $[H, K]$ denotes the subspace of L spanned by commutators $[x, y]$, $x \in H$, $y \in K$. See p.3.) So

$$\begin{aligned}
[\mathfrak{t}(n, \mathbb{F}), \mathfrak{t}(n, \mathbb{F})] &= [\mathfrak{n}(n, \mathbb{F}) \oplus \mathfrak{d}(n, \mathbb{F}), \mathfrak{n}(n, \mathbb{F}) \oplus \mathfrak{d}(n, \mathbb{F})] \\
&\subset [\mathfrak{n}(n, \mathbb{F}), \mathfrak{n}(n, \mathbb{F})] + [\mathfrak{d}(n, \mathbb{F}), \mathfrak{n}(n, \mathbb{F})] + [\mathfrak{n}(n, \mathbb{F}), \mathfrak{d}(n, \mathbb{F})] + [\mathfrak{d}(n, \mathbb{F}) \oplus \mathfrak{d}(n, \mathbb{F})] \\
&= [\mathfrak{n}(n, \mathbb{F}), \mathfrak{n}(n, \mathbb{F})] \subset \mathfrak{n}(n, \mathbb{F})
\end{aligned}$$

since $\mathfrak{d}(n,\mathbb{F})$ is abelian. The rest is to show that $\mathfrak{n}(n,\mathbb{F}) \subset [\mathfrak{t}(n,\mathbb{F}),\mathfrak{t}(n,\mathbb{F})]$, i.e., $e_{ij} \in [\mathfrak{t}(n,\mathbb{F}),\mathfrak{t}(n,\mathbb{F})]$ for all $i<j$. Since

$$[e_{ij}, e_{k\ell}] = \delta_{jk}e_{i\ell} - \delta_{\ell i}e_{kj}$$

(equation (*) on p.2) it follows that $e_{ij} = [e_{ii}, e_{ij}]$ if $i \neq j$ and we have the desired result.

1. $\mathfrak{t}(n,\mathbb{F})$ has basis e_{ij}, $1 \le i \le j \le n$. So $\dim \mathfrak{t}(n,\mathbb{F}) = 1 + \cdots + n = n(n+1)/2$.
2. $\mathfrak{n}(n,\mathbb{F})$ has basis e_{ij}, $1 \le i < j \le n$. So $\dim \mathfrak{n}(n,\mathbb{F}) = 1 + \cdots + (n-1) = n(n-1)/2$.
3. $\mathfrak{d}(n,\mathbb{F})$ has basis e_{ii}, $1 \le i \le n$. So $\dim \mathfrak{d}(n,\mathbb{F}) = n$.

6. Let $x \in \mathfrak{gl}(n,\mathbb{F})$ be a matrix of distinct eigenvalues. It is well known from matrix theory that a matrix with distinct eigenvalues is diagonalizable. Let $y^{-1}xy = d$ where d is a diagonal matrix, i.e., $x = ydy^{-1}$.

   Recall $\operatorname{ad} x \in \operatorname{End}(\mathfrak{gl}(n,\mathbb{F}))$. Consider a slightly more general case. Let $\sigma \in \operatorname{Aut}(\mathfrak{gl}(n,\mathbb{F}))$ defined by $\sigma(x) := yxy^{-1}$, $x \in \mathfrak{gl}(n,\mathbb{F})$, where $y \in \mathfrak{gl}(n,\mathbb{F})$ is nonsingular. Then

$$\operatorname{ad}\sigma(x) = \sigma(\operatorname{ad} x)\sigma^{-1}$$

   (indeed it is true for all $\sigma \in \operatorname{Aut}(\mathfrak{gl}(n,\mathbb{F}))$. See p.9). It is because

$$\operatorname{ad}(\sigma(x))(z) = \operatorname{ad}(yxy^{-1})(z) = [yxy^{-1}, z] = yxy^{-1}z - zyxy^{-1} = y[x, y^{-1}zy]y^{-1}.$$

   Hence $\operatorname{ad}\sigma(x)$ and $\operatorname{ad} x$ have the same eigenvalues since they are similar.

   Let us get back to our case. As $\sigma(d) := ydy^{-1} = x$, $\operatorname{ad} x$ and $\operatorname{ad} d$ are similar and have the same eigenvalues. So it is sufficient to consider $\operatorname{ad} d$ where $d = \operatorname{diag}(a_1,\ldots,a_n)$ with $a_1,\ldots,a_n$ distinct eigenvalues of x. Now

$$\operatorname{ad} d(e_{ij}) = (a_i - a_j)e_{ij}, \quad 1 \le i,j \le n.$$

   So e_{ij} $(1 \le i,j \le n)$ are eigenvectors of $\operatorname{ad} d$ with eigenvalue $a_i - a_j$ and there are n^2 such eigenvalues. They need not be distinct.

7. Each $a \in \mathfrak{gl}(n,\mathbb{F})$ can be decomposed as

$$a = \frac{\operatorname{Tr} a}{n} I_n + (a - \frac{\operatorname{Tr} a}{n}) I_n \in \mathfrak{sl}(n,\mathbb{F}) + \mathfrak{s}(n,\mathbb{F}).$$

   Moreover if $b \in \mathfrak{s}(n,\mathbb{F})$ (the algebra of scalar matrices), i.e., $b = \lambda I_n$, and $c \in \mathfrak{gl}(n,\mathbb{F})$, then

$$[b,c] = bc - cb = \lambda I_n c - c\lambda I_n = 0,$$

   i.e., $[\mathfrak{s}(n,\mathbb{F}), \mathfrak{gl}(n,\mathbb{F})] = 0$.

   It remains to show that the intersection $\mathfrak{sl}(n,\mathbb{F}) \cap \mathfrak{s}(n,\mathbb{F})$ is trivial with the assumption that the characteristic of $\mathbb{F}$, denoted by $\operatorname{char}\mathbb{F}$, is zero or else a prime not dividing n (note that in general $\operatorname{char}\mathbb{F}$ is either zero or a prime for a field $\mathbb{F}$). Let $a \in \mathfrak{sl}(n,\mathbb{F}) \cap \mathfrak{s}(n,\mathbb{F})$. So $a = \lambda I_n$ for some $\lambda \in \mathbb{F}$ and $0 = \operatorname{Tr} a = n\lambda$. Hence $\lambda = 0$ and thus $a = 0$.

8. Similar to $\mathfrak{o}(2\ell,\mathbb{F})$,

$$\mathfrak{o}(2\ell,\mathbb{F}) = \left\{ \begin{pmatrix} m & n \\ p & -m^t \end{pmatrix} : n^t = -n,\ p^t = \mathfrak{p}, m \in \mathbb{F}_{\ell\times\ell} \right\}.$$

So

$$\{e_{ij} - e_{\ell+j,\ell+i}, 1 \le i \le j \le \ell\} \cup \{e_{\ell+i,j} - e_{\ell+j,i}, e_{i,\ell+j} - e_{j,\ell+i} : 1 \le i < j \le \ell\}$$

is a basis of $\mathfrak{o}(2\ell,\mathbb{F})$, where $e_{ij} \in \mathbb{F}_{\ell\times\ell}$ is the standard basis element with (i,j) entry 1 and zero elsewhere. Hence

$$\dim \mathfrak{o}(2\ell,\mathbb{F}) = \ell^2 + \ell(\ell-1)/2 + \ell(\ell-1)/2 = 2\ell^2 - \ell.$$

9. Clearly $[L,L] \subset L$. It suffices to show that $L \subset [L,L]$ where L is a classical algebra with the assumption char $\mathbb{F} = 0$.

   1. $\mathfrak{sl}(\ell+1,\mathbb{F})$, $\ell \ge 1$: Notice

   $$\begin{aligned} 2e_{ij} &= [e_{ii} - e_{jj}, e_{ij}] \in [L,L], \quad 1 \le i \ne j \le \ell+1 \\ e_{ii} - e_{jj} &= [e_{ij}, e_{ji}] \in [L,L], \quad 1 \le i \ne j \le \ell+1, \end{aligned}$$

   motivated from Ex. 3.

   2. $\mathfrak{sp}(2\ell,\mathbb{F})$: Notice that

   $$\mathfrak{sp}(2\ell,\mathbb{F}) = \left\{ \begin{pmatrix} m & n \\ p & -m^t \end{pmatrix} : n^t = n,\ p^t = p,\ m \in \mathbb{F}_{\ell\times\ell} \right\}.$$

   To get $m \oplus (-m^t)$ consider $1 \le i \ne j \le \ell$ and

   $$\begin{aligned} 2(e_{ij} - e_{\ell+j,\ell+i}) &= [e_{ii} - e_{jj} - e_{\ell+i,\ell+i} + e_{\ell+j,\ell+j}, e_{ij} - e_{\ell+j,\ell+i}] \in [L,L] \\ e_{ii} - e_{\ell+i,\ell+i} &= [e_{i,\ell+i}, e_{\ell+i,i}] \in [L,L]. \end{aligned}$$

   To get n consider $1 \le i \ne j \le \ell$ and

   $$\begin{aligned} 2(e_{i,\ell+j} + e_{j,\ell+i}) &= [e_{ii} + e_{jj} - e_{\ell+i,\ell+i} - e_{\ell+j,\ell+j}, e_{i,\ell+j} + e_{j,\ell+i}] \in [L,L] \\ 2(e_{i,\ell+i} \mp e_{\ell+i,i}) &= [e_{ii} - e_{\ell+i,\ell+i}, e_{i,\ell+i} \pm e_{\ell+i,i}] \in [L,L]. \end{aligned}$$

   So $e_{i,\ell+i} \in [L,L]$. To get p consider $1 \le i \ne j \le \ell$

   $$2(e_{\ell+i,j} + e_{\ell+j,i}) = [e_{\ell+i,\ell+i} + e_{\ell+j,\ell+j} - e_{i,i} - e_{j,j}, e_{\ell+i,j} + e_{\ell+j,i}] \in [L,L].$$

   3. $\mathfrak{o}(2\ell,\mathbb{F})$: Recall

   $$\mathfrak{o}(2\ell,\mathbb{F}) = \left\{ \begin{pmatrix} m & n \\ p & -m^t \end{pmatrix} \in \mathfrak{sl}(2\ell,\mathbb{F}) : n^t = -n,\ p^t = -p, m \in \mathbb{F}_{\ell\times\ell} \right\}.$$

   In view of the computation for $\mathfrak{sp}(2\ell,\mathbb{F})$ we only need to consider n and p. For n, if $1 \le i \ne j \le \ell$, then

   $$2(e_{i,\ell+j} - e_{j,\ell+i}) = [e_{ii} + e_{jj} - e_{\ell+i,\ell+i} - e_{\ell+j,\ell+j}, e_{i,\ell+j} - e_{j,\ell+i}] \in [L,L].$$

   To get p consider $1 \le i \ne j \le \ell$

   $$2(e_{\ell+i,j} - e_{\ell+j,i}) = [e_{\ell+i,\ell+i} + e_{\ell+j,\ell+j} - e_{i,i} - e_{j,j}, e_{\ell+i,j} - e_{\ell+j,i}] \in [L,L].$$

4. $\mathfrak{o}(2\ell+1,\mathbb{F})$: Recall

$$\mathfrak{o}(2\ell+1,\mathbb{F}) = \left\{ \begin{pmatrix} 0 & b_1 & b_2 \\ -b_2^t & m & n \\ -b_1^t & p & -m^t \end{pmatrix} \in \mathfrak{sl}(2\ell+1,\mathbb{F}) : n^t = -n,\ p^t = -p,\ m \in \mathbb{F}_{\ell\times\ell} \right\}$$

and the computation is similar.

Remark: We will see in Corollary 5.4 that $L = [L, L]$ is a property of semisimple Lie algebra and $A_\ell - D_\ell$ are simple.

11. Let $\delta, \delta' \in \text{Der}\,\mathfrak{A}$. We are going to show that $[\delta, \delta'] \in \text{Der}\,\mathfrak{A}$. For $a, b \in L$,

$$\begin{aligned} & [\delta,\delta'](ab) \\ = \;& (\delta\delta' - \delta'\delta)(ab) \\ = \;& \delta\delta'(ab) - \delta'\delta(ab) \\ = \;& \delta(a\delta'(b) + \delta'(a)b) - \delta'(a\delta(b) + \delta(a)b) \\ = \;& \delta(a)\delta'(b) + a\delta\delta'(b) + \delta\delta'(a)b + \delta'(a)\delta(b) - (\delta'(a)\delta(b) + a\delta'\delta(b) + \delta'\delta(a)b + \delta(a)\delta'(b)) \\ = \;& a\delta\delta'(b) + \delta\delta'(a)b - a\delta'\delta(b) - \delta'\delta(a)b \\ = \;& a[\delta,\delta'](b) + [\delta,\delta'](a)b, \end{aligned}$$

i.e., $[\delta, \delta'] \in \text{Der}\,\mathfrak{A}$. So Der $\mathfrak{A}$ is a subalgebra of $\mathfrak{gl}(\mathfrak{A})$. Thus if L is a Lie algebra in which the product of x, y is just the bracket of x, y, then Der L is a subalgebra of L.

Notice that

$$(\delta\delta')(ab) = \delta(a\delta'(b) + \delta'(a)b) = \delta(a)\delta'(b) + a\delta\delta'(b) + \delta\delta'(a)b + \delta'(a)\delta(b).$$

Now set $\delta = \delta'$, $a = b \in \mathfrak{A}$ so that $\delta^2(a^2) = 2(\delta(a))^2 + a\delta^2(a) + \delta^2(a)a \neq a\delta^2(a) + \delta^2(a)a$, unless $(\delta(a))^2 = 0$.

12. It is sufficient to show that if $y, z \in L$ are eigenvectors of $\operatorname{ad} x$ corresponding to λ and μ respectively, then $[y, z] \in L$ is also an eigenvector of $\operatorname{ad} x$ or zero. By the Jacobi identity,

$$\operatorname{ad} x([y,z]) = [x,[y,z]] = [[x,y],z] + [y,[x,z]] = [\lambda y, z] + [y, \mu z] = (\lambda+\mu)[y,z].$$