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ON THE GELFAND-NAIMARK DECOMPOSITION OF A NONSINGULAR MATRIX

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Abstract. Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ and $A \in \mathrm{GL}_n(\mathbb{F})$. Let $s(A) \in \mathbb{R}^n_+$ be the singular values of A, $\lambda(A) \in \mathbb{C}^n$ the unordered n-tuple of eigenvalues of A, $a(A) := \operatorname{diag} R \in \mathbb{R}^n_+$, where $A = QR$ is the QR decomposition of A, $u(A) := \operatorname{diag} U \in \mathbb{C}^n$, where $A = L\omega U$ is any Gelfand-Naimark decomposition. We obtain complete relations between (1) $u(A)$ and $a(A)$, (2) $u(A)$ and $s(A)$, (3) $u(A)$ and $\lambda(A)$, and (4) $a(A)$ and $\lambda(A)$. We also study the relations between any three elements among u, λ, a, s.

1. Introduction

Let $\mathrm{GL}_n(\mathbb{F})$ denote the group of $n \times n$ matrices over $\mathbb{F}$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$. There are several sets of important scalars associated with $A \in \mathrm{GL}_n(\mathbb{F})$. The well-known QR decomposition asserts that

$$A = QR,$$

where $Q \in \mathrm{U}_n(\mathbb{F})$ and R is upper triangular with positive diagonal entries. Here $\mathrm{U}_n(\mathbb{F})$ denotes the group of unitary matrices if $\mathbb{F} = \mathbb{C}$ and orthogonal matrices if $\mathbb{F} = \mathbb{R}$. The decomposition is unique and is the matrix version of the Gram-Schmidt orthonormalization process. The first set of scalars is

$$a(A) := \operatorname{diag} R = (r_{11}, \cdots, r_{nn}) \in \mathbb{R}^n_+. \tag{1.1}$$

Here $\operatorname{diag} X$ denotes the diagonal of $X \in \mathbb{F}_{n\times n}$. Notice that $a_i(A) := r_{ii}$ is the distance in terms of the 2-norm of the i-th column of A to the span of the previous $i-1$ columns of A, $i = 1, \ldots, n$ (we adopt the convention that the span of the empty set is the zero space).

Eigenvalues are certainly important and are often associated with the Schur triangularization, which asserts that there exists $U \in \mathrm{U}_n(\mathbb{C})$ and complex upper triangular T such that

$$A = U^*TU,$$

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where $\operatorname{diag} T = (\lambda_1, \ldots, \lambda_n)$ and the λ's are the eigenvalues of A. Moreover the order of the λ's can be prefixed. We denote by

$$\lambda(A) := (\lambda_1, \ldots, \lambda_n) \tag{1.2}$$

the *unordered* n-tuple of eigenvalues of A. However the real analog for $A \in \mathrm{GL}_n(\mathbb{R})$: $A = O^T TO$, for some $O \in \mathrm{U}_n(\mathbb{R})$ and T real upper triangular, is not true in general, since eigenvalues and eigenvectors may not be real.

Needless to say, singular values are important scalars. From the well-known Singular Value Decomposition (SVD) there exist $U, V \in \mathrm{U}_n(\mathbb{F})$ such that

$$A = U \operatorname{diag}(s_1, \ldots, s_n) V,$$

where $s_1 \geq \cdots \geq s_n$ are the singular values of $A \in \mathrm{GL}_n(\mathbb{F})$. Here $\operatorname{diag} v$ means the diagonal matrix with diagonal $v \in \mathbb{F}^n$. We write

$$s(A) := \operatorname{diag}(s_1, \ldots, s_n). \tag{1.3}$$

The Gelfand-Naimark decomposition [1] asserts that

$$A = L\omega U,$$

where $L \in \mathrm{GL}_n(\mathbb{F})$ is unit lower triangular, $U \in \mathrm{GL}_n(\mathbb{F})$ is upper triangular and ω is a permutation matrix. It is different from the Gauss elimination $A = \omega' L' U'$. Though the Gelfand-Naimark decomposition is less well-known, it has very nice properties. For example, the permutation matrix ω and

$$u(A) := \operatorname{diag} U \in \mathbb{F}^n$$

in $A = L\omega U$ are uniquely determined by A (see Section 2). However none of the components in $A = \omega' L' U'$ (Gauss elimination) is unique.

We already mentioned four sets of scalars associated with $A \in \mathrm{GL}_n(\mathbb{F})$, namely, $a(A)$, $\lambda(A)$, $s(A)$ and $u(A)$. When $\mathbb{F} = \mathbb{C}$, the relation between $s(A)$ and $\lambda(A)$ is completely determined by Weyl [7] and Horn [2] in terms of log majorization $|\lambda(A)| \prec_{\log} s(A)$:

$$\begin{aligned} \prod_{i=1}^k |\lambda_i'(A)| &\leq \prod_{i=1}^k s_i(A), \quad k = 1, \ldots, n-1, \\ \prod_{i=1}^n |\lambda_i'(A)| &= \prod_{i=1}^n s_i(A), \end{aligned} \tag{1.4}$$

where $\lambda_1'(A), \ldots, \lambda_n'(A)$ is a rearrangement of $\lambda_1(A), \ldots, \lambda_n(A)$ such that $|\lambda_1'(A)| \geq \cdots \geq |\lambda_n'(A)|$; and conversely if $|\lambda| \prec_{\log} s$ ($\lambda \in \mathbb{C}^n, s \in \mathbb{R}^n_+, s_1 \geq \cdots \geq s_n$), then there exists $A \in \mathrm{GL}_n(\mathbb{C})$ such that $\lambda(A) = \lambda$ and $s(A) = s$. Moreover A may be chosen to be a real matrix if the non-real numbers among $\lambda_1, \ldots, \lambda_n$ occur in complex conjugate pairs [6]. See [5, Theorem 5.4] for a nice generalization. Likewise, given $A \in \mathbb{F}_{n \times n}$, the relation between $s(A)$ and $a(A)$ is completely determined by log majorization, that is,

$$a(A) \prec_{\log} s(A), \tag{1.5}$$

and conversely if $a \prec_{\log} s$ ($a, s \in \mathbb{R}^n_+$), then there exists $A \in \mathrm{GL}_n(\mathbb{F})$ such that $s(A) = s$ and $a(A) = a$. The result is a special case of the nonlinear Kostant convexity theorem [5, Theorem 4.1] on the Iwasawa decomposition of a semisimple Lie group. We want to find complete relations

(1) between $u(A)$ and $a(A)$,
(2) between $u(A)$ and $s(A)$,
(3) between $u(A)$ and $\lambda(A)$, and
(4) between $a(A)$ and $\lambda(A)$.

Relations (1) and (2) will be given by the following partial order $\trianglelefteq$ on the set of positive n-tuples $\mathbb{R}^n_+$: Given $a, b \in \mathbb{R}^n_+$, $a \trianglelefteq b$ means

$$\begin{aligned} \prod_{i=1}^k a_i &\leq \prod_{i=1}^k b_i, \quad k = 1, \dots, n-1, \\ \prod_{i=1}^n a_i &= \prod_{i=1}^n b_i. \end{aligned}$$

It turns out the relation (1) is given by $|u(A)| \trianglelefteq a(A)$ and the relation (2) is given by $|u(A)| \trianglelefteq s(A)$. The partial order $\trianglelefteq$ looks very similar to log majorization $a \prec_{\log} b$. However, they are different since $\trianglelefteq$ does not require the entries of a and b in the above inequalities having descending order.

We will show that relation (3) is given by

$$\pm \prod_{i=1}^n u_i(A) = \prod_{i=1}^n \lambda_i(A),$$

and relation (4) is given by

$$\prod_{i=1}^n a_i(A) = \prod_{i=1}^n |\lambda_i(A)|.$$

We organize the paper in the following way. We first review some basic facts on the Gelfand-Naimark decomposition in Section 2. In Section 3 we obtain complete relations between $u(A)$ and $a(A)$, and between $u(A)$ and $s(A)$. In Section 4 a relation between $u(A)$ and $\lambda(A)$ is given. In Section 5 a relation between $a(A)$ and $\lambda(A)$ is also given. Together with the results of Weyl-Horn-Thompson and Kostant, namely (1.4) and (1.5), the relations between any two among u, s, λ, a are completely known. Finally in Section 6 we make some remarks on the relations between any three among u, s, λ, a.

2. Basics of Gelfand-Naimark decomposition

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{F}^n$, that is, $\mathbf{e}_i$ has 1 as the only nonzero entry at the i-th position. We identify a permutation $\omega \in S_n$ with the unique permutation matrix (also written as ω) in the general linear group $\mathrm{GL}_n(\mathbb{F})$, where $\omega \mathbf{e}_i = \mathbf{e}_{\omega(i)}$. The matrix representation of ω under the standard basis is

$$\omega = \left[\, \mathbf{e}_{\omega(1)}, \cdots, \mathbf{e}_{\omega(n)} \,\right].$$

So if $A = \begin{bmatrix} \mathbf{a}_1, \cdots, \mathbf{a}_n \end{bmatrix}$ in column form, then $A\omega = \begin{bmatrix} \mathbf{a}_{\omega(1)}, \cdots, \mathbf{a}_{\omega(n)} \end{bmatrix}$. Moreover, if $x_1, \ldots, x_n \in \mathbb{F}$, then

$$\omega^{-1}\mathrm{diag}\,(x_1, \ldots, x_n)\omega = \mathrm{diag}\,(x_{\omega(1)}, \ldots, x_{\omega(n)}). \tag{2.1}$$

Given a matrix $A \in \mathbb{F}_{n\times n}$, let $A(i|j)$ denote the submatrix formed by the first i rows and the first j columns of A, $1 \le i, j \le n$. The following proposition establishes the existence of the Gelfand-Naimark decomposition.

Proposition 2.1. Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$. Each $A \in \mathrm{GL}_n(\mathbb{F})$ has $A = L\omega U$, for a permutation matrix ω, a unit lower triangular matrix $L \in \mathrm{GL}_n(\mathbb{F})$, and an upper triangular $U \in \mathrm{GL}_n(\mathbb{F})$. The permutation matrix ω is uniquely determined by A:

$$\mathrm{rank}\,\omega(i|j) = \mathrm{rank}\,A(i|j) \qquad \text{for} \qquad 1 \le i, j \le n.$$

Moreover $\mathrm{diag}\, U$ is uniquely determined by A.

Proof. We first prove the existence of the decomposition $A = L\omega U$, which is indeed a matrix version of some sequence of elementary row and column operations applied to A.

Let a_{k1} be the first nonzero entry of the first column of A. By multiplying the first column of A by $1/a_{k1}$, we turn the $(k, 1)$ entry to 1. Using the 1 as a pivot, we consecutively eliminate other nonzero entries on the first column (using row operations) and the kth row (using column operations).

The above operations are equivalent to the following post- and pre-matrix multiplications: Let $D_1 = \mathrm{diag}\,(1/a_{k1}, 1, \ldots, 1) \in \mathrm{GL}_n(\mathbb{F})$. Denote by $A' = AD_1$. Let $E_{ij} \in \mathbb{R}_{n\times n}$ with (i, j) entry 1 as the only nonzero entry. Let

$$L_1 = (I - a'_{k+1,1}E_{k+1,k})(I - a'_{k+2,1}E_{k+2,k}) \cdots (I - a'_{n1}E_{nk}) \in \mathrm{GL}_n(\mathbb{F}),$$

a unit lower triangular matrix, and

$$U_1 = (I - a'_{k2}E_{12})(I - a'_{k3}E_{13}) \cdots (I - a'_{kn}E_{1n}) \in \mathrm{GL}_n(\mathbb{F}),$$

a unit upper triangular matrix. Then

$$A \qquad \to \qquad L_1AD_1 \qquad \to \qquad L_1AD_1U_1$$

$$\begin{bmatrix} 0 & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * & * & * & \cdots & * \\ a_{k1} & * & \cdots & * & * & * & \cdots & * \\ * & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & * & \cdots & * & * & * & \cdots & * \end{bmatrix} \to \begin{bmatrix} 0 & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * & * & * & \cdots & * \\ 1 & * & \cdots & * & * & * & \cdots & * \\ 0 & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * & * & * & \cdots & * \end{bmatrix} \to \begin{bmatrix} 0 & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * & * & * & \cdots & * \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * & * & * & \cdots & * \end{bmatrix}$$

Repeat the procedure on the second column of $L_1AD_1U_1$ and so on. Eventually we obtain a permutation matrix ω, unit lower triangular matrices $L_1, \ldots, L_n \in \mathrm{GL}_n(\mathbb{F})$, diagonal matrices $D_1, \ldots, D_n \in \mathrm{GL}_n(\mathbb{F})$, and unit upper triangular matrices $U_1, \ldots, U_n \in \mathrm{GL}_n(\mathbb{F})$ such that

$$L_n \cdots L_1AD_1U_1 \cdots D_nU_n = \omega.$$

Put

$$\begin{aligned} L^{-1} &:= L_n \cdots L_1, \qquad \text{and} \\ U^{-1} &:= D_1 U_1 \cdots D_n U_n. \end{aligned}$$

Then $A = L\omega U$ as desired. Since the group of nonsingular diagonal matrices normalizes the group of unit upper triangular matrices, we have $U^{-1} = U'D$ for some unit upper triangular matrix U', where $D := D_1 \cdots D_n$. So $U = D^{-1}U'^{-1}$. In other words, the i-th diagonal entry u_{ii} of U is indeed the first nonzero entry of the i-th column in the i-th elimination step.

By block multiplication we notice that

$$\begin{aligned} A(i|j) &= \begin{bmatrix} L(i|i) & 0 \end{bmatrix} \begin{bmatrix} \omega(i|j) & * \\ * & * \end{bmatrix} \begin{bmatrix} U(j|j) \\ 0 \end{bmatrix} \\ &= L(i|i)\omega(i|j)U(j|j). \end{aligned}$$

So $\operatorname{rank}\omega(i|j) = \operatorname{rank} A(i|j)$, $1 \le i,j \le n$. Obviously $\operatorname{rank}\omega(i|j)$ is the number of nonzero entries in $\omega(i|j)$. Thus it is easy to verify that ω_{ij} is nonzero if and only if

$$\operatorname{rank}\omega(i|j) - \operatorname{rank}\omega(i|j-1) - \operatorname{rank}\omega(i-1|j) + \operatorname{rank}\omega(i-1|j-1) = 1.$$

So the permutation matrix ω is uniquely determined by $\operatorname{rank}\omega(i|j)$, $1 \le i,j \le n$. Hence ω is uniquely determined by A.

If $L\omega U = L'\omega U'$ for another unit lower triangular L' and upper triangular U', then $\omega^{-1}L'^{-1}L\omega = U'U^{-1}$. Clearly the diagonal entries of $\omega^{-1}L'^{-1}L\omega$ are ones and thus $\operatorname{diag} U = \operatorname{diag} U'$. □

Remark 2.2. Although ω and $u(A)$ are unique in the Gelfand-Naimark decomposition $A = L\omega U$ of A, the components L and U may be not unique. For example,

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

So the elimination given in the proof of Proposition 2.1 corresponds to one but not all Gelfand-Naimark decompositions. Nevertheless $\operatorname{diag} U$ is unique and its entries can be thought of the "pivots" in the elimination.

In contrast, the permutation ω' in a Gauss elimination $A = \omega' L'U'$ may be not unique, but L' and U' are uniquely determined by ω'. For example,

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}.$$

Moreover, the ω in a Gelfand-Naimark decomposition $A = L\omega U$ of A can also be a permutation in a Gauss elimination $A = \omega L'U'$ of A. To see this, we notice that $\omega^{-1}A = (\omega^{-1}L\omega)U$ and $\det[(\omega^{-1}L\omega)(k|k)] = 1$ since $(\omega^{-1}L\omega)(k|k)$ is the submatrix formed by choosing the $\omega(1), \cdots, \omega(k)$ rows and columns of L. Therefore, by the LU algorithm [3], $\omega^{-1}L\omega = L_1U_1$ for some unit lower triangular L_1 and unit upper triangular U_1, and

$$A = L\omega U = \omega(\omega^{-1}L\omega)U = \omega L_1(U_1U) = \omega L'U', \tag{2.2}$$

where $L' := L_1$ and $U' := U_1 U$.

From (2.2) we also have $\omega^{-1} A = L_1 U_1 U$. Then $u(A)$ can be computed by

$$\det[(\omega^{-1}A)(k|k)] = \det[(L_1U_1U)(k|k)] = \det[U(k|k)] = \prod_{i=1}^{k} u_{ii}. \tag{2.3}$$

Remark 2.3. The above proof constructs a Gelfand-Naimark decomposition of A via an elimination process. Indeed each Gelfand-Naimark decomposition of $A = L\omega U$ is a matrix version of some elimination process. This is because $L^{-1}AU^{-1} = \omega$, where L^{-1} corresponds to a sequences of elementary row operations and U^{-1} corresponds to a sequences of elementary column operations.

Remark 2.4. When ω is the identity, it is well-known that [3] the decomposition $A = LU$ is unique.

Now the following is well-defined:

$$u(A) := \operatorname{diag} U = \operatorname{diag}(u_{11}, \dots, u_{nn}), \tag{2.4}$$

where $A = L\omega U$ is *any* Gelfand-Naimark decomposition of A.

3. u-components, a-components and singular values

Let $\mathbb{R}^n_+$ be the set of positive n-tuples. If $a, b \in \mathbb{R}^n_+$, we write $a \trianglelefteq b$ if

$$\prod_{i=1}^{k} a_i \le \prod_{i=1}^{k} b_i, \quad k = 1, \dots, n-1, \tag{3.1}$$

$$\prod_{i=1}^{n} a_i = \prod_{i=1}^{n} b_i \tag{3.2}$$

The partial order $\trianglelefteq$ is different from log majorization. For example, if $a = (3, 2)$ and $b = (1, 6)$, then $a \prec_{\log} b$ but $a \not\trianglelefteq b$. Indeed $b \trianglelefteq a$.

The following theorem gives a complete relation between the a-component and the u-component of $A \in \mathrm{GL}_n(\mathbb{F})$, in terms of the partial order $\trianglelefteq$. We denote by $C_k(A)$ the k-compound of $A \in \mathrm{GL}_n(\mathbb{F})$, where $1 \le k \le n$.

Theorem 3.1. Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. If $A \in \mathrm{GL}_n(\mathbb{F})$, then $|u(A)| \trianglelefteq a(A)$. Conversely if $a := (a_1, \dots, a_n)$, where $a_1, \dots, a_n > 0$, and $u := (u_1, \dots, u_n)$, where $u_1, \dots, u_n \in \mathbb{F}$, are nonzero numbers such that $|u| \trianglelefteq a$, then there exists $A \in \mathrm{GL}_n(\mathbb{F})$ such that $a(A) = a$ and $u(A) = u$ and A has an LU decomposition. Indeed, if $u \trianglelefteq a$, where $u_1, \dots, u_n > 0$, then there exists $Q \in \mathrm{SO}(n)$ such that $u((Q \operatorname{diag} a)) = u$ and $Q \operatorname{diag} a$ has an LU decomposition.

Proof. Suppose $A \in \mathbb{C}_{n \times n}$ and $A = QR = L\omega U$. Since $u_1(A)$ is the first nonzero entry of the first column of A and $a_1(A)$ is the 2-norm of the first column of A, we have

$$|u_1(A)| \le a_1(A). \tag{3.3}$$

Since

$$C_k(A) = C_k(Q)C_k(R), \qquad C_k(A) = C_k(L)C_k(\omega)C_k(U)$$

are the QR decomposition and the Gelfand-Naimark decomposition of $C_k(A)$, respectively, we apply (3.3) to $C_k(A)$. Since $u_1(C_k(A)) = \prod_{i=1}^k u_i(A)$ and $a_1(C_k(A)) = \prod_{i=1}^k a_i(A)$, $k = 1, \ldots, n-1$ and $\prod_{i=1}^n |u_i(A)| = \prod_{i=1}^n a_i(A)$ (follows from determinant considerations), we have $|u(A)| \trianglelefteq a(A)$.

Given $u \in \mathbb{C}^n, a \in \mathbb{R}_+^n$, we say that the pair (u, a) is $\mathbb{F}$-*realizable* if there exists $A \in \mathrm{GL}_n(\mathbb{F})$ such that $u(A) = u$ and $a(A) = a$. It is not hard to see that (u, a) is $\mathbb{F}$-realizable if and only if there exists $Q \in \mathrm{U}_n(\mathbb{F})$ such that $u = u(Q \operatorname{diag} a)$. We remark that

1. if (u, a) is $\mathbb{C}$-realizable, so is $(D_\theta u, a)$, where $D_\theta := \operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$, $\theta_1, \ldots, \theta_n \in \mathbb{R}$.
2. if (u, a) is $\mathbb{R}$-realizable, so is (Du, a), where $D := \operatorname{diag}(\pm 1, \ldots, \pm 1)$.

The reason is that (1) if $Q \operatorname{diag} a = L\omega U$, from (2.1) we get

$$\begin{aligned}
\operatorname{diag}(e^{i\theta_{\omega^{-1}(1)}}, \ldots, e^{i\theta_{\omega^{-1}(n)}})Q \operatorname{diag} a &= \operatorname{diag}(e^{i\theta_{\omega^{-1}(1)}}, \ldots, e^{i\theta_{\omega^{-1}(n)}})L\omega U \\
&= L'\operatorname{diag}(e^{i\theta_{\omega^{-1}(1)}}, \ldots, e^{i\theta_{\omega^{-1}(n)}})\omega U \\
&= L'(\omega\operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})\omega^{-1})\omega U \\
&= L'\omega\operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})U,
\end{aligned}$$

where $L' := \operatorname{diag}(e^{i\theta_{\omega^{-1}(1)}}, \ldots, e^{i\theta_{\omega^{-1}(n)}})L \operatorname{diag}(e^{-i\theta_{\omega^{-1}(1)}}, \ldots, e^{-i\theta_{\omega^{-1}(n)}})$ is still unit lower triangular. The real case (2) is similar.

We now prove the converse by induction on n. Because of the above remark it is sufficient to consider $u \in \mathbb{R}_+^n$ and prove the last statement in Theorem 3.1. Suppose $(u_1, u_2) \trianglelefteq (a_1, a_2)$, that is, $u_1 \le a_1$ and $u_1u_2 = a_1a_2$. Let

$$Q := \begin{bmatrix} p & -(1-p^2)^{1/2} \\ (1-p^2)^{1/2} & p \end{bmatrix} \in \mathrm{SO}(2),$$

where $p := u_1/a_1 \in (0, 1]$. The first column of

$$A := Q \operatorname{diag}(a_1, a_2) = \begin{bmatrix} a_1 p & -a_2(1-p^2)^{1/2} \\ a_1(1-p^2)^{1/2} & a_2 p \end{bmatrix}$$

has the first nonzero entry $a_1 p = u_1$ so that A has an LU decomposition. Clearly $a(A) = a$. Since $u_1u_2 = a_1a_2$, we have $u(A) = u$. Suppose that the statement is true for $n \le k$. Let $a = (a_1, \ldots, a_k, a_{k+1}), u = (u_1, \ldots, u_k, u_{k+1}) \in \mathbb{R}_+^{k+1}$ and $u \trianglelefteq a$. Then

$$u' := (u_1, \ldots, u_{k-1}, a_1 \cdots a_k/u_1 \cdots u_{k-1}) \trianglelefteq a' := (a_1, \ldots, a_k).$$

By the induction hypothesis there exists $Q' \in \mathrm{SO}(k)$ such that

$$A' := Q' \operatorname{diag}(a_1, \ldots, a_k)$$

has an LU decomposition satisfying $u(A') = u'$ and $a(A') = a'$. Set

$$Q_2 := \begin{bmatrix} t & -(1-t^2)^{1/2} \\ (1-t^2)^{1/2} & t \end{bmatrix} \in \mathrm{SO}(2),$$

where $t := u_1 \cdots u_k / a_1 \cdots a_k \leq 1$. The first $(k-1)$ rows of

$$A := (I_{k-1} \oplus Q_2)(Q' \oplus 1) \operatorname{diag} a$$

are those of $(Q' \oplus 1) \operatorname{diag} a$, so that the first $k-1$ rows of the matrices $A(k|k)$ and A' are identical and the last row of $A(k|k)$ is t times the last row of A'. So

$$\det A(k|k) = t a_1 \cdots a_k = u_1 \cdots u_k.$$

Moreover $A(k|k)$ has an LU decomposition and $u_i(A) = u_i$ for all $i = 1, \dots, k$. Hence $A = Q \operatorname{diag} a$ has an LU decomposition and is the required matrix, where $Q := (I_{k-1} \oplus Q_2)(Q' \oplus 1) \in \mathrm{SO}(k+1)$. □

Theorem 3.2. *Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. Let $A \in \mathrm{GL}_n(\mathbb{F})$. Then $|u(A)| \trianglelefteq a(A) \prec_{\log} s(A)$. Conversely, if $a := (a_1, \dots, a_n)$, $s := (s_1, \dots, s_n)$, $u := (u_1, \dots, u_n)$, where $a_1, \dots, a_n > 0$, $s_1 \geq \cdots \geq s_n > 0$, $u_1, \dots, u_n \in \mathbb{F}$ are nonzero numbers such that $|u| \trianglelefteq a \prec_{\log} s$, then there exists $A \in \mathrm{GL}_n(\mathbb{F})$ such that $u(A) = u$, $a(A) = a$ and $s(A) = s$.*

Proof. The relations $|u(A)| \trianglelefteq a(A) \prec_{\log} s(A)$ follow from Theorem 3.1 and (1.5).

Suppose $s_1 \geq \cdots \geq s_n$. By the nonlinear Kostant convexity theorem [5, Theorem 4.1] and the SVD, there exist $Q_1, Q_2 \in \mathrm{U}_n(\mathbb{F})$ such that $Q_1 (\operatorname{diag} s) Q_2$ has QR-decomposition $Q_1 (\operatorname{diag} s) Q_2 = Q_3 R$, where $Q_3 \in \mathrm{U}_n(\mathbb{F})$, R is upper triangular and $\operatorname{diag} R = a$. Write $R = (\operatorname{diag} a) V$, where V is unit upper triangular. By Theorem 3.1, there exists $Q \in \mathrm{U}_n(\mathbb{F})$ such that $Q \operatorname{diag} a = LU$, where $u(Q \operatorname{diag} a) = u$. Set

$$A := L(UV) = (LU)V = Q (\operatorname{diag} a) V = QR = Q Q_3^{-1} Q_1 s Q_2.$$

Then $s(A) = s$, $u(A) = \operatorname{diag}(UV) = u$ and $a(A) = a$. □

The following theorem gives a complete relation between the u-component and the singular values of $A \in \mathrm{GL}_n(\mathbb{F})$, also in terms of the partial order $\trianglelefteq$.

Corollary 3.3. *Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. If $A \in \mathrm{GL}_n(\mathbb{F})$, then $|u(A)| \trianglelefteq s(A)$. Conversely, if $s := (s_1, \dots, s_n)$ and $u := (u_1, \dots, u_n)$, where $s_1 \geq \cdots \geq s_n > 0$ and $u_1, \dots, u_n \in \mathbb{F}$ are nonzero numbers such that $|u| \trianglelefteq s$, then there exists $A \in \mathrm{GL}_n(\mathbb{F})$ such that A has an LU decomposition and $u(A) = u$ and $s(A) = s$.*

Proof. By Theorem 3.2, $|u(A)| \trianglelefteq s(A)$. For the converse, choose $a = s$ and apply Theorem 3.2. □

Remark 3.4. There is a slight difference between Theorem 3.1 and Theorem 3.3, that is, the entries of a may not be in descending order but the entries of s are already in descending order.

4. u-COMPONENTS AND EIGENVALUES

The following theorem gives a complete relation between the u-component and the eigenvalues of $A \in \mathrm{GL}_n(\mathbb{C})$.

Theorem 4.1. If $A \in \mathrm{GL}_n(\mathbb{C})$, then $\pm u_1(A) \cdots u_n(A) = \lambda_1(A) \cdots \lambda_n(A)$. Conversely, when $n \geq 2$, if $\pm u_1 \cdots u_n = \lambda_1 \cdots \lambda_n \neq 0$, where $u_1, \ldots, u_n \in \mathbb{C}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are nonzero numbers, then there exists $A \in \mathrm{GL}_n(\mathbb{C})$ such that $u(A) = u$ and $\lambda(A) = \lambda$. Moreover A may be chosen so that

(a) if $u_1 \cdots u_n = \lambda_1 \cdots \lambda_n$, then A has an LU decomposition,
(b) if $-u_1 \cdots u_n = \lambda_1 \cdots \lambda_n$, then A has a Gelfand-Naimark decomposition $L\omega U$, where ω is the transposition $(n-1, n)$.

When $n = 1$, only (a) is true.

Proof. Suppose that $A = L\omega U$ and $A = Q^*TQ$ are a Gelfand-Naimark decomposition and a Schur Triangularization of $A \in \mathrm{GL}_n(\mathbb{C})$, where $Q \in \mathrm{U}_n(\mathbb{C})$ and T is upper triangular with $\operatorname{diag} T = \lambda(A)$. Taking determinants yields $\pm u_1(A) \cdots u_n(A) = \lambda_1(A) \cdots \lambda_n(A)$.

We will prove the converse by induction on n for both (a) and (b).

When $n = 1$, (a) is obviously true and (b) will not happen.

Now consider $n = 2$.

Case (a): $u_1 u_2 = \lambda_1 \lambda_2$. Set

$$A := \begin{bmatrix} u_1 & \lambda_1 - u_1 \\ -\lambda_2 + u_1 & \lambda_1 + \lambda_2 - u_1 \end{bmatrix}.$$

Then $\operatorname{tr} A = \lambda_1 + \lambda_2$ and $\det A = \lambda_1 \lambda_2$. So $\lambda(A) = (\lambda_1, \lambda_2)$. Obviously A has an LU decomposition since $u_1 \neq 0$. Moreover $u_1(A) = u_1$ so that $u_2(A) = u_2$.

Case (b): $-u_1 u_2 = \lambda_1 \lambda_2$. Set

$$A := \begin{bmatrix} 0 & u_2 \\ u_1 & \lambda_1 + \lambda_2 \end{bmatrix}.$$

Then $u(A) = (u_1, u_2)$ and $\lambda(A) = (\lambda_1, \lambda_2)$ as desired and ω is the transposition.

Suppose that the statements are true for $n \leq k$ where $k \geq 2$. Let

$$\lambda = (\lambda_1, \ldots, \lambda_k, \lambda_{k+1}), \quad u = (u_1, \ldots, u_k, u_{k+1}) \in \mathbb{C}^{k+1}$$

such that $\lambda_1 \cdots \lambda_k \lambda_{k+1} = \pm u_1 \cdots u_k u_{k+1} \neq 0$. Then

$$u' := (u_1, \ldots, u_{k-1}, \frac{\lambda_1 \cdots \lambda_k}{u_1 \cdots u_{k-1}}), \quad \lambda' := (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$$

satisfy $\lambda'_1 \cdots \lambda'_k = u'_1 \cdots u'_k \neq 0$. So by the induction hypothesis there exists $A' \in \mathrm{GL}_k(\mathbb{C})$ such that $u(A') = u'$, $\lambda(A') = \lambda'$, and A' has an LU decomposition. Let

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \in \mathrm{SO}(2).$$

Set

$$A \;:=\; \begin{bmatrix} I_{k-1} & 0 \\ 0 & Q^* \end{bmatrix} \begin{bmatrix} A' & y \\ 0 & \lambda_{k+1} \end{bmatrix} \begin{bmatrix} I_{k-1} & 0 \\ 0 & Q \end{bmatrix} \in \mathrm{GL}_{k+1}(\mathbb{C}), \tag{4.1}$$

where $y \in \mathbb{C}^k$ is an indeterminate vector. Clearly $\lambda(A) = \lambda$. Moreover, $A(k-1|k-1) = A'(k-1|k-1)$. Since A' has an LU decomposition, we have $u_i(A) = u_i(A') = u_i$ for $i = 1, \cdots, k-1$.

Case (a): $u_1 \cdots u_{k+1} = \lambda_1 \cdots \lambda_{k+1}$. Let $y := t\mathbf{e}_k \in \mathbb{C}^k$, where $t \in \mathbb{C}$ is to be determined. From (4.1) and the fact that $\det A(k-1|k-1) = u_1 \cdots u_{k-1} \neq 0$, $\det A(k|k)$ is a polynomial of t of degree one. By choosing an appropriate $t \in \mathbb{C}$, we can make $\det A(k|k) = u_1 \cdots u_k \neq 0$, which implies that

$$u_k(A) = \det A(k|k) / \det A(k-1|k-1) = u_k.$$

From (4.1), $\det A = \lambda_{k+1} \det A' = \lambda_1 \cdots \lambda_{k+1} = u_1 \cdots u_{k+1}$. So

$$u_{k+1}(A) = \det A / \det A(k|k) = u_{k+1}$$

and A has an LU decomposition.

Case (b): $-u_1 \cdots u_{k+1} = \lambda_1 \cdots \lambda_{k+1}$. Put $A'' := A'(k-1|k-1)$. Write

$$A' = \begin{bmatrix} A'' & \beta \\ \alpha^T & \mu \end{bmatrix}, \qquad y = \begin{bmatrix} y' \\ y_k \end{bmatrix} \in \mathbb{C}^k,$$

where $\alpha, \beta, y' \in \mathbb{C}^{k-1}$ and $\mu, y_k \in \mathbb{C}$. Direct computation on (4.1) yields

$$\begin{aligned} A \;&=\; \begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} A'' & \beta & y' \\ \alpha^T & \mu & y_k \\ 0 & 0 & \lambda_{k+1} \end{bmatrix} \begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &=\; \begin{bmatrix} A'' & \frac{1}{\sqrt{2}}(\beta - y') & \frac{1}{\sqrt{2}}(\beta + y') \\ \frac{1}{\sqrt{2}}\alpha^T & \frac{1}{2}(\mu - y_k + \lambda_{k+1}) & \frac{1}{2}(\mu + y_k - \lambda_{k+1}) \\ \frac{1}{\sqrt{2}}\alpha^T & \frac{1}{2}(\mu - y_k - \lambda_{k+1}) & \frac{1}{2}(\mu + y_k + \lambda_{k+1}) \end{bmatrix}. \end{aligned}$$

We want to have a Gelfand-Naimark decomposition $A = L\omega U$, where ω is the transposition $(k, k+1)$. Set $y' = \beta$ and $y_k = \mu + \lambda_{k+1}$ so that

$$\beta - y' = 0 \in \mathbb{C}^k, \qquad \mu - y_k + \lambda_{k+1} = 0 \in \mathbb{C},$$

and thus

$$A = \begin{bmatrix} A'' & 0 & \sqrt{2}\beta \\ \frac{1}{\sqrt{2}}\alpha^T & 0 & \mu \\ \frac{1}{\sqrt{2}}\alpha^T & -\lambda_{k+1} & \mu + \lambda_{k+1} \end{bmatrix}.$$

By Proposition 2.1, the permutation ω in the Gelfand-Naimark decomposition of $A = L\omega U$ is the transposition $(k, k+1)$. However,

$$u(A) = (u_1, \cdots, u_{k-1}, -\lambda_{k+1}, \frac{\lambda_1 \cdots \lambda_k}{u_1 \cdots u_{k-1}}).$$

Let $D := I_k \oplus (-\lambda_{k+1}/u_k)$ and $A_0 := D^{-1}AD$. Then A_0 has the same ω as A in its Gelfand-Naimark decomposition, and clearly $\lambda(A_0) = \lambda$ and $u(A_0) = u$. Reset $A = A_0$ and we are done. □

If $A \in \mathrm{GL}_n(\mathbb{R})$, the non-real eigenvalues of A occur in complex conjugate pairs. It turns out that this is the only additional requirement for the real case but the proof is more involved.

Theorem 4.2. If $A \in \mathrm{GL}_n(\mathbb{R})$, then $\pm u_1(A)\cdots u_n(A) = \lambda_1(A)\cdots\lambda_n(A)$. Conversely, when $n \geq 2$, if $\pm u_1 \cdots u_n = \lambda_1 \cdots \lambda_n \neq 0$, where $u_1, \ldots, u_n \in \mathbb{R}$, and the non-real numbers of $\lambda_1, \ldots, \lambda_n$ occur in complex conjugate pairs, then there exists $A \in \mathrm{GL}_n(\mathbb{R})$ such that $u(A) = u$ and $\lambda(A) = \lambda$, where $u := (u_1, \ldots, u_n)$ and $\lambda := (\lambda_1, \ldots, \lambda_n)$. Moreover A may be chosen so that

(a) if $u_1 \cdots u_n = \lambda_1 \cdots \lambda_n$, then A has an LU decomposition,

(b) if $-u_1 \cdots u_n = \lambda_1 \cdots \lambda_n$, then A has a Gelfand-Naimark decomposition $A = L\omega U$, where ω is the transposition $(1,2)$, provided that $n \geq 2$.

When $n = 1$, only (a) is true.

Proof. The relation $\pm u_1(A)\cdots u_n(A) = \lambda_1(A)\cdots\lambda_n(A)$ is contained in Theorem 4.1. We now prove the converse by induction.

When $n = 1$, (a) is obviously true and (b) will not happen.

When $n = 2$, suppose $\pm u_1 u_2 = \lambda_1\lambda_2 (\neq 0)$. Let

$$A := \begin{cases} \begin{bmatrix} u_1 & \lambda_1+\lambda_2-u_1-u_2 \\ u_1 & \lambda_1+\lambda_2-u_1 \end{bmatrix} & \text{if } u_1u_2 = \lambda_1\lambda_2, \\[2ex] \begin{bmatrix} 0 & u_2 \\ u_1 & \lambda_1+\lambda_2 \end{bmatrix} & \text{if } -u_1u_2 = \lambda_1\lambda_2. \end{cases} \tag{4.2}$$

Then $A \in \mathrm{GL}_2(\mathbb{R})$ since $\lambda_1 + \lambda_2$ is real. Clearly $\lambda(A) = (\lambda_1, \lambda_2)$, $u(A) = (u_1, u_2)$, and (a) and (b) are true.

When $n = 3$, suppose $\pm u_1u_2u_3 = \lambda_1\lambda_2\lambda_3 (\neq 0)$ and the non-real numbers in $\lambda_1, \lambda_2, \lambda_3$ form a complex conjugate pair. Then at least one of $\lambda_1, \lambda_2, \lambda_3$ is real. Without loss of generality, suppose $\lambda_3 \in \mathbb{R}$. Put

$$\rho_1 := \lambda_1+\lambda_2+\lambda_3, \quad \rho_2 := \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1, \quad \rho_3 := \lambda_1\lambda_2\lambda_3. \tag{4.3}$$

Set

$$A := \begin{cases} \begin{bmatrix} u_1 & -u_2 & -\frac{\rho_2}{u_1}+\rho_1-u_1+u_2 \\ u_1 & 0 & \rho_1-u_1-u_3 \\ u_1 & 0 & \rho_1-u_1 \end{bmatrix} & \text{if } u_1u_2u_3 = \lambda_1\lambda_2\lambda_3, \\[2ex] \begin{bmatrix} 0 & u_2 & -\frac{\lambda_1\lambda_2}{u_1}-u_2 \\ u_1 & 0 & \lambda_1+\lambda_2 \\ u_1 & -\lambda_3 & \rho_1 \end{bmatrix} & \text{if } -u_1u_2u_3 = \lambda_1\lambda_2\lambda_3. \end{cases} \tag{4.4}$$

Then $A \in \mathrm{GL}_3(\mathbb{R})$ and $\lambda(A) = (\lambda_1, \lambda_2, \lambda_3)$, $u(A) = (u_1, u_2, u_3)$, and (a) and (b) are true.

Suppose that the statements are true for $n \le k$, where $k \ge 3$. Let

$$\lambda = (\lambda_1, \dots, \lambda_k, \lambda_{k+1}) \in \mathbb{C}^{k+1}, \quad u = (u_1, \dots, u_k, u_{k+1}) \in \mathbb{R}^{k+1}$$

such that $\pm u_1 \cdots u_k u_{k+1} = \lambda_1 \cdots \lambda_k \lambda_{k+1} (\ne 0)$ and the non-real λ's occur in complex conjugate pairs. We now consider two situations.

Situation A: $\lambda_i \in \mathbb{R}$ for some $i = 1, \cdots, k+1$. Without loss of generality we assume that $\lambda_{k+1} \in \mathbb{R}$.

Case (a): If $u_1 \cdots u_{k+1} = \lambda_1 \cdots \lambda_{k+1}$, let

$$u' := (u_1, \dots, u_{k-1}, \frac{\lambda_1 \cdots \lambda_k}{u_1 \cdots u_{k-1}}), \quad \lambda' := (\lambda_1, \dots, \lambda_k), \tag{4.5}$$

then $u'_1 \cdots u'_k = \lambda'_1 \cdots \lambda'_k \ne 0$. By the induction hypothesis there exists $A' \in \mathrm{GL}_k(\mathbb{R})$ such that $u(A') = u'$, $\lambda(A') = \lambda'$, and A' has an LU decomposition.

Case (b): If $-u_1 \cdots u_{k+1} = \lambda_1 \cdots \lambda_{k+1}$, let

$$u' := (u_1, \dots, u_{k-1}, -\frac{\lambda_1 \cdots \lambda_k}{u_1 \cdots u_{k-1}}), \quad \lambda' := (\lambda_1, \dots, \lambda_k), \tag{4.6}$$

then $-u'_1 \cdots u'_k = \lambda'_1 \cdots \lambda'_k \ne 0$. By the induction hypothesis there exists $A' \in \mathrm{GL}_k(\mathbb{R})$ such that $u(A') = u'$, $\lambda(A') = \lambda'$, and A' has a Gelfand-Naimark decomposition $L\omega U$, where ω is the transposition $(1, 2)$.

In both cases, write $A' = \begin{bmatrix} A'' & \beta \\ \alpha^T & \mu \end{bmatrix}$, where $A'' := A'(k-1|k-1)$, $\alpha, \beta \in \mathbb{R}^{k-1}$ and $\mu \in \mathbb{R}$. Let $t \in \mathbb{R}$ be an indeterminate and set

$$A := \begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}^{-1} \begin{bmatrix} A'' & \beta & 0 \\ \alpha^T & \mu & t \\ 0 & 0 & \lambda_{k+1} \end{bmatrix} \begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \tag{4.7}$$

$$= \begin{bmatrix} A'' & \frac{1}{2}\beta & \beta \\ \alpha^T & \frac{1}{2}(\mu - t + \lambda_{k+1}) & \mu + t - \lambda_{k+1} \\ \frac{1}{2}\alpha^T & \frac{1}{4}(\mu - t - \lambda_{k+1}) & \frac{1}{2}(\mu + t + \lambda_{k+1}) \end{bmatrix}. \tag{4.8}$$

Then $\lambda(A) = \lambda$ and $A(k-1|k-1) = A'' = A'(k-1|k-1)$, where $k-1 \ge 2$. By Proposition 2.1 (about ω) and (2.3), in both cases $u_i(A) = u_i(A') = u_i$ for $i = 1, \cdots, k-1$. From (4.8) and the fact that $\det A'' = \pm u_1 \cdots u_{k-1} \ne 0$, $\det A(k|k)$ is a polynomial of t of degree 1. Choose $t \in \mathbb{R}$ such that

$$\det A(k|k) = \begin{cases} u_1 \cdots u_k & \text{in case (a)} \\ -u_1 \cdots u_k & \text{in case (b).} \end{cases} \tag{4.9}$$

Then $u_k(A) = u_k$ and $A(k|k)$ has the same ω as A'. From (4.7), $\det A = \lambda_1 \cdots \lambda_{k+1}$. So $u_{k+1}(A) = u_{k+1}$ by (4.9). Moreover, A has an LU decomposition in case (a), and A has a Gelfand-Naimark decomposition $L\omega U$ in case (b), where ω is the transposition $(1, 2)$.

In the above two cases $\lambda(A) = \lambda$, $u(A) = u$.

Situation B: All $\lambda_1, \cdots, \lambda_{k+1}$ occur in complex conjugate pairs. So $k \geq 3$ is odd. Without loss of generality, we assume $\lambda_{k+1} = \overline{\lambda}_k$.

Case (a): If $u_1 \cdots u_{k+1} = \lambda_1 \cdots \lambda_{k+1}$, let

$$u' := (u_1, \cdots, u_{k-2}, \frac{\lambda_1 \cdots \lambda_{k-1}}{u_1 \cdots u_{k-2}}), \qquad \lambda' := (\lambda_1, \cdots, \lambda_{k-1}), \tag{4.10}$$

then $u'_1 \cdots u'_{k-1} = \lambda'_1 \cdots \lambda'_{k-1}$. By the induction hypothesis, there exists $A' \in \mathrm{GL}_{k-1}(\mathbb{R})$, where $u(A') = u'$, $\lambda(A') = \lambda'$, and A' has an LU decomposition.

Case (b): If $-u_1 \cdots u_{k+1} = \lambda_1 \cdots \lambda_{k+1}$, let

$$u' := (u_1, \ldots, u_{k-2}, -\frac{\lambda_1 \cdots \lambda_{k-1}}{u_1 \cdots u_{k-2}}), \quad \lambda' := (\lambda_1, \ldots, \lambda_{k-1}), \tag{4.11}$$

then $-u'_1 \cdots u'_{k-1} = \lambda'_1 \cdots \lambda'_{k-1} \neq 0$. By the induction hypothesis there exists $A' \in \mathrm{GL}_{k-1}(\mathbb{R})$ such that $u(A') = u'$, $\lambda(A') = \lambda'$, and A' has a Gelfand-Naimark decomposition $L\omega U$, where ω is the transposition $(1, 2)$.

In both cases, put $A'' := A'(k-2|k-2)$. Write $A' := \begin{bmatrix} A'' & \beta \\ \alpha^T & \mu \end{bmatrix}$, where $\alpha, \beta \in \mathbb{R}^{k-2}$ and $\mu \in \mathbb{R}$. Let $t \in \mathbb{R}$ and $0 \neq s \in \mathbb{R}$ be indeterminates. Set

$$p \;:=\; \lambda_k + \lambda_{k+1} - s - \frac{\lambda_k \lambda_{k+1}}{s}, \tag{4.12}$$

$$q \;:=\; \lambda_k + \lambda_{k+1} - s. \tag{4.13}$$

Then $\begin{bmatrix} s & p \\ s & q \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R})$ has eigenvalues λ_k and λ_{k+1}.

Possibility (i): Suppose $k > 3$, or $k = 3$ and $u_1u_2u_3u_4 = \lambda_1\lambda_2\lambda_3\lambda_4$ (case (a)). We have $\det A'' = \pm u_1 \cdots u_{k-2} \neq 0$. Set

$$A := \begin{bmatrix} I_{k-2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A'' & \beta & 0 & 0 \\ \alpha^T & \mu & t & 0 \\ 0 & 0 & s & p \\ 0 & 0 & s & q \end{bmatrix} \begin{bmatrix} I_{k-2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.14}$$

$$= \begin{bmatrix} A'' & \frac{1}{2}\beta & \beta & 0 \\ \alpha^T & \frac{1}{2}(\mu - t + s) & \mu + t - s & -p \\ \frac{1}{2}\alpha^T & \frac{1}{4}(\mu - t - s) & \frac{1}{2}(\mu + t + s) & \frac{1}{2}p \\ 0 & -\frac{1}{2}s & s & q \end{bmatrix}. \tag{4.15}$$

Then $A \in \mathrm{GL}_{k+1}(\mathbb{R})$ and $\lambda(A) = \lambda$ by (4.14). On the one hand,

$$\det A(k|k) \;=\; \det \begin{bmatrix} A'' & \beta & 0 \\ \alpha^T & \mu & t \\ 0 & 0 & s \end{bmatrix} = s\lambda_1 \cdots \lambda_{k-1} \tag{4.16}$$

so that we can choose $0 \neq s \in \mathbb{R}$ such that

$$\det A(k|k) = \begin{cases} u_1 \cdots u_k & \text{in case (a),} \\ -u_1 \cdots u_k & \text{in case (b).} \end{cases}$$

On the other hand, by (4.15), $A(k-2|k-2) = A'' = A'(k-2|k-2)$. Then

$$\det A(k-2|k-2) = \det A'' = \pm u_1 \cdots u_{k-2} \neq 0$$

and $u_i(A) = u_i$ for $i = 1, \cdots, k-2$. By (4.15), $\det A(k-1|k-1)$ is a real polynomial of t of degree 1. We can choose $t \in \mathbb{R}$ such that

$$\det A(k-1|k-1) = \begin{cases} u_1 \cdots u_{k-1} & \text{in case (a)} \\ -u_1 \cdots u_{k-1} & \text{in case (b).} \end{cases}$$

Then $u_i(A) = u_i$ for $i = k-1, k, k+1$. Clearly, A has an LU decomposition in case (a), and A has a Gelfand-Naimark decomposition $L\omega U$, where $\omega = (1, 2)$, in case (b).

Possibility (ii): Suppose $k = 3$ and $-u_1u_2u_3u_4 = \lambda_1\lambda_2\lambda_3\lambda_4$ (case (b)). Let $s := -\frac{u_1u_2u_3}{\lambda_1\lambda_2}$, and let p and q be defined as in (4.12) and (4.13). Set

$$\begin{aligned} A \ &:= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -\frac{\lambda_1\lambda_2}{u_1} & -\frac{\lambda_1\lambda_2+2u_1u_2}{u_1} & 0 \\ u_1 & \lambda_1+\lambda_2 & 0 & 0 \\ 0 & 0 & s & p \\ 0 & 0 & s & q \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & u_2 & \frac{-2(\lambda_1\lambda_2+u_1u_2)}{u_1} & 0 \\ u_1 & \frac{1}{2}(\lambda_1+\lambda_2+s) & \lambda_1+\lambda_2-s & -p \\ \frac{1}{2}u_1 & \frac{1}{4}(\lambda_1+\lambda_2-s) & \frac{1}{2}(\lambda_1+\lambda_2+s) & \frac{1}{2}p \\ 0 & -\frac{1}{2}s & s & q \end{bmatrix}. \end{aligned}$$

Then $A \in \mathrm{GL}_4(\mathbb{R})$, $\lambda(A) = \lambda$, $u_1(A) = u_1$ and $u_2(A) = u_2$. Notice that

$$\det A(3|3) = \det \begin{bmatrix} 0 & -\frac{\lambda_1\lambda_2}{u_1} & -\frac{\lambda_1\lambda_2+2u_1u_2}{u_1} \\ u_1 & \lambda_1+\lambda_2 & 0 \\ 0 & 0 & s \end{bmatrix} = -u_1u_2u_3.$$

So $u_3(A) = u_3$ and thus $u_4(A) = u_4$. Moreover, the ω in a Gelfand-Naimark decomposition of A is the transposition $(1, 2)$.

Therefore, we complete the proof by induction. □

5. a-COMPONENT, EIGENVALUES AND SINGULAR VALUES

Let $A \in \mathrm{GL}_n(\mathbb{F})$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. We first determine the relation among $s(A)$, $\lambda(A)$, and $a(A)$. Then we use the result to determine the relation between the a-component and the eigenvalues of A.

Theorem 5.1. If $A \in \mathrm{GL}_n(\mathbb{C})$, then $a(A) \prec_{\log} s(A)$ and $|\lambda(A)| \prec_{\log} s(A)$. Conversely, if $a := (a_1, \cdots, a_n)$, $s := (s_1, \cdots, s_n)$, $\lambda := (\lambda_1, \cdots, \lambda_n)$, where $a_1, \cdots, a_n > 0$, $s_1 \geq \cdots \geq s_n > 0$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are nonzero numbers such that $a \prec_{\log} s$ and $|\lambda| \prec_{\log} s$, then there exists $A \in \mathrm{GL}_n(\mathbb{C})$ such that $a(A) = a$, $\lambda(A) = \lambda$ and $s(A) = s$. Moreover, A may be chosen to be real if the non-real numbers among $\lambda_1, \ldots, \lambda_n$ occur in complex conjugate pairs.

Proof. The relation $a(A) \prec_{\log} s(A)$ and $|\lambda(A)| \prec_{\log} s(A)$ are known [5, 7].

We now establish the converse. Since $|\lambda| \prec_{\log} s$, by [2], there exists $A_0 \in \mathrm{GL}_n(\mathbb{C})$ such that $\lambda(A_0) = \lambda$ and $s(A_0) = s$. By the SVD write $A_0 = K_1 (\operatorname{diag} s) K_2$, where $K_1, K_2 \in \mathrm{U}_n(\mathbb{C})$. Since $a \prec_{\log} s$ by [5, Theorem 4.1] there exists $V \in \mathrm{U}_n(\mathbb{C})$ such that $a((\operatorname{diag} s) V) = a$. Let

$$\begin{aligned} A &:= V^{-1} K_2 K_1 (\operatorname{diag} s) V \\ &= (K_2^{-1} V)^{-1} (K_1 (\operatorname{diag} s) K_2)(K_2^{-1} V) \\ &= (K_2^{-1} V)^{-1} A_0 (K_2^{-1} V). \end{aligned} \tag{5.1}$$

Then $a(A) = a((\operatorname{diag} s) V) = a$, $\lambda(A) = \lambda(A_0) = \lambda$, and $s(A) = s(A_0) = s$, as desired.

Suppose that the non-real numbers among $\lambda_1, \ldots, \lambda_n$ occur in complex conjugate pairs. Since $|\lambda| \prec_{\log} s$, by Thompson's result [6] there exists $A_0 \in \mathrm{GL}_n(\mathbb{R})$ such that $\lambda(A_0) = \lambda$ and $s(A_0) = s$. Write $A_0 = K_1 (\operatorname{diag} s) K_2$, where $K_1, K_2 \in \mathrm{U}_n(\mathbb{R})$. Since $a \prec_{\log} s$ by [5, Theorem 4.1] there exists $V \in \mathrm{U}_n(\mathbb{R})$ such that $a((\operatorname{diag} s) V) = a$. Then follow the argument in (5.1) to get the desired result. □

Corollary 5.2. If $A \in \mathrm{GL}_n(\mathbb{C})$, then $a_1(A) \cdots a_n(A) = |\lambda_1(A) \cdots \lambda_n(A)|$. Conversely, if $a := (a_1, \cdots, a_n)$ and $\lambda := (\lambda_1, \cdots, \lambda_n)$, where $a_1, \cdots, a_n > 0$ and $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ are nonzero numbers such that $a_1 \cdots a_n = |\lambda_1 \cdots \lambda_n|$, then there exists $A \in \mathrm{GL}_n(\mathbb{C})$ such that $a(A) = a$ and $\lambda(A) = \lambda$. Moreover, A may be chosen to be real if the non-real numbers among $\lambda_1, \ldots, \lambda_n$ occur in complex conjugate pairs.

Proof. We only need to prove the converse. Choose $s_1 \geq \cdots \geq s_n > 0$ such that $s_1, s_2, \ldots, s_{n-1}$ are sufficiently large and $s_1 \cdots s_n = a_1 \cdots a_n$ so that $a \prec_{\log} s$ and $|\lambda| \prec_{\log} s$. By Theorem 5.1 there exists $A \in \mathrm{GL}_n(\mathbb{C})$ such that $a(A) = a$, $\lambda(A) = \lambda$, and $s(A) = s$. Moreover A may be chosen to be real if the non-real numbers among $\lambda_1, \ldots, \lambda_n$ occur in complex conjugate pairs. □

6. u, λ, s and u, λ, a

In Theorem 3.2 we see that $|u| \trianglelefteq a \prec_{\log} s$ is necessary and sufficient for the existence of $A \in \mathrm{GL}_n(\mathbb{F})$ such that $u(A) = u$, $a(A) = a$ and $s(A) = s$. Notice that the conditions $|u| \trianglelefteq a \prec_{\log} s$ are equivalent to the three conditions $|u| \trianglelefteq a$, $a \prec_{\log} s$ and $|u| \trianglelefteq s$, since the last is implied by the first two conditions (we assume $s_1 \geq \cdots \geq s_n$). In other words, the totality of the pairwise conditions (see Theorem 3.1, (1.5), Theorem 3.3) among u, a, s are the necessary and sufficient conditions for the existence of $A \in \mathrm{GL}_n(\mathbb{F})$ such that $u(A) = u$, $a(A) = a$ and $s(A) = s$.

Similarly in Theorem 5.1 the conditions $a \prec_{\log} s$ and $|\lambda| \prec_{\log} s$ are necessary and sufficient for the existence of $A \in \mathrm{GL}_n(\mathbb{C})$ such that $a(A) = a$, $\lambda(A) = \lambda$ and $s(A) = s$. Notice that $a \prec_{\log} s$ and $|\lambda| \prec_{\log} s$ imply

$a_1 \cdots a_n = |\lambda_1 \cdots \lambda_n|$. In other words, the totality of the pairwise conditions (see (1.5), (1.4), Corollary 5.2) among u, λ, a are the necessary and sufficient conditions for the existence of $A \in \mathrm{GL}_n(\mathbb{C})$ such that $u(A) = u$, $a(A) = a$ and $\lambda(A) = \lambda$. The real case is similar and the only difference is that the non-real numbers among $\lambda_1, \ldots, \lambda_n$ occur in complex conjugate pairs.

However the three pairwise conditions among u, λ, s: $|u| \trianglelefteq s$ (in Corollary 3.3), $|\lambda| \prec_{\log} s$ in (1.4) and $\pm u_1 \cdots u_n = \lambda_1 \cdots \lambda_n$ (in Theorem 4.1) do not suffice to ensure the existence of $A \in \mathrm{GL}_n(\mathbb{F})$ such that $u(A) = u$, $\lambda(A) = \lambda$ and $s(A) = s$. For example if we set $\lambda = s$, $s_1 \geq \cdots \geq s_n$, then A with $\lambda(A) = s(A) = s$ must be a positive definite matrix. By the Cholesky decomposition, we have $A = T^*T$ for some upper triangular matrix $T \in \mathrm{GL}_n(\mathbb{F})$ with positive diagonal entries. Notice that

$$u_i(A) = |\lambda_i(T)|^2, \quad s_i(A) = s_i^2(T), \qquad i = 1, \ldots, n$$

so that $u(A) \prec_{\log} s(A)$ by Weyl's result (1.4) on T. Clearly $u(A) \prec_{\log} s(A)$ is not necessarily implied by $|u| \trianglelefteq s$. For example $s = \lambda = (3, 2)$, $u = (1, 6)$.

If we consider u, λ, a, the pairwise conditions $\pm u_1 \cdots u_n = \lambda_1 \cdots \lambda_n$ (in Theorem 4.1), $|u| \trianglelefteq a$ (in Theorem 3.1) and $|\lambda_1 \cdots \lambda_n| = a_1 \cdots a_n$ (in Theorem 5.2) are not sufficient to ensure the existence of an $A \in \mathrm{GL}_n(\mathbb{C})$ such that $u(A) = u$, $\lambda(A) = \lambda$ and $a(A) = a$ (indeed the last condition $|\lambda_1 \cdots \lambda_n| = a_1 \cdots a_n$ is implied by the first two). For example, consider $n = 2$. Suppose $u_1 u_2 = \lambda_1 \lambda_2$. Choose u_1 such that $|u_1| = a_1$ (then $|u_2| = a_2$). If there existed $A \in \mathrm{GL}_2(\mathbb{C})$ such that $u(A) = u$, $\lambda(A) = \lambda$ and $a(A) = a$, then A would be of the form

$$A = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}$$

and thus λ_1, λ_2 would be u_1, u_2. This does not necessarily follow from $u_1 u_2 = \lambda_1 \lambda_2$, $(u_1, u_2) \trianglelefteq (a_1, a_2)$. For example, $u = a = (3, 2)$, $\lambda = (1, 6)$.

The following proposition is straightforward computation.

Proposition 6.1. Suppose that $u_1, u_2, \lambda_1, \lambda_2 \in \mathbb{C}$ are nonzero numbers and $a_1, a_2 > 0$.

(1) If $u_1 u_2 = \lambda_1 \lambda_2$ and $(|u_1|, |u_2|) \trianglelefteq (a_1, a_2)$ such that $|u_1| \neq a_1$, then

$$A = \begin{bmatrix} u_1 & \frac{u_1(\lambda_1 + \lambda_2 - u_1) - \lambda_1 \lambda_2}{\sqrt{a_1^2 - |u_1|^2}} \\ \sqrt{a_1^2 - |u_1|^2} & \lambda_1 + \lambda_2 - u_1 \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

satisfies $\lambda(A) = \lambda$, $u(A) = u$, $a(A) = a$. In addition, if $u_1, u_2 \in \mathbb{R}$ and if λ_1, λ_2 are real or form a complex conjugate pair, then $A \in \mathrm{GL}_2(\mathbb{R})$.

(2) If $u_1 u_2 = \lambda_1 \lambda_2$ and $(|u_1|, |u_2|) = (a_1, a_2)$, then $A \in \mathrm{GL}_2(\mathbb{C})$ satisfying $u(A) = u$, $\lambda(A) = \lambda$ and $a(A) = a$ must be of the form:

$$A = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}$$

so that λ_1, λ_2 are u_1, u_2.

(3) If $-u_1u_2 = \lambda_1\lambda_2$ and $(|u_1|, |u_2|) \trianglelefteq (a_1, a_2)$, then $A \in \mathrm{GL}_2(\mathbb{C})$ satisfying $u(A) = u$, $\lambda(A) = \lambda$ and $a(A) = a$ must be of the form:

$$A = \begin{bmatrix} 0 & u_2 \\ u_1 & \lambda_1 + \lambda_2 \end{bmatrix}$$

so that $(|u_1|, |u_2|) = (a_1, a_2)$. In addition, if $u_1, u_2 \in \mathbb{R}$ and if λ_1, λ_2 are real or form a complex conjugate pair, then $A \in \mathrm{GL}_2(\mathbb{R})$.

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