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On four kinds of scalars of a nonsingular matrix

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Colloquium

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Goal: Discuss four sets of scalars of a nonsingular matrix

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Outline

- 3 sets of scalars
- Gelfand-Naimark decomposition
- 2 parties
- 3 parties
1. 3 sets of scalars of a nonsingular matrix

\[ \text{GL}_n(F) = n \times n \text{ nonsingular matrices over } F \text{ where } F = \mathbb{C} \text{ or } F = \mathbb{R}. \]

\[ QR \text{ decomposition:} \]

\[ A = QR, \]

where \( Q \in \text{U}_n(F) \) and \( R \) is upper \( \Delta \) with positive diagonal entries.

Here \( \text{U}_n(F) \) denotes the group of unitary matrices if \( F = \mathbb{C} \) and orthogonal matrices if \( F = \mathbb{R} \).
The first set of scalars is

\[ a(A) := \text{diag } R = (r_{11}, \cdots, r_{nn}) \in \mathbb{R}^n_+. \]

**Remark:** \( a_i(A) \) is the distance in terms of 2-norm of the \( i \)-th column of \( A \) to the span of the previous \( i - 1 \) columns of \( A \), \( i = 1, \ldots, n \) (we adopt the convention that the span of the empty set is the zero space).
Eigenvalues

Schur triangularization: There is $U \in U_n(\mathbb{C})$ and upper $\Delta T$ such that

$$A = U^* T U,$$

where $\text{diag } T = (\lambda_1, \ldots, \lambda_n)$ and $\lambda$'s are the eigenvalues of $A$. Moreover the order of $\lambda$'s can be prefixed. We denote by

$$\lambda(A) := (\lambda_1, \ldots, \lambda_n)$$

the unordered $n$-tuple of eigenvalues of $A$. 
Singular values

SVD asserts that there are $U, V \in U_n(\mathbb{F})$ such that

$$A = U \text{diag} (s_1, \ldots, s_n) V,$$

where $s_1 \geq \cdots \geq s_n$ are the singular values of $A \in \text{GL}_n(\mathbb{F})$. Here $\text{diag} \, \nu$ means the diagonal matrix with diagonal $\nu \in \mathbb{F}^n$. We denote by

$$s(A) := \text{diag} (s_1, \ldots, s_n).$$
Relation between $\lambda(A)$ and $s(A)$

\textbf{Weyl: } $|\lambda(A)| \prec_{\log} s(A)$:

$$\prod_{i=1}^{k} |\lambda'_i(A)| \leq \prod_{i=1}^{k} s_i(A), \quad k = 1, \ldots, n - 1,$$

$$\prod_{i=1}^{n} |\lambda'_i(A)| = \prod_{i=1}^{n} s_i(A),$$

where $\lambda'_1(A), \ldots, \lambda'_n(A)$ are the rearrangements of $\lambda_1(A), \ldots, \lambda_n(A)$ such that

$$|\lambda'_1(A)| \geq \cdots \geq |\lambda'_n(A)|.$$
Horn: Conversely if $|\lambda| \prec_{\log} s$ ($\lambda \in \mathbb{C}^n$, $s \in \mathbb{R}_+^n$, $s_1 \geq \cdots \geq s_n$), then there exists $A \in \text{GL}_n(\mathbb{C})$ such that $\lambda(A) = \lambda$ and $s(A) = s$.

Thompson: $A$ may be chosen to be a real matrix if the non-real numbers among $\lambda_1, \ldots, \lambda_n$ occur in complex conjugate pairs.
Relation between $a(A)$ and $s(A)$

Kostant’s theorem:

$$a(A) \prec_{\log} s(A),$$

and conversely if $a \prec_{\log} s$ ($a, s \in \mathbb{R}_+^n$), then there exists $A \in \text{GL}_n(\mathbb{F})$ such that $s(A) = s$ and $a(A) = a$.

The result is a special case of Kostant’s nonlinear convexity theorem on Iwasawa decomposition of a semisimple Lie group.
2. Gelfand-Naimark decomposition and the 4th set of scalars

**Proposition 2.1.** Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$. Each $A \in \text{GL}_n(\mathbb{F})$ has $A = L\omega U$, for a permutation matrix $\omega$, a unit lower triangular matrix $L \in \text{GL}_n(\mathbb{F})$, and an upper triangular $U \in \text{GL}_n(\mathbb{F})$. The permutation matrix $\omega$ is uniquely determined by $A$:

$$\text{rank } \omega(i|j) = \text{rank } A(i|j) \quad \text{for} \quad 1 \leq i, j \leq n.$$ 

Moreover $\text{diag } U$ is uniquely determined by $A$. Here $A(i|j)$ denote the submatrix formed by the first $i$ rows and the first $j$ columns of $A$, $1 \leq i, j \leq n$. 
Proof:

\[
\begin{align*}
A & \quad \rightarrow \quad L_1AD_1 & \quad \rightarrow \quad L_1AD_1U_1 \\
\begin{bmatrix}
0 & \cdots & \ast & \cdots & \ast & \cdots & \ast \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & \ast & \cdots & \ast & \cdots & \ast \\
\ast & \cdots & \ast & \cdots & \ast & \cdots & \ast \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\ast & \cdots & \ast & \cdots & \ast & \cdots & \ast \\
\ast & \cdots & \ast & \cdots & \ast & \cdots & \ast \\
\end{bmatrix} & \quad \rightarrow \quad \begin{bmatrix}
0 & \cdots & \ast & \cdots & \ast & \cdots & \ast \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & \cdots & \ast & \cdots & \ast & \cdots & \ast \\
0 & \cdots & \ast & \cdots & \ast & \cdots & \ast \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & \ast & \cdots & \ast & \cdots & \ast \\
0 & \cdots & \ast & \cdots & \ast & \cdots & \ast \\
\end{bmatrix} & \quad \rightarrow \quad \begin{bmatrix}
0 & \cdots & \ast & \cdots & \ast & \cdots & \ast \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & \ast & \cdots & \ast & \cdots & \ast \\
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \ast & \cdots & \ast & \cdots & \ast \\
0 & \cdots & \ast & \cdots & \ast & \cdots & \ast \\
0 & \cdots & \ast & \cdots & \ast & \cdots & \ast \\
\end{bmatrix}
\end{align*}
\]

Repeat the procedure on the second column of \( L_1AD_1U_1 \) and so on. Eventually we obtain a permutation matrix \( \omega \), unit lower triangular matrices \( L_1, \ldots, L_n \in \text{GL}_n(\mathbb{F}) \), diagonal matrices \( D_1, \ldots, D_n \in \text{GL}_n(\mathbb{F}) \), and unit upper triangular matrices \( U_1, \ldots, U_n \in \text{GL}_n(\mathbb{F}) \) such that

\[
L_n \cdots L_1AD_1U_1 \cdots D_nU_n = \omega.
\]

Denote

\[
L^{-1} = L_n \cdots L_1, \quad \text{and} \quad U^{-1} = D_1U_1 \cdots D_nU_n.
\]

Then \( A = L\omega U \) as desired.
Since the group of nonsingular diagonal matrices normalizes the group of unit upper triangular matrices, $U^{-1} = U'D$ for some unit upper triangular matrix $U'$, where $D := D_1 \cdots D_n$. So $U = D^{-1}U'^{-1}$. In other words, the $i$-th diagonal entry $u_{ii}$ of $U$ is indeed the first nonzero entry of the $i$-th column in the $i$-th elimination step.

By block multiplication we notice that

$$A(i|j) = \begin{bmatrix} L(i|i) & 0 \\ \omega(i|j) & * \end{bmatrix} \begin{bmatrix} U(j|j) \end{bmatrix} = L(i|i)\omega(i|j)U(j|j).$$

So $\text{rank } \omega(i|j) = \text{rank } A(i|j), 1 \leq i, j \leq n$. Obviously $\text{rank } \omega(i|j)$ is the number of nonzero entries in $\omega(i|j)$. Thus it is easy to verify that $\omega_{ij}$ is nonzero if and only if

$$\text{rank } \omega(i|j) - \text{rank } \omega(i|j - 1) - \text{rank } \omega(i - 1|j) + \text{rank } \omega(i - 1|j - 1) = 1.$$

So the permutation matrix $\omega$ is uniquely determined by $\text{rank } \omega(i|j), 1 \leq i, j \leq n$. Hence $\omega$ is uniquely determined by $A$. 
The 4th set of scalars

If \( L\omega U = L'\omega U' \) for another unit lower triangular \( L' \) and upper triangular \( U' \), then

\[
\omega^{-1}L'^{-1}L\omega = U'U^{-1}.
\]

Clearly the diagonal entries of \( \omega^{-1}L'^{-1}L\omega \) are ones and thus

\[
\text{diag } U = \text{diag } U'
\]

So we define

\[
u (A) := \text{diag } U = \text{diag } (u_{11}, \ldots, u_{nn}) \in \mathbb{F}^n,
\]

where \( A = L\omega U \) is any Gelfand-Naimark decomposition of \( A \).
Remark: Although $\omega$ and $u(A)$ are unique in Gelfand-Naimark decomposition $A = L\omega U$ of $A$, the $L$ and $U$ components may be not unique.

We want to find complete relations

1. between $u(A)$ and $a(A)$,
2. between $u(A)$ and $s(A)$,
3. between $u(A)$ and $\lambda(A)$, and
4. between $a(A)$ and $\lambda(A)$.
Weyl-Horn-Thompson

\[ \lambda \leftrightarrow S \leftrightarrow \kappa \]

↓ × ↓ Kostant

\[ u \leftrightarrow a \]
A partial order $\leq$

Given $a, b \in \mathbb{R}^n_+$, $a \leq b$ means

\[
\prod_{i=1}^{k} a_i \leq \prod_{i=1}^{k} b_i, \quad k = 1, \ldots, n - 1,
\]

\[
\prod_{i=1}^{n} a_i = \prod_{i=1}^{n} b_i.
\]

The partial order $\leq$ looks very similar to log majorization $a \prec_{\log} b$. However they are different since $\leq$ does not require the entries of $a$ and $b$ in the above inequalities having descending order.
The partial order $\preceq$ is different from log majorization. For example, if

$$a = (3, 2), \quad b = (1, 6),$$

then

$$a \preceq_{\log} b$$

but

$$a \npreceq b.$$

Indeed $b \preceq a.$
3. Two Parties

\( u \) and \( a \)

**Theorem 3.1.** Let \( F = \mathbb{C} \) or \( \mathbb{R} \). If \( A \in \text{GL}_n(F) \), then
\[ |u(A)| \leq a(A). \]
Conversely if \( a := (a_1, \ldots, a_n) \) where \( a_1, \ldots, a_n > 0 \) and \( u := (u_1, \ldots, u_n) \) where \( u_1, \ldots, u_n \in F \) are nonzero numbers such that \( |u| \leq a \), then there exists \( A \in \text{GL}_n(F) \) such that \( a(A) = a \) and \( u(A) = u \) and \( A \) has \( LU \) decomposition.

Indeed, if \( u \leq a \), where \( u_1, \ldots, u_n > 0 \), then there exists \( Q \in \text{SO}(n) \) such that \( u(Q \text{ diag } a) = u \) and \( Q \text{ diag } a \) has \( LU \) decomposition.
\[ u \text{ and } s \]

**Theorem 3.2.** Let \( \mathbb{F} = \mathbb{C} \) or \( \mathbb{R} \). If \( A \in \text{GL}_n(\mathbb{F}) \), then
\[ |u(A)| \leq s(A). \]

Conversely, if \( s := (s_1, \ldots, s_n) \) and \( u := (u_1, \ldots, u_n) \), where \( s_1 \geq \cdots \geq s_n > 0 \) and \( u_1, \ldots, u_n \in \mathbb{F} \) are nonzero numbers such that \( |u| \leq s \), then there exists \( A \in \text{GL}_n(\mathbb{F}) \) such that \( A \) has \( LU \) decomposition and \( u(A) = u \) and \( s(A) = s \).
\( a \text{ and } \lambda \)

**Theorem 3.3.** If \( A \in \text{GL}_n(\mathbb{C}) \), then
\[
a_1(A) \cdots a_n(A) = |\lambda_1(A) \cdots \lambda_n(A)|.
\]
Conversely, if \( a := (a_1, \cdots, a_n) \) and \( \lambda := (\lambda_1, \cdots, \lambda_n) \), where \( a_1, \cdots, a_n > 0 \) and \( \lambda_1, \cdots, \lambda_n \in \mathbb{C} \) are nonzero numbers such that \( a_1 \cdots a_n = |\lambda_1 \cdots \lambda_n| \), then there exists \( A \in \text{GL}_n(\mathbb{C}) \) such that \( a(A) = a \) and \( \lambda(A) = \lambda \). Moreover, \( A \) may be chosen to be real if the non-real numbers among \( \lambda_1, \ldots, \lambda_n \) appear in complex conjugate pairs.
\textbf{Theorem 3.4.} If \( A \in \text{GL}_n(\mathbb{C}) \), then \( \pm u_1(A) \cdots u_n(A) = \lambda_1(A) \cdots \lambda_n(A) \).

Conversely, when \( n \geq 2 \), if \( \pm u_1 \cdots u_n = \lambda_1 \cdots \lambda_n \neq 0 \), where \( u_1, \ldots, u_n \in \mathbb{C}, \lambda_1, \ldots, \lambda_n \in \mathbb{C} \) are nonzero, then there is \( A \in \text{GL}_n(\mathbb{C}) \) such that \( u(A) = u \) and \( \lambda(A) = \lambda \). Moreover \( A \) may be chosen so that

(a) if \( u_1 \cdots u_n = \lambda_1 \cdots \lambda_n \), then \( A \) has \( LU \) decomposition,

(b) if \( -u_1 \cdots u_n = \lambda_1 \cdots \lambda_n \), then \( A \) has \( L\omega U \) decomposition where \( \omega \) is the transposition \( (n-1, n) \).

When \( n = 1 \), only (a) is true.
$u$ and $\lambda$: Real case

If $A \in \text{GL}_n(\mathbb{R})$, the non-real eigenvalues of $A$ appear in complex conjugate pairs. It turns out that this is the only additional requirement for the real case.

**Theorem 3.5.** If $A \in \text{GL}_n(\mathbb{R})$, then $\pm u_1(A) \cdots u_n(A) = \lambda_1(A) \cdots \lambda_n(A)$.

Conversely, when $n \geq 2$, if $\pm u_1 \cdots u_n = \lambda_1 \cdots \lambda_n \neq 0$, where $u_1, \ldots, u_n \in \mathbb{R}$, and the non-real numbers of $\lambda_1, \ldots, \lambda_n$ appear in complex conjugate pairs, then there exists $A \in \text{GL}_n(\mathbb{R})$ such that $u(A) = u$ and $\lambda(A) = \lambda$, where $u := (u_1, \ldots, u_n)$ and $\lambda := (\lambda_1, \ldots, \lambda_n)$. Moreover $A$ may be chosen so that

(a) if $u_1 \cdots u_n = \lambda_1 \cdots \lambda_n$, then $A$ has $LU$ decomposition,

(b) if $-u_1 \cdots u_n = \lambda_1 \cdots \lambda_n$, then $A$ has $L\omega U$ decomposition where $\omega$ is the transposition $(1, 2)$, provided that $n \geq 2$.

When $n = 1$, only (a) is true.
Conjecture 3.1. Let $u_1, \cdots, u_n$ and $\lambda_1, \cdots, \lambda_n$ be given in the last theorem.

(a) If $u_1 \cdots u_n = \lambda_1 \cdots \lambda_n$, then for every even permutation $\omega$, there exists $A \in \text{GL}_n(\mathbb{C})$ such that $\lambda(A) = \lambda$, $u(A) = u$, and $A$ has the Gelfand-Naimark decomposition $A = L\omega U$.

(b) If $u_1 \cdots u_n = -\lambda_1 \cdots \lambda_n$, then for every odd permutation $\omega$, there exists $A \in \text{GL}_n(\mathbb{C})$ such that $\lambda(A) = \lambda$, $u(A) = u$, and $A$ has the Gelfand-Naimark decomposition $A = L\omega U$. 
4. 3 parties

\( u, a \) and \( s \)

**Theorem 4.1.** Let \( \mathbb{F} = \mathbb{C} \) or \( \mathbb{R} \). Let \( A \in \text{GL}_n(\mathbb{F}) \). Then \( |u(A)| \leq a(A) \prec \log s(A) \).

Conversely, if \( a := (a_1, \ldots, a_n), s := (s_1, \ldots, s_n), u := (u_1, \ldots, u_n) \), where \( a_1, \ldots, a_n > 0, s_1 \geq \cdots \geq s_n > 0, u_1, \ldots, u_n \in \mathbb{C} \) are nonzero numbers such that \( |u| \leq a \prec \log s \), then there exists \( A \in \text{GL}_n(\mathbb{F}) \) such that \( u(A) = u, a(A) = a \) and \( s(A) = s \).
Theorem 4.2. If $A \in \text{GL}_n(\mathbb{C})$, then $a(A) \prec_{\log} s(A)$ and $|\lambda(A)| \prec_{\log} s(A)$.

Conversely, if $a := (a_1, \ldots, a_n)$, $s := (s_1, \ldots, s_n)$, $\lambda := (\lambda_1, \ldots, \lambda_n)$ where $a_1, \ldots, a_n > 0$, $s_1 \geq \cdots \geq s_n > 0$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are nonzero numbers such that $a \prec_{\log} s$ and $|\lambda| \prec_{\log} s$, then there exists $A \in \text{GL}_n(\mathbb{C})$ such that $a(A) = a$, $\lambda(A) = \lambda$ and $s(A) = s$. Moreover, $A$ may be chosen to be real if the non-real numbers among $\lambda_1, \ldots, \lambda_n$ appear in complex conjugate pairs.
However the three pairwise conditions on $u$, $\lambda$, $s$

\[ |u| \leq s, \quad |\lambda| < \log s \]

and

\[ \pm u_1 \cdots u_n = \lambda_1 \cdots \lambda_n \]

do not suffice to ensure the existence of $A \in \text{GL}_n(F)$
such that $u(A) = u$, $\lambda(A) = \lambda$ and $s(A) = s$. 
For example if we set \( \lambda = s, s_1 \geq \cdots \geq s_n > 0 \), then \( A \) with \( \lambda(A) = s(A) = s \) must be a positive definite matrix.

By **Cholesky decomposition**, \( A = T^*T \) for some upper triangular matrix \( T \in \text{GL}_n(\mathbb{F}) \) with positive diagonal entries. Notice that

\[
\begin{align*}
    u_i(A) &= |\lambda_i(T)|^2, \\
    s_i(A) &= s_i^2(T), \\
    i &= 1, \ldots, n
\end{align*}
\]

so that \( u(A) \prec_{\text{log}} s(A) \) by Weyl’s result. Clearly \( u(A) \prec_{\text{log}} s(A) \) is not necessarily implied by \( |u| \trianglelefteq s \).

For example

\[
    s = \lambda = (3, 2), \quad u = (1, 6).
\]
The pairwise conditions

\[ \pm u_1 \cdots u_n = \lambda_1 \cdots \lambda_n \]

\[ |u| \leq a, \quad |\lambda_1 \cdots \lambda_n| = a_1 \cdots a_n \]

are not sufficient to ensure the existence of an \( A \in \text{GL}_n(\mathbb{C}) \) such that \( u(A) = u, \lambda(A) = \lambda \) and \( a(A) = a \) (indeed the last condition \( |\lambda_1 \cdots \lambda_n| = a_1 \cdots a_n \) is implied by the first two).
Suppose $u_1 u_2 = \lambda_1 \lambda_2$. Choose $u_1$ such that $|u_1| = a_1$ (then $|u_2| = a_2$). If there is $A \in \text{GL}_2(\mathbb{C})$ such that $u(A) = u$, $\lambda(A) = \lambda$ and $a(A) = a$, then $A$ would be of the form

$$A = \text{diag} (u_1, u_2)$$

and thus $\lambda_1, \lambda_2$ would be $u_1, u_2$.

It does not necessarily follow from $u_1 u_2 = \lambda_1 \lambda_2$, $(u_1, u_2) \trianglelefteq (a_1, a_2)$, say,

$$u = a = (3, 2) \quad \lambda = (1, 6).$$
Proposition 4.1. Suppose that $u_1, u_2, \lambda_1, \lambda_2 \in \mathbb{C}$ are nonzero numbers and $a_1, a_2 > 0$.

1. If $u_1 u_2 = \lambda_1 \lambda_2$ and $(|u_1|, |u_2|) \preceq (a_1, a_2)$ such that $|u_1| \neq a_1$, then

$$A = \begin{bmatrix} u_1 & \frac{u_1(\lambda_1 + \lambda_2 - u_1) - \lambda_1 \lambda_2}{\sqrt{a_1^2 - |u_1|^2}} \\ \sqrt{a_1^2 - |u_1|^2} & \lambda_1 + \lambda_2 - u_1 \end{bmatrix} \in \text{GL}_2(\mathbb{C})$$

satisfies $\lambda(A) = \lambda$, $u(A) = u$, $a(A) = a$. In addition, if $u_1, u_2 \in \mathbb{R}$ and if $\lambda_1, \lambda_2$ are real or are complex conjugate pair, then $A \in \text{GL}_2(\mathbb{R})$.

2. If $u_1 u_2 = \lambda_1 \lambda_2$ and $(|u_1|, |u_2|) = (a_1, a_2)$, then $A \in \text{GL}_2(\mathbb{C})$ satisfying $u(A) = u$, $\lambda(A) = \lambda$ and $a(A) = a$ must be of the form:

$$A = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}$$

so that $\lambda_1, \lambda_2$ are $u_1, u_2$.

3. If $-u_1 u_2 = \lambda_1 \lambda_2$ and $(|u_1|, |u_2|) \preceq (a_1, a_2)$, then $A \in \text{GL}_2(\mathbb{C})$ satisfying $u(A) = u$, $\lambda(A) = \lambda$ and $a(A) = a$ must be of the form:

$$A = \begin{bmatrix} 0 & u_2 \\ u_1 & \lambda_1 + \lambda_2 \end{bmatrix}$$

so that $(|u_1|, |u_2|) = (a_1, a_2)$. In addition, if $u_1, u_2 \in \mathbb{R}$ and if $\lambda_1, \lambda_2$ are real or are complex conjugate pair, then $A \in \text{GL}_2(\mathbb{R})$. 
THANK YOU FOR YOUR ATTENTION