DETERMINANTS OF SUM OF ORBITS UNDER COMPACT LIE GROUP

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Abstract. We study the determinants on the sum of orbits of two elements in the Lie algebra of a compact connected subgroup in the unitary group. As an application, the extremal determinant expressions are obtained for the symplectic group.

1. Introduction

Let $U(n) \subset \mathbb{C}_{n \times n}$ be the unitary group. Recall a result of Fiedler [2].

Theorem 1.1. (Fiedler) Let $A, B \in \mathbb{C}_{n \times n}$ be Hermitian matrices with eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$. Then

$$\{ \det(UAU^{-1} + VBV^{-1}) : U, V \in U(n) \}$$

is the interval $[\min_{\sigma \in S_n} \prod_{i=1}^{n} (\alpha_i + \beta_{\sigma(i)}), \max_{\sigma \in S_n} \prod_{i=1}^{n} (\alpha_i + \beta_{\sigma(i)})]$, where $S_n$ denotes the symmetric group on $\{1, \ldots, n\}$.

The following is an immediate consequence of Theorem 1.1. It also follows from [6, Corollary 2.3].

Corollary 1.2. All elements in $\Delta(A, B)$ are singular if and only if there is $\mu \in \mathbb{R}$ such that $\mu$ is an eigenvalue of $A$ with multiplicity $m_1$ and $-\mu$ is an eigenvalues of $B$ with multiplicity $m_2$ such that $m_1 + m_2 > n$.

Remark 1.3. The permutation that yields the maximum or minimum may not do the same if we translate $A$ or $B$ when $n \geq 3$. For example, if $A = \text{diag}(5, 2, 1)$ and $B = \text{diag}(6, 3, 1)$, then the maximum is $\det M_1 = 210$, where $M_1 := \text{diag}(5, 2, 1) + \text{diag}(1, 3, 6) = 210$. Set $M_2 := \text{diag}(5, 2, 1) + \text{diag}(1, 6, 3)$ so that $\det M_2 = 192$. But $\det(M_1 - 6.5I) = 0.375 < 1.875 = \det(M_2 - 6.5)$.

Let $G \subset U(n)$ be a compact connected subgroup of $U(n)$. We want to extend Theorem 1.1 in the context of compact connected subgroups of $U(n)$.

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Let $\mathfrak{g}$ denote the Lie algebra of $G$. Suppose $A, B \in i\mathfrak{g} \subset H_n$, where $H_n := \mathfrak{u}(n)$ is the space of $n \times n$ Hermitian matrices. Set

$$D(A, B) := \{ \det(UAU^{-1} + VBV^{-1}) : U, V \in G \} \subset \mathbb{R}. $$

Let $T$ be a maximal torus of $G$ with Lie algebra $\mathfrak{t}$. So $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ \cite[p.98]{7}. Since $G$ is a matrix group, adjoint action is merely conjugation so that \cite[p.288]{1} each $X \in \mathfrak{g}$ is conjugate to some element in $\mathfrak{t}$, i.e., there is $g \in G$ such that $gXg^{-1} \in \mathfrak{t}$. Thus we may assume that $A, B \in i\mathfrak{t}$. Moreover, since $G$ is compact connected and the determinant function is continuous, $D(A, B) = \{ \det(A + VBV^{-1}) : V \in G \} = [m, M]$.

Let $W$ be the Weyl group of $(G, T)$ \cite[p.136]{7}, i.e., $W = N_G(T)/T$, where $N_G(T)$ denotes the normalizer of $T$ in $G$. The Weyl group acts on $\mathfrak{t}$ (thus on $i\mathfrak{t}$) via the adjoint action (and thus conjugation) and we denote the action of $\omega \in W$ on $A \in i\mathfrak{t}$ by $\omega \cdot A$.

Given $A, B \in i\mathfrak{t}$, denote by

$$\Delta(A, B) := \{ UAU^{-1} + VBV^{-1} : U, V \in G \} \subset i\mathfrak{g} \subset H_n,$$

the sum of the orbits of $A$ and $B$.

In Section 2 we determine $m$ and $M$ under some conditions. Then the result is used in Section 3 for the symplectic group $\text{Sp}(k)$.

### 2. Determinant and sum of orbits

Recall a lemma of Fiedler \cite{2}.

Lemma 2.1. (Fiedler) Let $P, Q \in \mathbb{C}^{n \times n}$ and $P$ be nonsingular. Then

$$\det(P + \epsilon Q) = (\det P)(1 + \epsilon \text{tr } QP^{-1}) + o(\epsilon^2).$$

The following is a slight extension of Lemma 2.1. It is of independent interest and the proof is short.

Proposition 2.2. Let $P, Q \in \mathbb{C}^{n \times n}$. Then

$$\det(P + \epsilon Q) = \det P + \epsilon \text{tr } (Q \text{ adj } P) + o(\epsilon^2),$$

where adj $P$ denotes the adjugate of $P$.

Proof. By Cauchy-Binet formula \cite[p.22]{3} \[\det(P + \epsilon Q) = \det \begin{bmatrix} P & I \end{bmatrix} \begin{bmatrix} I \\ \epsilon Q \end{bmatrix} = \det P + \epsilon \text{tr } (Q \text{ adj } P) + o(\epsilon^2). \] $\square$
Theorem 2.3. Let \( A, B \in i \mathfrak{t} \). Assume that \([X^{-1}, Y] \in \mathfrak{g}\) for any \( X, Y \in \mathfrak{g} \), whenever \( X \) is nonsingular. Suppose that the extremum \( \xi = m \) or \( M \) is nonzero. Then \( \xi = \det(A + \omega \cdot B) \) for some \( \omega \in W \).

Proof. By continuity argument we may assume that \( iA \in \mathfrak{t} \) is regular, i.e., the centralizer \( Z_{\mathfrak{g}}(iA) \) of \( iA \) in \( \mathfrak{g} \) is a Cartan subalgebra of \( \mathfrak{g} \), since regular elements of \( \mathfrak{t} \) form an open dense set in \( \mathfrak{t} \) \([7, p.156]\) and the determinant function is continuous.

Suppose that the extremum \( \xi \neq 0 \) occurs at the optimizing matrix

\[
C_0 := A + B_0 \in i\mathfrak{g},
\]

where

\[
B_0 = V_0 B V_0^{-1}
\]

for some \( V_0 \in G \). Let \( S \in \mathfrak{g} \). So the matrix exponential

\[
e^{-\epsilon S} := I + \epsilon S + \frac{1}{2!} \epsilon^2 S^2 + \cdots \in G,
\]

for all \( \epsilon \in \mathbb{R} \) \([4, Prop 1.74, p.45]\). By Lemma 2.1

\[
\det(A + e^{\epsilon S} B_0 e^{-\epsilon S}) = \det(A + e^{\epsilon \text{ad} S} B_0)
\]

\[
= \det(A + B_0 + \epsilon [S, B_0]) + o(\epsilon^2)
\]

\[
= \det C_0(1 + \epsilon \text{tr} [S, B_0] C_0^{-1}) + o(\epsilon^2)
\]

\[
= \det C_0(1 + \epsilon \text{tr} S [B_0, C_0^{-1}]) + o(\epsilon^2).
\]

So \( \text{tr} S [B_0, C_0^{-1}] = 0 \) for all \( S \in \mathfrak{g} \). Thus \( [B_0, C_0^{-1}] \) is in the orthogonal complement \( \mathfrak{g}^\perp \) in \( H_n \) with respect to the inner product \( (X, Y) := \text{tr} X Y \) on \( H_n \). By the assumption \([B_0, C_0^{-1}] \in \mathfrak{g}\) so we obtain \([B_0, C_0^{-1}] = 0\), i.e., \( B_0 \) commutes with \( C_0^{-1} \) and thus with \( C_0 \). Hence

\[
B_0(A + B_0) = (A + B_0)B_0,
\]

i.e., \( B_0 A = A B_0 \). Since \( iA \in \mathfrak{t} \) is regular \([7, p.101]\), \( Z_{\mathfrak{g}}(iA) = \mathfrak{t} \) so that \( iB_0 \in \mathfrak{t} \). Since the intersection with \( \mathfrak{t} \) of the orbit of \( G \) is the orbit of \( W \) on \( \mathfrak{t} \) \([1] \ p.294\), \( B_0 = \omega \cdot B \) for some \( \omega \in W \).

Remark 2.4. We can derive Theorem 1.1 via Theorem 2.3. Let \( A, B \in H_n \). The Weyl group for \( U(n) \) is the symmetric group with \( T \subset U(n) \) being the set of diagonal matrices in \( U(n) \). Clearly the inverse of a nonsingular Hermitian matrix remains Hermitian. So the condition \([X^{-1}, Y] \in \mathfrak{u}(n)\) in Theorem 2.3 is readily satisfied. If the extremum \( \xi \) (\( m \) or \( M \)) is nonzero, then apply Theorem 2.3 to yield Theorem 1.1. Suppose \( \xi = 0 \), say \( m = \det C_0 = 0 \) with \( U C_0 C^{-1} = \).
diag \((c_1, \ldots, c_r, 0, \ldots, 0)\) for some \(U \in U(n)\), where \(c_1, \ldots, c_r \neq 0\). We can perturb \(A\) to \(A + UDU^{-1}\) \((D := \text{diag} (\delta_1, \ldots, \delta_n)\), where \(\delta_i \in \mathbb{R}\) \(i = 1, \ldots, n)\) so that \(\det(UDU^{-1} + C_0) < 0\). However \(\det(UDU^{-1} + C_0) \in D(A + UDU^{-1}, B)\) so that the optimizing matrix \(C_0\) for \(D(A + UDU^{-1}, B)\) is nonsingular. Then use Theorem 2.3 and apply continuity argument since \(\det C_0\) depends on \(A\) continuously [2, p.29].

**Remark 2.5.**

1. The condition \([X^{-1}, Y] \subset \mathfrak{g}\) follows immediately if \(\mathfrak{g}\) is an ideal of \(\mathfrak{u}(n)\).

2. The condition \([X^{-1}, Y] \subset \mathfrak{g}\) is satisfied for the classical groups \(\text{SU}(n), \text{U}(n), \text{O}(n)\) and \(\text{Sp}(n)\). The stronger condition \(X^{-1} \in \mathfrak{g}\) is true for \(\text{U}(n), \text{O}(n)\) and \(\text{Sp}(n)\) but not for \(\text{SU}(n)\).

**Remark 2.6.** When \(m = 0\) or \(M = 0\), the conclusion in Theorem 2.3 may not hold in general, for example \(G = \text{SO}(2^k)\) [8], i.e., the extremal determinant is not given by any Weyl group element. We will see in the next section that \(\text{Sp}(k)\) exhibits similar behavior.

### 3. Symplectic group \(\text{Sp}(k)\)

The symplectic group \(\text{Sp}(k) \subset \text{U}(2k)\) consists of matrices of the form

\[
\begin{bmatrix}
A & -B \\
B & A
\end{bmatrix} \in \text{U}(2k).
\]

It is a compact connected Lie group. We may choose

\[
\mathfrak{u}t = \{ \text{diag} (\alpha_1, \ldots, \alpha_k, -\alpha_1, \ldots, -\alpha_k) : \alpha_1 \geq \cdots \geq \alpha_k \geq 0 \}.
\]

The Weyl group acts on \(\mathfrak{u}t\):

\[
\text{diag} (\alpha_1, \ldots, \alpha_k, -\alpha_1, \ldots, -\alpha_k) \rightarrow \text{diag} (\pm \alpha_{\sigma(1)}, \ldots, \pm \alpha_{\sigma(k)}, \mp \alpha_{\sigma(1)}, \ldots, \mp \alpha_{\sigma(k)})
\]

where \(\sigma \in S_n\).

**Lemma 3.1.** Let \(A, B \in \mathfrak{is}p(k) \subset \mathfrak{is}u(2k)\) with eigenvalues \(\alpha_1 \geq \cdots \geq \alpha_k \geq -\alpha_k \geq \cdots \geq -\alpha_1\) and \(\beta_1 \geq \cdots \geq \beta_k \geq -\beta_k \geq \cdots \geq -\beta_1\) respectively. Then

\[
\Delta(A, B) := \{ UAU^{-1} + VBV^{-1} : U, V \in \text{Sp}(k) \} \subseteq \text{GL}_{2k}(\mathbb{C})
\]

if and only if \([\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] = \phi\).

**Proof.** Suppose that \([\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] \neq \phi\). Then there are \(1 \leq i, j \leq k\) such that \(\alpha_i \leq \beta_j \leq \alpha_1\) or \(\beta_i \leq \alpha_j \leq \beta_1\). For definiteness we assume
$\alpha_i \leq \beta_j \leq \alpha_1$ and the other case follows by symmetry. Now choose $V \in \text{Sp}(k)$ so that

$$V B V^{-1} = \text{diag}(\beta_1, \beta_2, \ldots, -\beta_j, \ldots, -\beta_i, \ldots, -\beta_k),$$

where $-\beta_j$ is in the $i$th position and $\beta_i$ is in the $j$th position. Then consider the leading principal submatrices of $A$ and $V B V^{-1}$

$A_1 := \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_k)$, $B_1 := \text{diag}(\beta_1, \beta_2, \ldots, -\beta_j, \ldots, \beta_i, \ldots, -\beta_k)$.

Then

$$\det(A_1 + B_1) = (\alpha_i - \beta_j)(\alpha_j + \beta_i) \prod_{l \neq i,j} (\alpha_l + \beta_l) \leq 0$$

since $\alpha_i \leq \beta_j$.

On the other hand, we can find $U \in \text{U}(k)$ so that $UB_1 U^{-1} = \text{diag}(-\beta_j, \beta_1, \beta_2, \ldots, \beta_k)$. Then

$$\det(A_1 + UB_1 U^{-1}) = (\alpha_1 - \beta_j) \prod_{l > 1} (\alpha_l + \beta_{l-1}).$$

Since $\alpha_1 \geq \beta_j$, $\det(A_1 + UB_1 U^{-1}) \geq 0$. Since $\text{U}(k)$ is connected, by continuity, there is a $U_0 \in \text{U}(k)$ so that $\det(A_1 + U_0 B_1 U_0^{-1}) = 0$. Set $U_1 := U_0 \oplus \overline{U_0} \in \text{Sp}(k)$. Then

$$\det(A + U_1 V B V^{-1} U_1^{-1}) = \det(A_1 + U_0 B_1 U_0^{-1})^2 = 0.$$

Suppose $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] = \emptyset$, i.e., either $\alpha_k > \beta_1$ or $\beta_k > \alpha_1$. For definiteness assume $\alpha_k > \beta_1$. Since $A' := UA U^{-1}$ and $B' := V B V^{-1}$ $(U, V \in \text{U}(n))$ are Hermitian, by Rayleigh-Ritz theorem $[3]$, for any unit vector $x \in \mathbb{C}^{2k}$, we have

$$\|A' x\|_2 \geq \alpha_k > \beta_1 \geq \|B' x\|_2.$$

As a result,

$$\|(A' + B') x\|_2 \geq \|A' x\|_2 - \|B' x\|_2 > 0$$

for any unit vector $x \in \mathbb{C}^{2k}$. Hence $A' + B'$ is nonsingular. By symmetry, the case $\beta_k > \alpha_1$ follows as well.

\[\square\]

**Theorem 3.2.** Let $A, B \in i\text{sp}(k) \subset i\text{u}(2k)$ with eigenvalues

$$\alpha_1 \geq \cdots \geq \alpha_k \geq -\alpha_k \geq \cdots \geq -\alpha_1$$

and

$$\beta_1 \geq \cdots \geq \beta_k \geq -\beta_k \geq \cdots \geq -\beta_1$$
respectively. Let

\[ D(A, B) := \{ \det(UA^{-1} + VBV^{-1}) : U, V \in \text{Sp}(k) \}. \]

A. Suppose that \([\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] = \phi.\]

(a) If \(k\) is even, then

\[ D(A, B) = \left[ \prod_{i=1}^{k}(\alpha_i - \beta_{k-i+1})^2, \prod_{i=1}^{k}(\alpha_i + \beta_{k-i+1})^2 \right]. \]

(b) If \(k\) is odd, then

\[ D(A, B) = \left[ -\prod_{i=1}^{k}(\alpha_i + \beta_{k-i+1})^2, -\prod_{i=1}^{k}(\alpha_i - \beta_{k-i+1})^2 \right]. \]

B. Suppose that \([\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] \neq \phi.\]

(a) If \(k\) is even, then \(D(A, B) = [0, \prod_{i=1}^{k}(\alpha_i + \beta_{k-i+1})^2].\)

(b) If \(k\) is odd, then \(D(A, B) = [-\prod_{i=1}^{k}(\alpha_i + \beta_{k-i+1})^2, 0].\)

**Proof.** The condition \([X^{-1}, Y] \in \text{sp}(k)\) in Theorem 2.3 is satisfied for \(\text{sp}(k)\) since nonsingular elements in \(\text{sp}(k)\) have inverses in \(\text{sp}(k)\). A. Suppose \([\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] = \phi.\)

(a) If \(k\) is even, then \(m \geq 0\) since each element of \(\text{sp}(k)\) is \(\text{Sp}(k)\)-conjugate to some element of \(t\). By Lemma 3.1 all matrices in \(\Delta(A, B)\) are nonsingular. Thus \(m > 0\) and so \(M > 0\). By Theorem 2.3 and because of the Weyl group action,

\[ B_0 = \text{diag}(\pm \beta_{\sigma(1)}, \ldots, \pm \beta_{\sigma(k)}, \mp \beta_{\sigma(1)}, \ldots, \mp \beta_{\sigma(k)}), \]

for some \(\sigma \in S_n\), where \(B_0 = \omega \cdot B\) yields \(m = \det(A + \omega \cdot B)\). Thus \(m = \min_{\sigma \in S_n} \prod_{i=1}^{k}(\alpha_i + \beta_{\sigma(i)})^2\). Similarly \(M = \max_{\sigma \in S_n} \prod_{i=1}^{k}(\alpha_i + \beta_{\sigma(i)})^2\). Clearly,

\[ m = \min_{\sigma \in S_k} \prod_{i=1}^{k}(\alpha_i - \beta_{\sigma(i)})^2, \quad M = \max_{\sigma \in S_k} \prod_{i=1}^{k}(\alpha_i + \beta_{\sigma(i)})^2. \]

The expression \(M = \max_{\sigma \in S_k} \prod_{i=1}^{k}(\alpha_i + \beta_{\sigma(i)})^2\) can be identified. Notice that [2] p.29 for \(i < j\) and \(\sigma(i) < \sigma(j)\),

\[ (\alpha_i + \beta_{\sigma(i)})(\alpha_j + \beta_{\sigma(j)}) - (\alpha_i + \beta_{\sigma(j)})(\alpha_j + \beta_{\sigma(i)}) = -(\alpha_i - \alpha_j)(\beta_{\sigma(i)} - \beta_{\sigma(j)}) \leq 0 \]

for each \(\sigma \in S_k\). So

\[ M = \max_{\sigma \in S_k} \prod_{i=1}^{k}(\alpha_i + \beta_{\sigma(i)})^2 = \prod_{i=1}^{k}(\alpha_i + \beta_{k-i+1})^2. \]
Similarly the expression \( m = \min_{\sigma \in S_k} \prod_{i=1}^{k} (\alpha_i - \beta_{\sigma(i)})^2 \) can be identified as \( m = \prod_{i=1}^{k} (\alpha_i - \beta_{k-i+1})^2 \). It is because

\[
\min_{\sigma \in S_k} \prod_{i=1}^{k} (\alpha_i - \beta_{\sigma(i)})^2 = \prod_{i=1}^{k} (\alpha_i - \beta_{k-i+1})^2
\]

since for definiteness we may assume that \( \alpha_k > \beta_1 \) and for \( i < j \) and \( \sigma(i) < \sigma(j) \),

\[
(\alpha_i - \beta_{\sigma(i)})(\alpha_j - \beta_{\sigma(j)}) - (\alpha_i - \beta_{\sigma(j)})(\alpha_j - \beta_{\sigma(i)}) = (\alpha_i - \alpha_j)(\beta_{\sigma(i)} - \beta_{\sigma(j)}) \geq 0
\]

(b) If \( k \) is odd, then \( M < 0 \) and similar argument leads to the desired conclusion.

B. Suppose \([\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] \neq \phi\).

(a) If \( k \) is even, then all elements in \( D(A, B) \) are nonnegative. So \( m = 0 \) by Lemma \[3.1\] and \( M \geq 0 \). We may assume that \( M > 0 \), otherwise vary \( \alpha_i \) to \( \alpha_i + \delta \) for all \( i \) \((\delta \in \mathbb{R})\) and

\[
0 < \prod_{i=1}^{k} (\alpha_i + \delta + \beta_{k-i+1})^2 \in D(A + \delta(I \oplus (-I)), B)
\]

and use continuity argument. Theorem \[2.3\] and the previous argument yield

\[
M = \max_{\sigma \in S_k} \prod_{i=1}^{k} (\alpha_i + \beta_{\sigma(i)})^2 = \prod_{i=1}^{k} (\alpha_i + \beta_{k-i+1})^2.
\]

(b) If \( k \) is even, all elements in \( D(A, B) \) are nonpositive. So \( M = 0 \) by Lemma \[3.1\]. The previous argument yields

\[
m = \min_{\sigma \in S_k} \left[ (-1)^k \prod_{i=1}^{k} (\alpha_i + \beta_{\sigma(i)})^2 \right] = - \max_{\sigma \in S_k} \prod_{i=1}^{k} (\alpha_i + \beta_{\sigma(i)})^2 = - \prod_{i=1}^{k} (\alpha_i + \beta_{k-i+1})^2.
\]

Remark 3.3. When \([\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] \neq \phi\), \( m \) (with \( k \) even) and \( M \) (with \( k \) odd) are zero, not given by any Weyl group element.

Corollary 3.4. All elements in \( \Delta(A, B) \) are singular if and only if the total number of zeros among \( \alpha_1, \ldots, \alpha_n \) and \( \beta_1, \ldots, \beta_n \) is greater than \( n \).
References


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