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DETERMINANTS OF SUM OF ORBITS UNDER COMPACT LIE GROUP

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Abstract. We study the determinants on the sum of orbits of two elements in the Lie algebra of a compact connected subgroup in the unitary group. As an application, the extremal determinant expressions are obtained for the symplectic group.

1. Introduction

Let $\mathrm{U}(n) \subset \mathbb{C}_{n\times n}$ be the unitary group. Recall a result of Fiedler [2].

Theorem 1.1. (Fiedler) Let $A, B \in \mathbb{C}_{n\times n}$ be Hermitian matrices with eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$. Then

$$\{\det(UAU^{-1} + VBV^{-1}) : U, V \in \mathrm{U}(n)\}$$

is the interval $[\min_{\sigma\in S_n} \prod_{i=1}^n (\alpha_i+\beta_{\sigma(i)}), \max_{\sigma\in S_n} \prod_{i=1}^n (\alpha_i+\beta_{\sigma(i)})]$, where S_n denotes the symmetric group on $\{1, \ldots, n\}$.

The following is an immediate consequence of Theorem 1.1. It also follows from [6, Corollary 2.3].

Corollary 1.2. All elements in $\Delta(A, B)$ are singular if and only if there is $\mu \in \mathbb{R}$ such that μ is an eigenvalue of A with multiplicity m_1 and $-\mu$ is an eigenvalues of B with multiplicity m_2 such that $m_1 + m_2 > n$.

Remark 1.3. The permutation that yields the maximum or minimum may not do the same if we translate A or B when $n \geq 3$. For example, if $A = \mathrm{diag}\,(5, 2, 1)$ and $B = \mathrm{diag}\,(6, 3, 1)$, then the maximum is $\det M_1 = 210$, where $M_1 := \mathrm{diag}\,(5, 2, 1) + \mathrm{diag}\,(1, 3, 6) = 210$. Set $M_2 := \mathrm{diag}\,(5, 2, 1) + \mathrm{diag}\,(1, 6, 3)$ so that $\det M_2 = 192$. But $\det(M_1 - 6.5I) = 0.375 < 1.875 = \det(M_2 - 6.5)$.

Let $G \subset \mathrm{U}(n)$ be a compact connected subgroup of $\mathrm{U}(n)$. We want to extend Theorem 1.1 in the context of compact connected subgroups of $\mathrm{U}(n)$.

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Let $\mathfrak{g}$ denote the Lie algebra of G. Suppose $A, B \in i\mathfrak{g} \subset H_n$, where $H_n := i\mathfrak{u}(n)$ is the space of $n \times n$ Hermitian matrices. Set

$$D(A,B) := \{\det(UAU^{-1} + VBV^{-1}) : U, V \in G\} \subset \mathbb{R}.$$

Let T be a maximal torus of G with Lie algebra $\mathfrak{t}$. So $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ [7, p.98]. Since G is a matrix group, adjoint action is merely conjugation so that [1, p.288] each $X \in \mathfrak{g}$ is conjugate to some element in $\mathfrak{t}$, i.e., there is $g \in G$ such that $gXg^{-1} \in \mathfrak{t}$. Thus we may assume that $A, B \in i\mathfrak{t}$. Moreover, since G is compact connected and the determinant function is continuous,

$$D(A,B) = \{\det(A + VBV^{-1}) : V \in G\} = [m, M].$$

Let W be the Weyl group of (G, T) [7, p.136], i.e., W is $N_G(T)/T$, where $N_G(T)$ denotes the normalizer of T in G. The Weyl group acts on $\mathfrak{t}$ (thus on $i\mathfrak{t}$) via the adjoint action (and thus conjugation) and we denote the action of $\omega \in W$ on $A \in i\mathfrak{t}$ by $\omega \cdot A$.

Given $A, B \in i\mathfrak{t}$, denote by

$$\Delta(A,B) := \{UAU^{-1} + VBV^{-1} : U, V \in G\} \subset i\mathfrak{g} \subset H_n,$$

the sum of the orbits of A and B.

In Section 2 we determine m and M under some conditions. Then the result is used in Section 3 for the symplectic group $\mathrm{Sp}(k)$.

2. Determinant and sum of orbits

Recall a lemma of Fiedler [2].

Lemma 2.1. (Fiedler) Let $P, Q \in \mathbb{C}_{n\times n}$ and P be nonsingular. Then

$$\det(P + \epsilon Q) = (\det P)(1 + \epsilon \operatorname{tr} QP^{-1}) + o(\epsilon^2).$$

The following is a slight extension of Lemma 2.1. It is of independent interest and the proof is short.

Proposition 2.2. Let $P, Q \in \mathbb{C}_{n\times n}$. Then

$$\det(P + \epsilon Q) = \det P + \epsilon \operatorname{tr}(Q \operatorname{adj} P) + o(\epsilon^2),$$

where $\operatorname{adj} P$ denotes the adjugate of P.

Proof. By Cauchy-Binet formula [3, p.22]

$$\det(P + \epsilon Q) = \det\left(\begin{bmatrix} P & I \end{bmatrix} \begin{bmatrix} I \\ \epsilon Q \end{bmatrix}\right) = \det P + \epsilon \operatorname{tr}(Q \operatorname{adj} P) + o(\epsilon^2).$$

□

Theorem 2.3. Let $A, B \in i\mathfrak{t}$. Assume that $[X^{-1}, Y] \in \mathfrak{g}$ for any $X, Y \in \mathfrak{g}$, whenever X is nonsingular. Suppose that the extremum $\xi = m$ or M is nonzero. Then $\xi = \det(A + \omega \cdot B)$ for some $\omega \in W$.

Proof. By continuity argument we may assume that $iA \in \mathfrak{t}$ is regular, i.e., the centralizer $Z_{\mathfrak{g}}(iA)$ of iA in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$, since regular elements of $\mathfrak{t}$ form an open dense set in $\mathfrak{t}$ [7, p.156] and the determinant function is continuous.

Suppose that the extremum $\xi \neq 0$ occurs at the optimizing matrix

$$C_0 := A + B_0 \in i\mathfrak{g}, \tag{1}$$

where

$$B_0 = V_0 B V_0^{-1}$$

for some $V_0 \in G$. Let $S \in \mathfrak{g}$. So the matrix exponential

$$e^{\epsilon S} := I + \epsilon S + \frac{1}{2!}\epsilon^2 S^2 + \cdots \in G,$$

for all $\epsilon \in \mathbb{R}$ [4, Prop 1.74, p.45]. By Lemma 2.1

$$\begin{aligned}
\det(A + e^{\epsilon S} B_0 e^{-\epsilon S}) &= \det(A + e^{\epsilon \operatorname{ad} S} B_0) \\
&= \det(A + B_0 + \epsilon\,[S, B_0]) + o(\epsilon^2) \\
&= \det C_0 (1 + \epsilon \operatorname{tr}\,[S, B_0] C_0^{-1}) + o(\epsilon^2) \\
&= \det C_0 (1 + \epsilon \operatorname{tr} S[B_0, C_0^{-1}]) + o(\epsilon^2).
\end{aligned}$$

So $\operatorname{tr} S[B_0, C_0^{-1}] = 0$ for all $S \in \mathfrak{g}$. Thus $[B_0, C_0^{-1}]$ is in the orthogonal complement $\mathfrak{g}^{\perp}$ in H_n with respect to the inner product $(X, Y) := \operatorname{tr} XY$ on H_n. By the assumption $[B_0, C_0^{-1}] \in \mathfrak{g}$ so we obtain $[B_0, C_0^{-1}] = 0$, i.e., B_0 commutes with C_0^{-1} and thus with C_0. Hence

$$B_0(A + B_0) = (A + B_0)B_0,$$

i.e., $B_0 A = A B_0$. Since $iA \in \mathfrak{t}$ is regular [7, p.101], $Z_{\mathfrak{g}}(iA) = \mathfrak{t}$ so that $iB_0 \in \mathfrak{t}$. Since the intersection with $\mathfrak{t}$ of the orbit of G is the orbit of W on $\mathfrak{t}$ [1, p.294], $B_0 = \omega \cdot B$ for some $\omega \in W$.

□

Since the Weyl group W is finite, the problem is reduced to a finite optimization problem.

Remark 2.4. We can derive Theorem 1.1 via Theorem 2.3. Let $A, B \in H_n$. The Weyl group for $\mathrm{U}(n)$ is the symmetric group with $T \subset \mathrm{U}(n)$ being the set of diagonal matrices in $\mathrm{U}(n)$. Clearly the inverse of a nonsingular Hermitian matrix remains Hermitian. So the condition $[X^{-1}, Y] \in \mathfrak{u}(n)$ in Theorem 2.3 is readily satisfied. If the extremum ξ (m or M) is nonzero, then apply Theorem 2.3 to yield Theorem 1.1. Suppose $\xi = 0$, say $m = \det C_0 = 0$ with $UC_0C^{-1} =$

$\operatorname{diag}(c_1, \ldots, c_r, 0, \ldots, 0)$ for some $U \in \mathrm{U}(n)$, where $c_1, \ldots, c_r \neq 0$. We can perturb A to $A + UDU^{-1}$ ($D := \operatorname{diag}(\delta_1, \ldots, \delta_n)$, where $\delta_i \in \mathbb{R}$ $i = 1, \ldots, n$) so that $\det(UDU^{-1} + C_0) < 0$. However $\det(UDU^{-1} + C_0) \in D(A + UDU^{-1}, B)$ so that the optimizing matrix C_0 for $D(A + UDU^{-1}, B)$ is nonsingular. Then use Theorem 2.3 and apply continuity argument since $\det C_0$ depends on A continuously [2, p.29].

Remark 2.5. (1) The condition $[X^{-1}, Y] \subset \mathfrak{g}$ follows immediately if $\mathfrak{g}$ is an ideal of $\mathfrak{u}(n)$.

(2) The condition $[X^{-1}, Y] \subset \mathfrak{g}$ is satisfied for the classical groups $\mathrm{SU}(n)$, $\mathrm{U}(n)$, $\mathrm{O}(n)$ and $\mathrm{Sp}(n)$. The stronger condition $X^{-1} \in \mathfrak{g}$ is true for $\mathrm{U}(n)$, $\mathrm{O}(n)$ and $\mathrm{Sp}(n)$ but not for $\mathrm{SU}(n)$.

Remark 2.6. When $m = 0$ or $M = 0$, the conclusion in Theorem 2.3 may not hold in general, for example $G = \mathrm{SO}(2k)$ [8], i.e., the extremal determinant is not given by any Weyl group element. We will see in the next section that $\mathrm{Sp}(k)$ exhibits similar behavior.

3. Symplectic group $\mathrm{Sp}(k)$

The symplectic group $\mathrm{Sp}(k) \subset \mathrm{U}(2k)$ consists of matrices of the form

$$\begin{bmatrix} A & -\bar{B} \\ B & \bar{A} \end{bmatrix} \in \mathrm{U}(2k).$$

It is a compact connected Lie group. We may choose

$$i\mathfrak{t} = \{\operatorname{diag}(\alpha_1, \ldots, \alpha_k, -\alpha_1, \ldots, -\alpha_k) : \alpha_1 \geq \cdots \geq \alpha_k \geq 0\}.$$

The Weyl group acts on $i\mathfrak{t}$:

$$\begin{aligned} & \operatorname{diag}(\alpha_1, \ldots, \alpha_k, -\alpha_1, \ldots, -\alpha_k) \\ \to\ & \operatorname{diag}(\pm\alpha_{\sigma(1)}, \ldots, \pm\alpha_{\sigma(k)}, \mp\alpha_{\sigma(1)}, \ldots, \mp\alpha_{\sigma(k)}) \end{aligned}$$

where $\sigma \in S_n$.

Lemma 3.1. Let $A, B \in i\mathfrak{sp}(k) \subset i\mathfrak{su}(2k)$ with eigenvalues $\alpha_1 \geq \cdots \geq \alpha_k \geq -\alpha_k \geq \cdots \geq -\alpha_1$ and $\beta_1 \geq \cdots \geq \beta_k \geq -\beta_k \geq \cdots \geq -\beta_1$ respectively. Then

$$\Delta(A, B) := \{UAU^{-1} + VBV^{-1} : U, V \in \mathrm{Sp}(k)\} \subseteq \mathrm{GL}_{2k}(\mathbb{C})$$

if and only if $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] = \phi$.

Proof. Suppose that $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] \neq \phi$. Then there are $1 \leq i, j \leq k$ such that $\alpha_i \leq \beta_j \leq \alpha_1$, or $\beta_i \leq \alpha_j \leq \beta_1$. For definiteness we assume

$\alpha_i \le \beta_j \le \alpha_1$ and the other case follows by symmetry. Now choose $V \in \mathrm{Sp}(k)$ so that

$$\begin{aligned} & VBV^{-1} \\ = \quad & \mathrm{diag}\,(\beta_1, \beta_2, \ldots, -\beta_j, \ldots, \beta_i, \ldots, \beta_k, -\beta_1, -\beta_2, \ldots, \beta_j, \ldots, -\beta_i, \ldots, -\beta_k), \end{aligned}$$

where $-\beta_j$ is in the ith position and β_i is in the jth position. Then consider the leading principal submatrices of A and VBV^{-1}

$$A_1 := \mathrm{diag}\,(\alpha_1, \alpha_2, \ldots, \alpha_k), \quad B_1 := \mathrm{diag}\,(\beta_1, \beta_2, \ldots, -\beta_j, \ldots, \beta_i, \ldots, \beta_k).$$

Then

$$\det(A_1 + B_1) = (\alpha_i - \beta_j)(\alpha_j + \beta_i) \prod_{l \ne i,j} (\alpha_l + \beta_l) \le 0$$

since $\alpha_i \le \beta_j$.

On the other hand, we can find $U \in \mathrm{U}(k)$ so that

$$UB_1U^{-1} = \mathrm{diag}\,(-\beta_j, \beta_1, \beta_2, \ldots, \beta_k).$$

Then

$$\det(A_1 + UB_1U^{-1}) = (\alpha_1 - \beta_j) \prod_{l>1} (\alpha_l + \beta_{l-1}).$$

Since $\alpha_1 \ge \beta_j$, $\det(A_1 + UB_1U^{-1}) \ge 0$. Since $\mathrm{U}(k)$ is connected, by continuity, there is a $U_0 \in \mathrm{U}(k)$ so that $\det(A_1 + U_0B_1U_0^{-1}) = 0$. Set $U_1 := U_0 \oplus \overline{U}_0 \in \mathrm{Sp}(k)$. Then

$$\det(A + U_1VBV^{-1}U_1^{-1}) = \det(A_1 + U_0B_1U_0^{-1})^2 = 0.$$

Suppose $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] = \phi$, i.e., either $\alpha_k > \beta_1$ or $\beta_k > \alpha_1$. For definiteness assume $\alpha_k > \beta_1$. Since $A' := UAU^{-1}$ and $B' := VBV^{-1}$ ($U, V \in \mathrm{U}(n)$) are Hermitian, by Rayleigh-Ritz theorem [3], for any unit vector $x \in \mathbb{C}^{2k}$, we have

$$\|A'x\|_2 \ge \alpha_k > \beta_1 \ge \|B'x\|_2.$$

As a result,

$$\|(A' + B')x\|_2 \ge \|A'x\|_2 - \|B'x\|_2 > 0$$

for any unit vector $x \in \mathbb{C}^{2k}$. Hence $A' + B'$ is nonsingular. By symmetry, the case $\beta_k > \alpha_1$ follows as well.

□

Theorem 3.2. Let $A, B \in i\mathfrak{sp}(k) \subset i\mathfrak{u}(2k)$ with eigenvalues

$$\alpha_1 \ge \cdots \ge \alpha_k \ge -\alpha_k \ge \cdots \ge -\alpha_1$$

and

$$\beta_1 \ge \cdots \ge \beta_k \ge -\beta_k \ge \cdots \ge -\beta_1$$

respectively. Let

$$D(A,B) := \{\det(UAU^{-1} + VBV^{-1}) : U, V \in \mathrm{Sp}(k)\}.$$

A. Suppose that $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] = \phi$.

(a) If k is even, then

$$D(A,B) = [\prod_{i=1}^{k}(\alpha_i - \beta_{k-i+1})^2, \prod_{i=1}^{k}(\alpha_i + \beta_{k-i+1})^2].$$

(b) If k is odd, then

$$D(A,B) = [-\prod_{i=1}^{k}(\alpha_i + \beta_{k-i+1})^2, -\prod_{i=1}^{k}(\alpha_i - \beta_{k-i+1})^2].$$

B. Suppose that $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] \neq \phi$.

(a) If k is even, then $D(A,B) = [0, \prod_{i=1}^{k}(\alpha_i + \beta_{k-i+1})^2]$.

(b) If k is odd, then $D(A,B) = [-\prod_{i=1}^{k}(\alpha_i + \beta_{k-i+1})^2, 0]$.

Proof. The condition $[X^{-1}, Y] \in \mathfrak{sp}(k)$ in Theorem 2.3 is satisfied for $\mathfrak{sp}(k)$ since nonsingular elements in $\mathfrak{sp}(k)$ have inverses in $\mathfrak{sp}(k)$.

A. Suppose $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] = \phi$.

(a) If k is even, then $m \geq 0$ since each element of $\mathfrak{sp}(k)$ is $\mathrm{Sp}(k)$-conjugate to some element of $\mathfrak{t}$. By Lemma 3.1 all matrices in $\Delta(A,B)$ are nonsingular. Thus $m > 0$ and so $M > 0$. By Theorem 2.3 and because of the Weyl group action,

$$B_0 = \mathrm{diag}\,(\pm\beta_{\sigma(1)}, \ldots, \pm\beta_{\sigma(k)}, \mp\beta_{\sigma(1)}, \ldots, \mp\beta_{\sigma(k)}),$$

for some $\sigma \in S_n$, where $B_0 = \omega \cdot B$ yields $m = \det(A + \omega \cdot B)$. Thus $m = \min_{\sigma \in S_n} \prod_{i=1}^{k}(\alpha_i \pm \beta_{\sigma(i)})^2$. Similarly $M = \max_{\sigma \in S_n} \prod_{i=1}^{k}(\alpha_i \pm \beta_{\sigma(i)})^2$. Clearly,

$$m = \min_{\sigma \in S_n} \prod_{i=1}^{k}(\alpha_i - \beta_{\sigma(i)})^2, \quad M = \max_{\sigma \in S_n} \prod_{i=1}^{k}(\alpha_i + \beta_{\sigma(i)})^2.$$

The expression $M = \max_{\sigma \in S_k} \prod_{i=1}^{k}(\alpha_i + \beta_{\sigma(i)})^2$ can be identified. Notice that [2, p.29] for $i < j$ and $\sigma(i) < \sigma(j)$,

$$\begin{aligned} & (\alpha_i + \beta_{\sigma(i)})(\alpha_j + \beta_{\sigma(j)}) - (\alpha_i + \beta_{\sigma(j)})(\alpha_j + \beta_{\sigma(i)}) \\ = \;& -(\alpha_i - \alpha_j)(\beta_{\sigma(i)} - \beta_{\sigma(j)}) \leq 0 \end{aligned}$$

for each $\sigma \in S_k$. So

$$M = \max_{\sigma \in S_k} \prod_{i=1}^{k}(\alpha_i + \beta_{\sigma(i)})^2 = \prod_{i=1}^{k}(\alpha_i + \beta_{k-i+1})^2.$$

Similarly the expression $m = \min_{\sigma\in S_k}\prod_{i=1}^k(\alpha_i-\beta_{\sigma(i)})^2$ can be identified as $m = \prod_{i=1}^k(\alpha_i - \beta_{k-i+1})^2$. It is because

$$\min_{\sigma\in S_k}\prod_{i=1}^k(\alpha_i-\beta_{\sigma(i)})^2 = \prod_{i=1}^k(\alpha_i-\beta_{k-i+1})^2$$

since for definiteness we may assume that $\alpha_k > \beta_1$ and for $i < j$ and $\sigma(i) < \sigma(j)$,

$$\begin{aligned} & (\alpha_i - \beta_{\sigma(i)})(\alpha_j - \beta_{\sigma(j)}) - (\alpha_i - \beta_{\sigma(j)})(\alpha_j - \beta_{\sigma(i)}) \\ = \; & (\alpha_i - \alpha_j)(\beta_{\sigma(i)} - \beta_{\sigma(j)}) \geq 0 \end{aligned}$$

(b) If k is odd, then $M < 0$ and similar argument leads to the desired conclusion.

B. Suppose $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] \neq \phi$.

(a) If k is even, then all elements in $D(A, B)$ are nonnegative. So $m = 0$ by Lemma 3.1 and $M \geq 0$. We may assume that $M > 0$, otherwise vary α_i to $\alpha_i + \delta$ for all i ($\delta \in \mathbb{R}$) and

$$0 < \prod_{i=1}^k(\alpha_i + \delta + \beta_{k-i+1})^2 \in D(A + \delta(I \oplus (-I)), B)$$

and use continuity argument. Theorem 2.3 and the previous argument yield

$$M = \max_{\sigma\in S_k}\prod_{i=1}^k(\alpha_i+\beta_{\sigma(i)})^2 = \prod_{i=1}^k(\alpha_i+\beta_{k-i+1})^2.$$

(b) If k is even, all elements in $D(A, B)$ are nonpositive. So $M = 0$ by Lemma 3.1. The previous argument yields

$$\begin{aligned} m &= \min_{\sigma\in S_k}\left[(-1)^k\prod_{i=1}^k(\alpha_i+\beta_{\sigma(i)})^2\right] \\ &= -\max_{\sigma\in S_k}\prod_{i=1}^k(\alpha_i+\beta_{\sigma(i)})^2 = -\prod_{i=1}^k(\alpha_i+\beta_{k-i+1})^2. \end{aligned}$$

□

Remark 3.3. When $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] \neq \phi$, m (with k even) and M (with k odd) are zero, not given by any Weyl group element.

Corollary 3.4. All elements in $\Delta(A, B)$ are singular if and only if the total number of zeros among $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ is greater than n.

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