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**On Bertram Kostant's paper: On convexity, the  
Weyl group and the Iwasawa decomposition  
Ann. Sci. École Norm. Sup. (4) 6 (1973), 413–455**

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*2011 AU Summer Seminar*

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# 1. Some matrix results

Notation: Given  $A \in \mathbb{C}_{n \times n}$

$\lambda(A)$  denotes the  $n$ -tuple of eigenvalues of  $A$ ,

$s(A)$  denotes the  $n$ -tuple of singular values of  $A$ ,

$\text{diag } A$  denotes the diagonal of  $A$ .

**Theorem 1.1.** (Schur-Horn) Let  $\lambda, d \in \mathbb{R}^n$ . Then there exists a Hermitian  $A \in \mathbb{C}_{n \times n}$  such that  $\lambda(A) = \lambda$  and  $\text{diag } A = d$  if and only if

$$\sum_{j=1}^k d_j \leq \sum_{j=1}^k \lambda_j, \quad k = 1, \dots, n-1,$$
$$\sum_{j=1}^n d_j = \sum_{j=1}^n \lambda_j$$

after rearranging the entries of  $d$  and  $\lambda$  in descending order, respectively. It is equivalent to

$$d \in \text{conv} S_n \lambda,$$

where “conv” denotes the convex hull of the underlying set, and  $S_n \lambda$  denotes the orbit of  $\lambda$  under the action of the symmetric group  $S_n$ .

The concept is known as **majorization**, denoted by  $d \prec \lambda$ .



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- I. Schur, Über eine Klasse von Mittelbildungen mit Anwendungen auf der Determinantentheorie, Sitzungsberichte der Berliner Mathematischen Gesellschaft, 22 (1923), 9–20.
- A. Horn, Doubly stochastic matrices and the diagonal of a rotation matrix, Amer. J. Math. 76 (1954), 620–630.

Rewriting Schur-Horn's result in orbit terms:

$$\text{diag} \{U \text{diag} (\lambda_1, \dots, \lambda_n) U^{-1} : U \in U(n)\} = \text{conv} S_n \lambda.$$

Mirsky asked the analog for the singular values and diagonal entries of  $A \in \mathbb{C}_{n \times n}$ .

- L. Mirsky, Inequalities and existence theorems in the theory of matrices, J. Math. Anal. Appl. 9 (1964), 99–118.

Partial results were obtained.

- G. de Oliveira, Matrices with prescribed principal elements and singular values, Canad. Math. Bull. 14 (1971), 247–249.
- F.Y. Sing, Some results on matrices with prescribed diagonal elements and singular values, Canad. Math. Bulletin, 19 (1976), 89–92.



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**Theorem 1.2.** (Thompson-Sing) Let  $s \in \mathbb{R}_+^n$  and let  $d \in \mathbb{C}^n$ . Then there exists  $A \in \mathbb{C}_{n \times n}$  with  $s(A) = s$  and  $\text{diag } A = d$  if and only if

$$\sum_{i=1}^k |d_i| \leq \sum_{i=1}^k s_i, \quad k = 1, \dots, n,$$
$$\sum_{i=1}^{n-1} |d_i| - |d_n| \leq \sum_{i=1}^{n-1} s_i - s_n,$$

after rearranging the entries of  $s$  and  $d$  in descending order with respect to modulus.

- F.Y. Sing, Some results on matrices with prescribed diagonal elements and singular values, *Canad. Math. Bulletin*, 19 (1976), 89–92.
- R.C. Thompson, Singular values, diagonal elements and convexity, *SIAM J. Appl. Math.* 32 (1977), 39–63.

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MR0424847 (54 #12805) 15A18 (65F15)

**Thompson, R. C.**

**Singular values, diagonal elements, and convexity.**

*SIAM J. Appl. Math.* **32** (1977), no. 1, 39–63.

Rather more than twenty years ago, A. Horn [Amer. J. Math. **76** (1954), 620–630; MR0063336 (16,105c)] found, inter alia, necessary and sufficient conditions for the existence of a Hermitian matrix with prescribed diagonal elements and characteristic roots. (An alternative treatment was given subsequently by the reviewer [J. London Math. Soc. **33** (1958), 14–21; MR0091931 (19,1034c)].) Horn's initiative (in the paper cited above and in a few other notes) became the starting point of a distinctive tendency within matrix theory, and the work reviewed here belongs to the tradition so established. Probably the most arresting conclusion reached by the author, which settles a difficult problem of long standing, is the following: Let  $d_1, \dots, d_n$  be complex numbers arranged so that  $|d_1| \geq \dots \geq |d_n|$ , and let  $s_1 \geq \dots \geq s_n$  be non-negative real numbers; then there exists a (complex)  $n \times n$  matrix with diagonal elements  $d_1, \dots, d_n$  (in any prescribed order) and singular values  $s_1, \dots, s_n$  if and only if  $\sum_{i=1}^k |d_i| \leq \sum_{i=1}^k s_i$  ( $1 \leq k \leq n$ ) and  $\sum_{i=1}^{n-1} |d_i| - |d_n| \leq \sum_{i=1}^{n-1} s_i - s_n$ . These inequalities have a familiar look. **Nevertheless, the proof is neither easy nor very short and exhibits a high degree of technical ingenuity.** The author goes on to derive a variant of the theorem just stated for the case when all numbers are real and the matrix is also required to have a non-negative determinant.

In the final two sections of the paper, questions of convexity are considered; in this field, too, Horn [op. cit.] had made a start. The discussion is mainly concerned with a variety of characterizations involving singular values; it is again very solid and detailed and is not readily summarized in a few sentences. We content ourselves with recalling a result of Horn's which here emerges as a trivial corollary: The real numbers  $d_1, \dots, d_n$  are the diagonal elements of a proper orthogonal matrix if and only if  $(d_1, \dots, d_n)$  lies in the convex hull of those vectors  $(\pm 1, \dots, \pm 1)$  which have an even number of negative components [cf. the reviewer, Amer. Math. Monthly **66** (1959), 19–22; MR0098758 (20 #5213)].

**In the reviewer's opinion, the paper discussed here represents an advance of almost the same order of magnitude as the earlier work of Horn's.** In particular, our understanding of convexity properties of matrices is, as yet, very imperfect and it seems likely that the author's results will help us to gain a more complete insight. Anyone interested in matrix theory would do well to study the paper, to develop further the techniques introduced here, and (if possible) to simplify the rather intricate arguments.

Reviewed by *L. Mirsky*



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MR0424850 (54 #12808) 15A42

**Sing, Fuk Yum**

**Some results on matrices with prescribed diagonal elements and singular values.**

*Canad. Math. Bull.* **19** (1976), no. 1, 89–92.

The author considers the relationship between the diagonal elements and singular values of a matrix. This problem originated with L. Mirsky in a well-known survey paper [*J. Math. Anal. Appl.* **9** (1964), 99–118; [MR0163918 \(29 #1217\)](#)] and was revived by G. N. de Oliveira [*Canad. Math. Bull.* **14** (1971), 247–249; [MR0311686 \(47 #248\)](#)] who gave a rather simple inequality relating the diagonal elements and singular values. In the paper under review the author obtains a different condition which implies de Oliveira’s condition, and he also completely discusses the case of  $2 \times 2$  matrices. In a note added in proof, he observes that he later proved his necessary condition to be sufficient for the existence of an  $n \times n$  matrix with prescribed diagonal elements and singular values, but that this fact had already been proved by the reviewer.

{Reviewer’s remark: The discovery of the necessary and sufficient conditions for the existence of a matrix with prescribed diagonal elements and singular values had been announced by the reviewer in an abstract in the Notices of the American Math. Society; **unfortunately, this announcement was overlooked by the author.** The reviewer’s paper [*SIAM J. Appl. Math.* **32** (1977), no. 1, 39–63] (not the *SIAM Journal of Mathematical Analysis* as stated by the author), contains not only this result but also many related theorems and corollaries. Three further papers of the reviewer exploiting these results are to be found [*Linear and Multilinear Algebra* **3** (1975/76), no. 1/2, 15–17; [MR0414581 \(54 #2682\)](#); *ibid.* **3** (1975/76), no. 1/2, 155–160; *ibid.* to appear]. **The author’s work apparently formed his thesis for his Master’s degree. It is unfortunate that his work was anticipated. He plainly is a talented mathematician from whom many more worthwhile results can be expected.**}

Reviewed by *R. C. Thompson*



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**Theorem 1.3.** (Thompson) Let  $s \in \mathbb{R}_+^n$  and let  $d \in \mathbb{R}^n$ . Then there exists  $A \in \mathbb{R}_{n \times n}$  with  $\det A \geq 0$  ( $\det A \leq 0$ ) having  $s(A) = s$  and  $\text{diag } A = d$  if and only if

$$\sum_{i=1}^k |d_i| \leq \sum_{i=1}^k s_i, \quad k = 1, \dots, n,$$
$$\sum_{i=1}^{n-1} |d_i| - |d_n| \leq \sum_{i=1}^{n-1} s_i - s_n,$$

and in addition, if the number of negative terms among  $d$  is odd (even, if nonpositive determinant)

$$\sum_{i=1}^n |d_i| \leq \sum_{i=1}^{n-1} s_i - s_n,$$

after rearranging the entries of  $s$  and  $d$  in descending order with respect to absolute value.

- The set is a convex set.

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**Theorem 1.4.** (Thompson) Let  $s \in \mathbb{R}_+^n$  and let  $d \in \mathbb{R}^n$ . Then there exists a real matrix with  $s(A) = s$  and  $\text{diag } A = d$  if and only if

$$\sum_{i=1}^k |d_i| \leq \sum_{i=1}^k s_i, \quad k = 1, \dots, n, \quad (1)$$

$$\sum_{i=1}^{n-1} |d_i| - |d_n| \leq \sum_{i=1}^{n-1} s_i - s_n, \quad (2)$$

after rearranging  $d$ 's in descending order with respect to absolute value.

Denote the relation by  $d \triangleleft s$  with respect to the inequalities.

$$\text{diag } (U \text{diag } (s_1, \dots, s_n) V) : U, V \in O(n) \} = \{d \in \mathbb{R}^n : d \triangleleft s\}$$

- The set is **not** a convex set.

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**Theorem 1.5.** (Weyl-Horn) Let  $\lambda \in \mathbb{C}^n$  and  $s \in \mathbb{R}_+^n$ . Then there exists  $A \in \mathbb{C}_{n \times n}$  such that  $\lambda(A) = \lambda$  and  $s(A) = s$  if and only if

$$\prod_{j=1}^k |\lambda_j| \leq \prod_{j=1}^k s_j, \quad k = 1, \dots, n-1,$$

$$\prod_{j=1}^n |\lambda_j| = \prod_{j=1}^n s_j.$$

after rearranging the entries of  $\lambda$  and  $s$  in descending order with respect to their moduli.

The conditions amounts to

$$\{\lambda(U \text{diag}(s_1, \dots, s_n)V) : U, V \in U(n)\} = \{\lambda \in \mathbb{C}^n : |\lambda| \prec_m s\}.$$

where  $\prec_m$  denotes the multiplicative majorization.

When  $A \in GL_n(\mathbb{C})$ , i.e.,  $s_i > 0$  for all  $i$ ,

$$\{\lambda(U \text{diag}(s_1, \dots, s_n)V) : U, V \in U(n)\} = \{\lambda \in \mathbb{C}^n : \log |\lambda| \prec \log s\}.$$

- A. Horn, On the eigenvalues of a matrix with prescribed singular values, Proc. Amer. Math. Soc., 5 (1954) 4–7.
- H. Weyl, Inequalities between the two kinds of eigenvalues of a linear transformation, Proc. Nat. Acad. Sci. U. S. A., 35 (1949) 408–411.



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Recall  $QR$  decomposition

$$A = QR$$

Set

$$a(A) := \text{diag}(r_{11}, \dots, r_{nn})$$

where  $A$  is written in column form

$$A = [A_1 | \dots | A_n]$$

Geometric interpretation of  $a(A)$ :

$r_{ii}$  is the **distance** between  $A_i$  and the span of  $A_1, \dots, A_{i-1}$ ,  $i = 2, \dots, n$ .

Weyl-Horn's (nonsingular) result

$$\{\lambda(U \text{diag}(s_1, \dots, s_n)V) : U, V \in U(n)\} = \{\lambda \in \mathbb{C}^n : \log |\lambda| \prec \log s\}.$$

is equivalent to the following

$$\begin{aligned} & \{a(\text{diag}(s_1, \dots, s_n)V) : U, V \in U(n)\} \\ &= \{a(U \text{diag}(s_1, \dots, s_n)V) : U, V \in U(n)\} \\ &= \{a \in \mathbb{R}_+^n : \log a \prec \log s\} \end{aligned}$$

Let us call it the **QR version of Weyl-Horn's result**.



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**Proof:** Let  $S := \text{diag}(s_1, \dots, s_n)$ . It suffices to show that

$$\{a(SV) : V \in U(n)\} = \{|\lambda|(USV) : U, V \in U(n)\}$$

Let  $SV = QR$ . Then

$$\begin{aligned} a(SV) &= a(QR) = a(R) = |\lambda|(R) = |\lambda|(Q^{-1}SV) \\ &\in \{|\lambda|(USV) : U, V \in U(n)\} \end{aligned}$$

On the other hand, let  $USV = WTW^{-1}$ ,  $W \in U(n)$  by Schur triangularization theorem, where  $T$  is upper triangular. So  $T = W^{-1}USVW$ . Thus

$$\begin{aligned} |\lambda|(USV) &= |\lambda|(WTW^{-1}) = a(T) = a(W^{-1}USVW) = a(SVW) \\ &\in \{a(SV) : V \in U(n)\} \end{aligned}$$

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## Some matrix decompositions

**Spectral decomposition:** Given  $A \in \mathbb{C}_{n \times n}$  Hermitian,

$$A = U \text{diag } \Lambda U^{-1}$$

for some  $U \in U(n)$ ,  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ .

**SVD:** Given  $A \in \mathbb{C}_{n \times n}$ .

$$A = USV$$

for some  $U, V \in U(n)$ ,  $S = \text{diag}(s_1, \dots, s_n)$ .

**real analog of SVD** Given  $A \in \mathbb{R}_{n \times n}$ .

$$A = USV$$

for some  $U, V \in O(n)$ ,  $S = \text{diag}(s_1, \dots, s_n)$ .

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## 2. Lie groups

- $\mathfrak{g}$  = **real semisimple** Lie algebra with connected noncompact Lie group  $G$ .
- $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  a fixed (algebra) Cartan decomposition of  $\mathfrak{g}$
- $K \subset G$  the connected subgroup with Lie algebra  $\mathfrak{k}$ .
- $\mathfrak{a} \subset \mathfrak{p}$  a maximal abelian subspace.
- Fix a *closed Weyl chamber*  $\mathfrak{a}_+$  in  $\mathfrak{a}$  and set

$$A_+ := \exp \mathfrak{a}_+, \quad A := \exp \mathfrak{a}$$

It is well known that for any  $X \in \mathfrak{p}$

$$\text{Ad}K(X) \cap \mathfrak{a} \neq \emptyset$$

and

$$\text{Ad}K(X) \cap \mathfrak{a}_+ = \text{singleton set}$$

Such a result is a unified extension of spectral decomposition for Hermitian matrices ( $\mathfrak{sl}_n(\mathbb{C})$ ) and real symmetric matrices ( $\mathfrak{sl}_n(\mathbb{R})$ ), SVD ( $\mathfrak{su}_{n,n}$ ) for complex matrices and SVD ( $\mathfrak{so}_{n,n}$ ) for real matrices. All are at the algebra level.

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### 3. QR $\rightarrow$ Iwasawa

For semisimple Lie group  $G$ , we have an extension called Iwasawa decomposition:

$$G = KAN$$

Let  $g \in G$ . There are unique  $k \in K$ ,  $a \in A$ ,  $n \in N$  such that  $g = kan = k(g)a(g)n(g)$ .

Kostant's **Theorem 4.1**: Let  $b \in A$ . Then

$$\{a(bv) : v \in K\} = \exp(\text{conv}W(\log b)),$$

where  $W$  is the Weyl group of  $(\mathfrak{a}, \mathfrak{g})$  which may be defined as the quotient of the normalizer of  $A$  (or  $\mathfrak{a}$ ) in  $K$  modulo the centralizer of  $A$  (or  $\mathfrak{a}$ ) in  $K$ .

**QR version of Weyl-Horn  $\rightsquigarrow$  Theorem 4.1 (Group level)**

Equivalently:

$$\{\log a(bv) : v \in K\} = \text{conv}W(\log b)$$

**Remark:**  $a(ubv) = a(bv)$  for any  $u, v \in K$ .

- C.J. Thompson, Inequalities and partial orders on matrix spaces. Indiana Univ. Math. J. 21 (1971/72), 469–480.
- A. Horn, On the eigenvalues of a matrix with prescribed singular values, Proc. Amer. Math. Soc., 5 (1954) 4–7.



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## 4. Schur-Horn $\rightarrow$ Theorem 8.2

Theorem 8.2 extends Schur-Horn's result.

Kostant's **Theorem 8.2** Let  $\pi : \mathfrak{p} \rightarrow \mathfrak{a}$  be the orthogonal projection with respect to the Killing form. Then for any  $Y \in \mathfrak{p}$ ,

$$\pi(\text{Ad}K(Y)) = \text{conv}WY.$$

Without loss of generality, we may assume that  $Y \in \mathfrak{a}$  since

$$\text{Ad}K(Y) \cap \mathfrak{a}_+$$

is a singleton set.

**Schur-Horn  $\rightsquigarrow$  Theorem 8.2** (algebra level)

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Kostant's paper generated a lot of research activities:

G.J. Heckman, Projections of orbits and asymptotic behaviour of multiplicities for compact Lie groups, thesis Rijksuniversiteit Leiden, 1980

G.J. Heckman, Projections of Orbits and Asymptotic Behavior of Multiplicities for Compact Connected Lie Groups, *Invent. Math.*, 67 (1982) 333-356.

M.F. Atiyah, Convexity and commuting Hamiltonians, *Bull. London Math. Soc.*, 308 (1982) 1–15.

V. Guillemin and S. Sternberg Convexity properties of the moment mapping, *Invent. Math.*, 67 (1982) 491–513.

V. Guillemin and S. Sternberg, Convexity properties of the moment mapping II, *Invent. Math.*, 77 (1984) 533–546.

Theorem 4.1 and Theorem 8.2 can be unified (group-algebra)

- J.J. Duistermaat, On the similarity between the Iwasawa projection and the diagonal part. Harmonic analysis on Lie groups and symmetric spaces (Kleebach, 1983). *Mém. Soc. Math. France (N.S.)* No. 15 (1984), 129–138.

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## 5. Kostant $\Rightarrow$ Thompson

In Thompson's 1988 Johns Hopkins Lecture 4 (p.57) he mentioned Kostant's paper and looked for the explanation of the subtracted term in his inequalities. The answer is Kostant's Theorem 8.2 and the simple Lie algebra  $\mathfrak{so}_{n,n}$ :

$$\mathfrak{so}_{n,n} = \left\{ \begin{pmatrix} X_1 & Y \\ Y^T & X_2 \end{pmatrix} : X_1^T = -X_1, X_2^T = X_2, Y \in \mathbb{R}_{n \times n} \right\},$$

$$K = \mathrm{SO}(n) \times \mathrm{SO}(n),$$

$$\mathfrak{k} = \mathfrak{so}(n) \oplus \mathfrak{so}(n),$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & Y \\ Y^T & 0 \end{pmatrix} : Y \in \mathbb{R}_{n \times n} \right\},$$

$$\mathfrak{a} = \bigoplus_{1 \leq j \leq n} \mathbb{R}(E_{j,n+j} + E_{n+j,j}),$$

where  $E_{i,j}$  is the  $2n \times 2n$  matrix and 1 at the  $(i, j)$  position is the only nonzero entry.

T.Y. Tam, A Lie theoretical approach of Thompson's theorems of singular values-diagonal elements and some related results, J. of London Math. Soc. (2), 60 (1999) 431-448.

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The projection  $\pi$  will send

$$\begin{aligned} \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}^T \begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} &= \begin{pmatrix} 0 & U^T S V \\ V^T S U & 0 \end{pmatrix} \\ &\mapsto \begin{pmatrix} 0 & \text{diag}(U^T S V) \\ \text{diag}(V^T S U) & 0 \end{pmatrix}, \end{aligned}$$

where  $X, Y \in \text{SO}(n)$ . The system of real roots of  $\mathfrak{so}_{n,n}$  is of type  $D_n$ . The action of  $W$  on  $\mathfrak{a}$  is given by the following:

$$\begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \in \mathfrak{a}, \quad (d_1, \dots, d_n) \mapsto (\pm d_{\sigma(1)}, \dots, \pm d_{\sigma(n)}),$$

where  $D = \text{diag}(d_1, \dots, d_n)$  and the number of negative signs is even.

T.Y. Tam, A Lie theoretical approach of Thompson's theorems of singular values-diagonal elements and some related results, J. of London Math. Soc. (2), 60 (1999) 431-448.

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## 6. CMJD for real semisimple $G$ :

How to extend Weyl-Horn's result (not  $QR$  version) to semisimple Lie groups?

Weyl-Horn's result is on the eigenvalue moduli and singular values of a (nonsingular) matrix.

eigenvalue moduli of a nonsingular  $A \in \mathbb{C}_{n \times n} \rightsquigarrow b(g)$ .

This requires the complete multiplicative Jordan decomposition (CMJD):

- $h \in G$  is **hyperbolic** if  $h = \exp(X)$  where  $X \in \mathfrak{g}$  is **real semisimple**, that is,  $\text{ad } X \in \text{End } \mathfrak{g}$  is diagonalizable over  $\mathbb{R}$ .
- $u \in G$  is **unipotent** if  $u = \exp(N)$  where  $N \in \mathfrak{g}$  is **nilpotent**, that is,  $\text{ad } N \in \text{End } \mathfrak{g}$  is nilpotent.
- $e \in G$  is **elliptic** if  $\text{Ad}(e) \in \text{Aut } \mathfrak{g}$  is **diagonalizable** over  $\mathbb{C}$  with eigenvalues of **modulus 1**.



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Apart from  $\pm 1$ , the elements of  $SL_2(\mathbb{R})$  fall into three types according to their Jordan forms. Let  $g \in SL_2(\mathbb{R})$

1.  $g$  is **elliptic**  $\Leftrightarrow g$  is conjugate to  $\text{diag}(e^{i\theta}, e^{-i\theta})$ ,  $0 < \theta < \pi \Leftrightarrow |\text{trace } g| < 2$ .
2.  $g$  is **hyperbolic**  $\Leftrightarrow g$  is conjugate to  $\text{diag}(\alpha, \alpha^{-1})$ ,  $\alpha \neq 0 \Leftrightarrow |\text{trace } g| > 2$ .
3.  $g$  is **unipotent** (parabolic)  $\Leftrightarrow g$  has repeated eigenvalues 1 or  $-1 \Leftrightarrow |\text{trace } g| = 2$ .

Kostant's **Proposition 2.1** (CMJD): Each  $g \in G$  can be **uniquely** written as

$$g = ehu,$$

where  $e, h, u$  commute.

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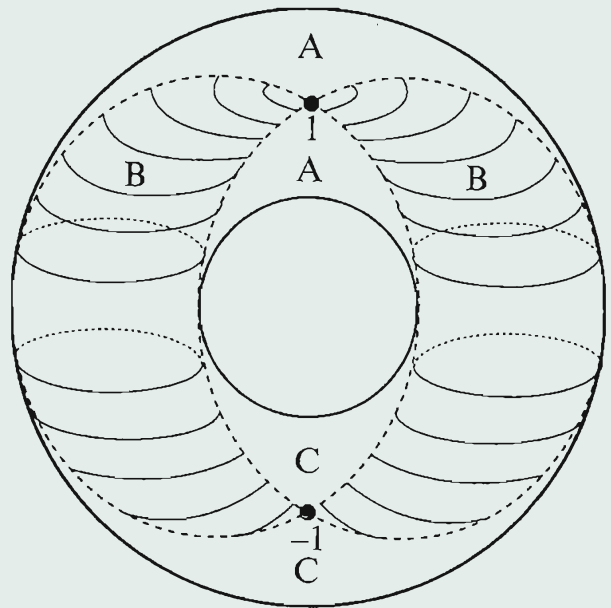
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## A picture for $SL_2(\mathbb{R})$



The elliptic elements form the **sausage-like** region B which is the union of all subgroups of  $SL_2(\mathbb{R})$  which are isomorphic to the circle group.

The closure of the region A consists of the elements with trace  $\geq 2$ .

The region C is the elements with trace  $< -2$ . These do not belong to any 1-parameter subgroup.

R. Carter, G. Segal, I. MacDonald, *Lectures on Lie Groups and Lie Algebras*, London Math. Soc. Student Texts, 1995, p.57.



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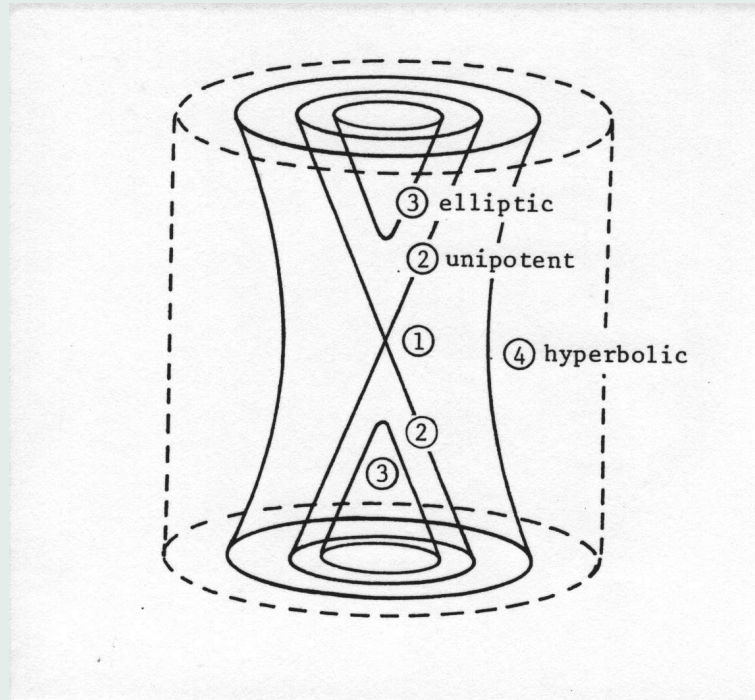
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## Infinitesimal picture at 1:



Orbits are

- (1) the origin,
- (2) each half of the cone excluding the origin (**unipotent elements**),
- (3) each sheet of the hyperboloid of two sheets (**elliptic elements**),
- (4) each hyperboloid of one sheet (**hyperbolic elements**).

M. Atiyah, *Representation of Lie Groups*, Proceedings of the SRC/LMS Research Symposium on Representations of Lie Groups, Oxford, 28 June-15 July 1977, London Mathematical Society Lecture Note Series, 1979 34, p.211



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## CMJD for $GL_n(\mathbb{C})$ :

Viewing  $g \in GL_n(\mathbb{C}) \subset \mathfrak{gl}_n(\mathbb{C})$  the **additive** Jordan decomposition for  $\mathfrak{gl}_n(\mathbb{C})$  yields

$$g = s + n_1$$

- $s \in GL_n(\mathbb{C})$  semisimple,
- $n_1 \in \mathfrak{gl}_n(\mathbb{C})$  nilpotent, and
- $sn_1 = n_1s$ .

Moreover the conditions determine  $s$  and  $n_1$  completely.

Put

$$u := 1 + s^{-1}n_1 \in GL_n(\mathbb{C})$$

and we have the **multiplicative** Jordan decomposition

$$g = su,$$

where  $s$  is semisimple,  $u$  is unipotent, and  $su = us$ , and  $s$  and  $u$  are completely determined by AJD.

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Since  $s$  is diagonalizable,

$$s = eh,$$

where  $e$  is elliptic,  $h$  is hyperbolic and

$$eh = he,$$

and these conditions completely determine  $e$  and  $h$ . Since

$$ehu = g = ugu^{-1} = ueu^{-1}uhu^{-1}u,$$

the uniqueness of  $s$ ,  $u$ ,  $e$  and  $h$  implies that  $e$ ,  $u$  and  $h$  commute.

### CMJD for $GL_n(\mathbb{R})$ :

Since  $g \in GL_n(\mathbb{R})$  is fixed under complex conjugation, the uniqueness of  $e$ ,  $h$  and  $u$  implies  $e, h, u \in GL_n(\mathbb{R})$ :

$$ehu = g = \bar{g} = \bar{e}\bar{h}\bar{u}$$

(unipotent, hyperbolic, elliptic are invariant under complex conjugation)

Thus  $g = eh u$ , when viewed as element in  $GL_n(\mathbb{C})$  is the CMJD for  $GL_n(\mathbb{R})$ .

S. Helgason, *Differential Geometry, Lie Groups and Symmetric Space*, New York: Academic, 1978, p.431.



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## 7. A pre-order $\leq$

**Proposition 2.4** An element  $h \in G$  is hyperbolic if and only if it is conjugate to an element in  $A$ .

**CMJD:**  $g = ehue \rightsquigarrow h(g) \rightsquigarrow b(g) \in A$

**Example:**  $G = \mathrm{SL}_n(\mathbb{R})$  (or  $\mathrm{SL}_n(\mathbb{C})$ ),  $b(g)$  is simply  $|\lambda|(g)$  where  $g \in \mathrm{SL}_n(\mathbb{R})$ . Define a relation on  $f, g \in G$ :  $f \leq g$  if  $\mathcal{A}(f) \subset \mathcal{A}(g)$  where

$$\mathcal{A}(g) := \exp \operatorname{conv} W(\log b(g)).$$

The pre-order  $\leq$  is independent of the choice of  $A$ .

Kostant's **Theorem 3.1** characterizes the pre-order order  $f \leq g$ :

**Theorem 7.1.** Let  $f, g \in G$ . Then  $f \leq g$  if and only if  $|\pi(f)| \leq |\pi(g)|$  for all finite dimensional representations  $\pi$  of  $G$ , where  $|\cdot|$  denotes the spectral radius.



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**Example:** We now are to describe the partial order  $\leq$  for  $SL_n(\mathbb{R})$ .

$$\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R}) = \mathfrak{so}(n) + \mathfrak{p}, \text{ Cartan decomposition}$$

$$\mathfrak{k} = \mathfrak{so}(n)$$

$$\mathfrak{p} = \text{space of real symmetric matrices of zero trace}$$

$$K = SO(n)$$

$$\mathfrak{a} \subset \mathfrak{p}, \text{ diagonal matrices of zero trace}$$

$$A = \text{positive diagonal matrices of determinant 1}$$

$$W = S_n$$

$W$  acts on  $A$  and  $\mathfrak{a}$  by permuting the diagonal entries of the matrices in  $A$  and  $\mathfrak{a}$ .

$$\begin{aligned} \mathcal{A}(f) &= \exp \text{conv} \{ \text{diag} (\log |\alpha_{\sigma(1)}|, \dots, \log |\alpha_{\sigma(n)}|) : \sigma \in S_n \} \\ &= \exp \text{conv} S_n (\log |\alpha|) \end{aligned}$$

where  $\alpha$ 's denote the eigenvalues of  $f \in SL_n(\mathbb{C})$ .

So  $f \leq g$ ,  $f, g \in SL_n(\mathbb{R})$  means that  $|\alpha| \prec_{\log} |\beta|$  where  $\beta$ 's are the eigenvalues of  $g$ .



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## 8. Weyl-Horn $\rightarrow$ Theorem 5.4

Kostant's **Theorem 5.4**:

Let  $p \in P := \exp \mathfrak{p}$ . Then for any  $k, v \in K$ ,  $kp v \leq p$ , i.e.  $h(kpv) \leq p$ . Conversely for every hyperbolic element  $h \leq p$ , there exists  $k, v \in K$  such that  $h(kpv)$  is conjugate to  $h$ . Moreover  $k$  and  $v$  can be chosen so that the elliptical component  $e(kpv) = 1$ .

**Weyl-Horn  $\rightsquigarrow$  Theorem 5.4.** (Group level)

Kostant's **Proposition 6.2**: The set of hyperbolic elements in  $G$  is  $P^2$ .

**Example:** When  $G = \mathrm{SL}_n(\mathbb{C})$ ,  $\mathfrak{p}$  is the space of Hermitian matrices of zero trace so that  $P = \exp \mathfrak{p}$  is the set of positive definite matrices  $A$  with  $\det A = 1$ . Thus the set of all diagonalizable matrices with positive eigenvalues in  $\mathrm{SL}_n(\mathbb{C})$  is  $P^2$  which is the set of products of two positive definite matrices in  $\mathrm{SL}_n(\mathbb{C})$ .

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## 9. Complex Symmetric matrices

Since a complex symmetric matrix  $A$  has decomposition

$$A = U^T S U$$

for some unitary matrix, where  $S = \text{diag}(s_1, \dots, s_n)$ .

**Theorem 9.1.** (Thompson) Let  $d \in \mathbb{C}^n$  and  $s \in \mathbb{R}_+^n$ . Then there exists a symmetric  $A \in \mathbb{C}_{n \times n}$  such that  $\text{diag } A = d$  and  $s(A) = s$  if and only if

$$\begin{aligned} \sum_{i=1}^k |d_i| &\leq \sum_{i=1}^k s_i, & k = 1, \dots, n, \\ \sum_{i=1}^{k-1} |d_i| - \sum_{i=k}^n |d_i| &\leq \sum_{i=1}^{k-1} s_i - s_k + \sum_{i=k+1}^n s_i, & k = 1, \dots, n, \\ \sum_{i=1}^{n-3} |d_i| - \sum_{i=n-2}^n |d_i| &\leq \sum_{i=1}^{n-2} s_i - s_{n-1} - s_n, \end{aligned} \quad (3)$$

after rearranging the entries of  $d$  and  $s$  in descending order with respect to modulus.

The three subtracted terms on the left, two on the right of (3) present for  $n \geq 3$  only.



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In other words, , the set  $\{\text{diag}(U^T S U) : U \in U(n)\}$  is completely described by the inequalities.

The proof given by Thompson is long and difficult and there is no conceptual proof so far.

R.C. Thompson, Singular values and diagonal elements of complex symmetric matrices, *Linear Algebra Appl.* 26 (1979), 65–106.

Motivation: Physics  $\rightarrow$  the study of diagonal entries and singular values of a complex symmetric matrix.

B. Tromberg and S. Waldenstrom, Bounds on the diagonal elements of a unitary matrix, *Linear Algebra and Appl.* 20 (1978) 189-195.

S. Waldenstrom, S-matrix and unitary bounds for three channel systems, with applications to low-energy photoproduction of pions from nucleons, *Nuclear Phys.* B77 (1974) 479-493.

**Challenge:** Conceptual proof of Thompson's theorem.

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