Restricted Secants of Grassmannians

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Tensors: Low Rank, High Rank

Given a tensor space like $\mathbb{C}^{n_1 \times \cdots \times n_k}$, how many rank-1 tensors fit in a sum?

$$T = \sum_{i=1}^{r} v_1^{(i)} \otimes \cdots \otimes v_k^{(i)}$$

1. If you insist all terms are completely independent?
2. If you just require identifiability?

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For now, we think about skew-symmetric tensors, so $n_i = n$ for all i and the tensors space is $\bigwedge^k \mathbb{C}^n$, and the basic rank-1 elements are square-free ordered monomials.

1. Things like $e_1e_2e_3 + e_4e_5e_6$ in $\bigwedge^3\mathbb{C}^6$.
2. Things like $e_2e_3e_4 + e_1e_3e_5 + e_1e_2e_6$ in $\bigwedge^3\mathbb{C}^6$, or $e_1e_2e_3 + e_1e_4e_5 + e_1e_6e_7$ in $\bigwedge^3\mathbb{C}^7$

Grassmannians

- Let V denote a finite dimensional vector space over a field $\mathbb{F}$.
- Indecomposable skew-symmetric tensors correspond to linear spaces.
- For instance, $e_1 e_2 e_3 \in \bigwedge^3 V$ corresponds to the 3-plane spanned by e_1, e_2, e_3 in V.
- Similarly $a \wedge b \wedge c$ might correspond to the span of a, b, c in V.
- In general the points $v_1 \wedge \cdots \wedge v_k$ in $\bigwedge^k V$ comprise the Grassmannian $\mathrm{Gr}(k, V)$ in the Plücker embedding in $\mathbb{P}\bigwedge^k V$.
- Plücker: take a $k \times n$ matrix to its list of maximal minors (up to scale).
- The points on the Grassmannian are the rank-1 tensors.

Secant Varieties

- Given a projective variety $X \subset \mathbb{P}^n$, let $\sigma_k(X)$ denote the k-secant variety:

  The closure of all points with X-rank $\leq k$, i.e. those of the form

$$[v] = [x_1 + \cdots + x_k] \text{ with all } [x_i] \in X$$

- An X-rank decomposition of $[v]$ might recover the information in the terms $[x_i]$.
- For coding theory, want to send messages (the elements of X, i.e. the rank-1 tensors as a sum and recover the summands on the other end.
- "How many rank 1 tensors can you recover?" $=$ channel capacity.
- requiring all rank-1 elements to be independent it very limiting.

r-Restricted Secant Varieties

Consider sums of monomials that have a mutual common factor, like $e_1e_2e_3 + e_1e_4e_5 + e_1e_6e_7$. The closure of such is a restricted secant variety.

- Restricted secants of Grassmannians appeared in [Fulton and Harris]. In general

Definition

The r-restricted s-secant variety of $\mathrm{Gr}(k, V)$, *is* $\sigma_s^r(\mathrm{Gr}(k, V)) =$

$$cl\{[E_1 + \cdots + E_s] \mid [E_i] \in \mathrm{Gr}(k, V), \dim(\bigcap_{i=1}^s E_i) \geq r\} \subset \mathbb{P}\bigwedge^k V.$$

- The first question we ask is what is the dimension of this variety? i.e. how many words can we pack into each message if they all share some letters?

Past Work on Dimensions of Secant Varieties

- Terracini's lemma reduces the dimension of the secant variety of a variety X to a dimension count for a sum of linear spaces. *Defectivity* is when this dimension count is not what we expect.

- Alexander and Hirshowitz settled the classification of defectivity for Veronese re-embeddings of projective space.

- Classifying defectivity for general tensors and for skew-symmetric tensors are still open and involve a long list of works [Catalisano, Geramita, Gimigliano, Abo,Ottaviani,Peterson, Vannieuwenhoven, Bernardi, Chiantini, Draisma...]

- Why study r-restricted secants? Usual secants $\sigma_s(\mathrm{Gr}(k, V))$ start having restrictions as soon as $k \cdot s \geq \dim V$. Study this methodically in the first case.

The Baur-Draisma-de Graaf Conjecture

Conjecture (BdDG 2007)

The secant varieties of Grassmannians $\sigma_s(\mathrm{Gr}(k, \mathbb{C}^n))$ are all non-defective except:

Secant Variety	*actual codimension*	*expected codimension*
$\sigma_s(\mathrm{Gr}(2, \mathbb{C}^n))$	$2s(s-1)$	*0*
$\sigma_3(\mathrm{Gr}(3, \mathbb{C}^7))$	*1*	*0*
$\sigma_3(\mathrm{Gr}(4, \mathbb{C}^8))$	*20*	*19*
$\sigma_4(\mathrm{Gr}(4, \mathbb{C}^8))$	*6*	*2*
$\sigma_4(\mathrm{Gr}(3, \mathbb{C}^9))$	*10*	*8*

Generating examples and Macaulay2

- To compute the dimension of a parametrized variety we may:
  1. Generate sufficiently many random points $[p]$ of the source.
  2. Compute the partial derivatives $\frac{\partial \phi_i}{\partial x_j}(p)$ and populate the matrix $d\phi_p$.
  3. Compute the rank of the matrix $d\phi_p$.
  4. Try to use the structure of the points (like cofactor expansion) to improve efficiency and generate a lot of examples to learn what's true.
- For points $p = A + B$ with A, B using some independent variables we can utilize the block structure of the Jacobian to improve computations:

$$\begin{pmatrix} \frac{\partial A_{m_1}}{\partial a_{00}} + \frac{\partial B_{m_1}}{\partial a_{00}} & \cdots & \frac{\partial A_{m_1}}{\partial a_{(r-1)(n-1)}} + \frac{\partial B_{m_1}}{\partial a_{(r-1)(n-1)}} & \frac{\partial A_{m_1}}{\partial a_{(r-1)(n-1)+1}} & \cdots & \frac{\partial A_{m_1}}{\partial a_{(k-1)(n-1)}} & \frac{\partial B_{m_1}}{\partial b_{00}} & \cdots & \frac{\partial B_{m_1}}{\partial b_{(k-r-1)(n-1)}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial A_{m_d}}{\partial a_{00}} + \frac{\partial B_{m_d}}{\partial a_{00}} & \cdots & \frac{\partial A_{m_d}}{\partial a_{(r-1)(n-1)}} + \frac{\partial B_{m_d}}{\partial a_{(r-1)(n-1)}} & \frac{\partial A_{m_d}}{\partial a_{(r-1)(n-1)+1}} & \cdots & \frac{\partial A_{m_d}}{\partial a_{(k-1)(n-1)}} & \frac{\partial B_{m_d}}{\partial b_{00}} & \cdots & \frac{\partial B_{m_d}}{\partial b_{(k-r-1)(n-1)}} \end{pmatrix}^{\top}. \tag{1}$$

Restricted Chordal Varieties

- Skew-symmetric matrices of rank $\leq r$ corresponds to the secant variety $\sigma_r(\mathrm{Gr}(2, V))$, which is always defective.

Proposition (Bidleman-Oeding)

Let $n = k + 2$ and $r = \max(r, 2k - n)$. Then the expected and virtual dimensions are:

$$\exp.\dim(\sigma_2^r(\mathrm{Gr}(k, n))) = \min\left\{\binom{n}{k} - 1, r(n-r) + 2((k-r)(n-k)) + 1\right\},$$

$$\mathrm{v}.\dim(\sigma_2^r(\mathrm{Gr}(k, n))) = \min\left\{\binom{n}{k} - 1, r(n-r) + 2((k-r)(n-k)) - 3\right\}.$$

Further $\sigma_2^{r+1}(\mathrm{Gr}(k, n))) = \mathrm{Gr}(k, n)$.

Abstract Secant Variety and Incidence Description

- The abstract s-secant variety of X is denoted $\Sigma_s(X) \subset (X)^{\times s} \times \mathbb{P}V$.
  It always has the expected dimension.

$$\Sigma_s(X) = \mathrm{cl}\{([x_1], [x_2], \ldots, [x_s], [p]) \mid p \in \mathrm{span}\{x_1, \ldots, x_s\}\} \subset \mathbb{P}V^{\times s} \times \mathbb{P}V.$$

  Projection to the last factor gives the embedded s-secant variety, $\sigma_s(X) \subset \mathbb{P}V$.
- The abstract r-restricted s-secant variety is the incidence variety

$$\mathcal{I} \subset \mathrm{Gr}(r, V) \times \mathrm{Gr}(k-r, V)^{\times s} \times \mathbb{P}\bigwedge^k V,$$

  defined by

$$\mathcal{I} := \mathrm{cl}\{([E], [F_1], \ldots, [F_s], [z]) \mid z \in \mathrm{span}\{E \wedge F_1, \ldots, E \wedge F_s\}\}.$$

Main Results

Theorem (Bidleman-Oeding)

Let $V = \mathbb{C}^n$ *with* $r, s, \geq 0$ *and* $0 \leq k \leq n$. *The restricted secant variety* $\sigma_s^r(\mathrm{Gr}(k, V))$ *is birationally isomorphic to the fiber bundle with base* $\mathrm{Gr}(r, V)$ *and whose fiber over a point* $[E] \in \mathrm{Gr}(r, V)$ *is* $\sigma_s(\mathrm{Gr}(k - r, V/E))$.

Corollary (Bidleman-Oeding)

If the BDdG conjecture is true, then $\sigma_s^r(\mathrm{Gr}(k, V))$ *has no additional defect other than the defect coming from (usual) secant varieties of Grassmannians.*

The tautological sequence of bundles over the Grassmannian $\mathrm{Gr}(r, V)$:

$$0 \longrightarrow \mathcal{S} \longrightarrow \underline{V} \longrightarrow \mathcal{Q} \longrightarrow 0$$

Over a point $E \in \mathrm{Gr}(r, V)$ the respective fibers are E, V and V/E.
Applying the Schur functor $\bigwedge^{k-r}$ we obtain a vector bundle:

$$\begin{array}{c} \bigwedge^{k-r} Q \\ \big\downarrow \\ \mathrm{Gr}(r, V) \end{array}$$

the fiber over E is $\bigwedge^{k-r}(V/E)$. In each fiber we have (a copy of) $\sigma_s(\mathrm{Gr}(k-r, V/E))$.
Our fiber bundle is depicted as:

$$\begin{array}{ccccc} \sigma_s(\mathrm{Gr}(k-r, V/E)) & \hookrightarrow & \mathbb{P}\bigwedge^{k-r} V/E & \hookrightarrow & \mathbb{P}\bigwedge^{k-r} Q \\ & \searrow & & & \big\downarrow \\ & & E & \in & \mathrm{Gr}(r, V) \end{array}$$

Applications to Coding Theory by Example

- Binary codes of $\mathrm{Gr}(3, \mathbb{F}_2^6) \subset \mathbb{P}\bigwedge^3 \mathbb{F}_2^6$. There are $1,395$ points in $\mathrm{Gr}(3, \mathbb{F}_2^6)$. The linear code has a $20 \times 1\,395$ generator matrix, M: columns are the Plücker coordinates of each of the $1\,395$ points. Encode a message $b \in \mathbb{F}_2^{1395}$ via Mb.
- Here we can completely describe the $\mathrm{SL}_6(\mathbb{F}_2)$-orbits in $\bigwedge^3 \mathbb{F}_2^6$.

X°	0	$\mathrm{Gr}(3,6)^\circ$	$\sigma_2^1(\mathrm{Gr}(3,6))^\circ$	$\tau(\mathrm{Gr}(3,6))^\circ$	$\sigma_2(\mathrm{Gr}(3,6))^\circ$	Z°
$\#X^\circ$	1	1,395	54,684	468,720	357,120	166,656

Normal forms:

  - $e_0 e_1 e_2 \in \mathrm{Gr}(3,6)^\circ$
  - $e_0 e_1 e_2 + e_0 e_3 e_4 \in \sigma_2^1(\mathrm{Gr}(3,6))^\circ$
  - $e_0 e_1 e_2 + e_1 e_2 e_4 + e_0 e_1 e_5 \in \tau(\mathrm{Gr}(3,6))^\circ$
  - $e_0 e_1 e_2 + e_3 e_4 e_5 \in \sigma_2(\mathrm{Gr}(3,6))^\circ$
  - $e_1 e_2 e_4 + e_0 e_3 e_4 + e_0 e_2 e_5 + e_0 e_3 e_5 + e_1 e_3 e_5 = (e_1 e_2 + e_0 e_3) e_4 + (e_0 e_2 + (e_0 + e_1) e_3) e_5 \in Z^\circ$
- An identifiability over $\mathbb{F}_2$ for $\sigma_2^1(\mathrm{Gr}(3,6))^\circ$: points correspond uniquely to pairs of a non-zero vector in $\mathbb{F}_2^6$ and a full rank skew-symmetric 5×5 matrix over $\mathbb{F}_2$.