Secant Cumulants and Toric Geometry

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Secant Varieties: $X \subset \mathbb{P}^n$, $\sigma_2(X) = \{ p \in \mathbb{P}^n \mid p \in \langle x, y \rangle, \text{ some } x, y \in X \}$

Classically: If $\sigma_2(X) \neq \mathbb{P}^n$, Then X can be isomorphically projected to $\mathbb{P}^{n-1}$.

Today: (Applied Algebraic Geometry)

(Tensors) & Secant Varieties Relate to

- Signal Processing
- Computer Vision
- Algebraic Statistics
- Computational Complexity ....

Want to know:

- Dimension (... degree, regularity, ...)
- Equations (Syzygies ...)
- Structure (Stratifications, Singular locus ...)
- Decomposition: Given $p \in \sigma_k(X)$, find $x_1, x_2, \dots, x_n \in X$ Such that $p = \sum_{i=1}^{n} x_i$
- Further algebraic Properties
  - Cohen Macaulay?
  - Gorenstein?
  - Rational Singularities?

Tensors, Rank & Border Rank

Let $A_1, \dots, A_n$ Vector Spaces $/\mathbb{C}$ (Think $\mathbb{C}^2$)

$A_1 \otimes \cdots \otimes A_n$ The Space of (Binary) tensors

Rank 1 tensors: $a_1 \otimes \cdots \otimes a_n$, $a_i \in A_i$

$Seg: \mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n \longrightarrow \mathbb{P}(A_1 \otimes \cdots \otimes A_n)$

$[a_1], \dots, [a_n] \longmapsto [a_1 \otimes \cdots a_n]$

- Dimension: $n_1 + \cdots + n_n$ inside $\mathbb{P}^{(n_1+1)\cdots(n_n+1)-1}$

- Smooth toric Variety

$(\mathbb{C}^*)^{n_i}$ acts in each A_i (also $GL(A_i)$)

- Also homogeneous for $GL(A_1) \times \cdots \times GL(A_n)$.

- Ideal Defined by binomial Quadrics
    - 2×2 determinants

- Decomposition found easially (consider all projections)

Today: Border Rank 2 tensors:

ie the Zariski closure of tensors of the form

$$a_1 \otimes a_2 \otimes \cdots \otimes a_n + b_1 \otimes b_2 \otimes \cdots \otimes b_n, \quad (a_i, b_i \in A_i)$$

- Not toric... or is it?

Toric Varieties (Bare bones)

<u>Defn</u> $Y \subseteq \mathbb{A}^n$ is <u>toric</u> if it is the image of a monomial map:

(<u>Normal</u>)

(has a dense torus whose action extends to Y.)

$$(\mathbb{C}^*)^k \longrightarrow \mathbb{A}^n$$

$$\bar{x} = (x_1 \ldots x_k) \longmapsto (m_1(\bar{x}), \ldots, m_n(\bar{x}))$$

- Described by Poly-topes / Polyhedral fans
- Algebraic properties are all "combinatorial"

Try to prove a given Variety is "not" toric ...

I dare you ...

Main Theorem:

The Secant variety $\sigma_2(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_n)$ is covered by normal affine toric varieties.

In particular, it has rational singularities

Corollary: The singular locus is the union of $\sigma_2(\mathbb{P}A_1 \times \mathbb{P}A_2) \times \mathbb{P}A_3 \times \cdots \times \mathbb{P}A_n$ + permutations.

Theorem: The only Gorenstein secants are

- $\sigma_2(\mathbb{P}^k \times \mathbb{P}^k)$
- $\sigma_2(\mathbb{P}^1 \times \mathbb{P}^k) = \mathbb{P}^{2k+1}$
- $\sigma_2(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3)$
- $\sigma_2(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{P}^7$
- $\sigma_2(\mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^3)$
- $\sigma_2(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$
- $\sigma_2(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$

Note: $\sigma_2(\mathbb{P}^3 \times \mathbb{P}^3) \cong \sigma_2(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3)$ is arith. Gorenstein.

What about the others?

A very special Cremona Transformation...

Cumulants (See also Zwiernik - Sturmfels) and Secant Cumulants (Binary Case)

For $I \subset [n]$ x_I give coordinants on $\mathbb{P}^{2^n-1}$

Change Coord's: Work on Affine patch $\{x_\emptyset = 1\}$

$$y_i = x_i \,, \quad y_I = \sum_{A \subseteq I} (-1)^{|I \setminus A|} x_A \prod_{i \in I \setminus A} x_i \,. \qquad \text{cumulants}$$

$$z_i = y_i \,, \quad z_I = \sum_{\pi \in \widetilde{IP}(I)} (-1)^{|\pi|-1} \prod_{B \in \pi} y_B \qquad \text{Secant Cumulants}$$

Where $IP(I)$ = interval set Partitions, $\widetilde{IP}$ = interval set partitions w/ no singletons

eg: $IP(\{1,2,3,4\}) = \{1234,\ 1|234,\ 12|34,\ 123|4,\ 1|2|34,\ 1|23|4,\ 12|3|4,\ 1|2|3|4\}$

eg: $\widetilde{IP}(\{1,2,3,4\}) = \{1234,\ \cancel{1|234},\ 12|34,\ \cancel{123|4},\ \cancel{1|2|34},\ \cancel{1|23|4},\ \cancel{12|3|4},\ \cancel{1|2|3|4}\}$

$\widetilde{IP}([5]) = \{12345,\ 12|345,\ 123|45\}$

ex. $n=3$ $y_i = x_i$

$y_{ij} = x_{ij} - x_i x_j$ $\qquad y_{123} = x_{123} - x_1 x_{23} - x_3 x_{12} - x_2 x_{13} + 2 x_1 x_2 x_3$

inverse: $x_{ij} = y_{ij} + y_i y_j$ $\qquad x_{123} = y_{123} + y_1 y_{23} + y_2 y_{13} + y_3 y_{12} + y_1 y_2 y_3$ $\qquad$ Möbius inv.

$n=4$:

$$\begin{aligned} y_{1234} = \ & x_{1234} - x_1 x_{234} - x_2 x_{134} - x_3 x_{124} - x_4 x_{123} \\ & + x_{12} x_3 x_4 + x_{13} x_2 x_4 + x_{14} x_2 x_3 \\ & + x_{23} x_1 x_4 + x_{24} x_1 x_3 + x_{34} x_1 x_2 - 3 x_1 x_2 x_3 x_4 \end{aligned}$$

$z_{ij} = y_{ij}$, $z_{ijk} = y_{ijk}$ $\qquad z_{1234} = y_{1234} - y_{12} y_{34}$, $\quad z_{12345} = y_{12345} - y_{12} y_{345} + y_{123} y_{45}$

These changes of coords are Δ-ular $\Rightarrow$ isomorphisms (in every characteristic)

Secants get nicer in Cumulants

Prop. [SZ'12] The y_I - cumulants Linearize $\widehat{Seg}(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1)$ in $U_\emptyset$

(defined by vanishing of $y_{ij} = x_{ij} - x_i x_j$)

Parametrizations on $U_\emptyset$:

Segre:

$\mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \to U_\emptyset$

$\begin{bmatrix} a_1 \\ 1 \end{bmatrix} \times \cdots \times \begin{bmatrix} a_n \\ 1 \end{bmatrix} \mapsto x_I = \prod_{i \in I} a_i$

Secant: $V = \sigma_2(Seg(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1)) \cap U_\emptyset$

$$x_I = (1-t) \prod_{i \in I} a_i + t \prod_{i \in I} b_i$$

Lemma: $V = \sigma_2(\mathbb{P}^{1 \times n}) \cap U_\emptyset$ is also parametrized by

$z_i = (1-t) a_i + t b_i$ Linear

$z_I = t(1-t)(1-2t)^{|I|-2} \prod_{i \in I} (b_i - a_i)$ monomial!

$\to$ z_I define a toric variety $T^{2,n}$

$|I| \geq 2$

So $V = \mathbb{C}^n \times T^{2,n}$ a vector bundle over a toric variety

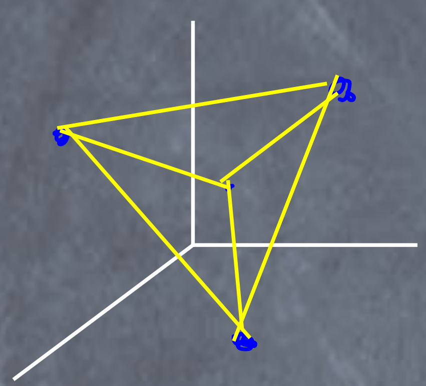
The toric varieties $T^{a,b}$ [Sturmfels '96 § 14A]

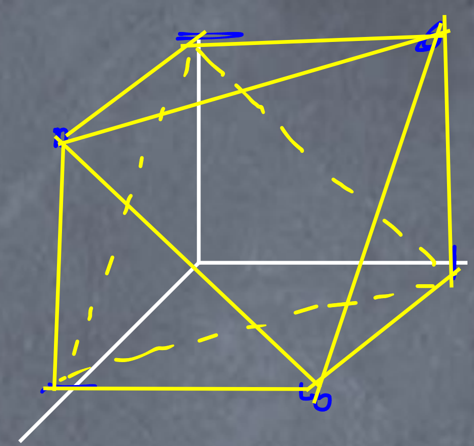
Lattice $\mathcal{L}_{a,b} = \{ I \subset \{1\} \times \{0,1\}^n \mid a+1 \le |I| \le b+1 \}$

Polytope: $P^{a,b} = \mathrm{Conv}(\mathcal{L}_{a,b})$

$P^{2,3} = \{1111, 1110, 1101, 1011\} \to \mathrm{Conv}(P^{2,3}) = \text{simplex}$

Example: $P^{1,2} = \{1110, 1101, 1011, 1100, 1010, 1001\} \to \mathrm{Conv}(P^{1,2}) = \text{Octahedron}$





Take Spec. of affine semigroup Alg. $\langle P^{a,b} \rangle \rightsquigarrow$ toric var. $T^{a,b}$

Def: **Bumping**: $p, q \in \mathcal{L}_{a,b-1}$. If $\exists\, i \in [n]$ $\begin{smallmatrix} p_i = 0 \\ q_i = 0 \end{smallmatrix}$ then $p, q, p+e_i, q+e_i \in \mathcal{L}_{a,b}$

$$(p+e_i)+q = p+(e_i+q)$$

Swapping: $p, q \in \mathcal{L}_{a-1,b-1}$. If $\exists\, i,j \in [n]$ $\begin{smallmatrix} p_i = 0 = p_j \\ q_i = 0 = q_j \end{smallmatrix}$ then $p, q, \begin{smallmatrix} p+e_i, \\ p+e_j \end{smallmatrix} \begin{smallmatrix} q+e_i \\ q+e_j \end{smallmatrix} \in \mathcal{L}_{a,b}$

$$(p+e_i)+(q+e_j) = (p+e_j)+(e_i+q)$$

Every relation $\sum_{i=1}^{d} p_i = \sum_{i=1}^{d} q_i$ in $\mathcal{L}_{a,b}$

induces $\prod_{i=1}^{d} z_{I_i} = \prod_{i=1}^{d} z_{J_i}$ binomial relation

Prop^n: The ideal of $T_{a,b}$ is gen'd by quadrics corresponding to bumping and swapping

Special case:

$$I(T_{2,n}) = \left\langle \begin{matrix} z_I\, z_{J \cup \{j\}} = z_J\, z_{I \cup \{j\}} & j \notin I \cup J,\ |I|, |J| > 1 \\ z_{ij}\, z_{kl} = z_{il}\, z_{jk} & ijkl \text{ distinct.} \end{matrix} \right\rangle$$

Prop^n $I(V)$ has a quadratic squarefree Gröbner basis
(in some monomial order)

follows from [Sturmfels '96, Thm 14.2 (Pulling triangulatons)]

Cor. The secant var. $\sigma_2(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1)$ has rational singularities

In particular it is normal, Cohen Macaulay,

w/ codim(singloc) ≥ 2.

Toric description tells precisely the singular locus.

$\sigma_2(\mathbb{P}A_1 \times \mathbb{P}A_2) \times \mathbb{P}A_3 \times \cdots \times \mathbb{P}A_n$ + permutations.

What about flattenings?

$$\{X_I\} \longrightarrow \begin{array}{c|cccc} & \emptyset & 2 & 4 & 24 \\ \hline \emptyset & X_\emptyset & X_2 & X_4 & X_{24} \\ 1 & X_1 & X_{12} & X_{14} & X_{124} \\ 3 & X_3 & X_{23} & X_{34} & X_{234} \\ 13 & X_{13} & X_{123} & X_{124} & X_{1234} \end{array}$$

A flattening corresp. to partition $13|24$

Set $X_\emptyset = 1$, Do row & col. ops

$$\begin{pmatrix} 1 & X_2 & X_4 & X_{24} \\ X_1 & X_{12} & X_{14} & X_{124} \\ X_3 & X_{23} & X_{34} & X_{234} \\ X_{13} & X_{123} & X_{124} & X_{1234} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & X_{12}-X_1X_2 & X_{14}-X_1X_4 & X_{124}-X_1X_{24} \\ 0 & X_{23}-X_3X_2 & X_{34}-X_3X_4 & X_{234}-X_3X_{24} \\ 0 & X_{123}-X_{13}X_2 & X_{124}-X_{13}X_4 & X_{1234}-X_{13}X_{24} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & y_{12} & y_{14} & z_{124} \\ 0 & y_{23} & y_{34} & z_{234} \\ 0 & z_{123} & z_{134} & z_{1234} \end{pmatrix}$$

$n \times n$ minors become $(n-1)\times(n-1)$ minors in secant cumulants

In particular, we show a weaker version of Raicu's Thm:

$I(V)$ is generated by 3x3 minors of flattenings.

Certain binomials in Secant cumulants

Conjecture (Garcia, Stillman, Sturmfels ~2005)
The ideal of $\sigma_2(Seg(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}))$ is generated by 2x2 minors of flatt.
Progress by Allman-Rhodes, Landsberg-Manivel, Landsberg-Weyman
Theorem (Raicu '12) The GSS conjecture holds even more generally for the secant of the Segre-Veronese variety.
Proof Relies on Rep'n Theory in Char. Zero. Deep Result

Techniques Also can be used for the Tangential variety, see (O___- Raicu '12)

A consequence of the Toric Description is
Thm (M.O.Z. '12) The 3x3 minors of flattenings define the Affine Scheme
$\sigma_2(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}) \cap U_\emptyset$ in any characteristic!

Wrap up:

Secant Cumulants provide a convient way to see local structure, Shows an open covering by normal affine toric varieties for

$\sigma_2(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s})$ secant variety
$\tau(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s})$ tangential variety also

Get info about singularities, Cohen Macaulay-ness determine when Gorenstein, get singular locus.

Where else can we use these techniques?