Set Theory Notation

Basic notation

1. $x \in A$ means x belongs to the set A. We also say "x is a member of A" or "x is an element of A".
2. $A \subset B$ or $A \subseteq B$ means the set A is a subset of the set B.

   (Both notations are used to indicate a subset, not necessarily a proper subset.)
3. $A \subsetneq B$ means the set A is a subset of the set B and $A \neq B$, i.e., A is a proper subset of the set B.
4. $\emptyset$ denotes the empty set, i.e., the set with no elements.

Special number sets

1. $\mathbb{Z}$ = the set of integers = $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
2. $\mathbb{Z}^+$= the set of positive integers = $\{1, 2, \ldots\} = \mathbb{N}$ = the set of natural numbers
3. $\mathbb{Z}^-$ = the set of negative integers = $\{\ldots, -2, -1\} = \{-1, -2, \ldots\}$
4. $\mathbb{R}$ = the set of real numbers = the real line = $\mathbb{R}^1$
5. $\mathbb{R}^+$ = the set of positive real numbers = $\{x \in \mathbb{R} \mid x > 0\}$
6. $\mathbb{R}^-$ = the set of negaitive real numbers = $\{x \in \mathbb{R} \mid x < 0\}$
7. $\mathbb{Q}$ = the set of rational numbers
8. $\mathbb{C}$ = the set of complex numbers = the complex plane

Examples of descriptions of a set

1. In words:
   (a) the upper half-plane
   (b) the students at Auburn University who play tennis
   (c) all complex numbers at distance one from zero
   (d) the integers between π and 2π
2. Mathematically:
   (a) $\{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$
   (b) $\{S \in \text{AU} \mid S \text{ plays tennis}\}$
   (c) $\{z \in \mathbb{C} \mid |z| = 1\}$
   (d) $\{4, 5, 6\}$ (The order of elements does not matter, e.g., $\{5, 6, 4\}$ describes the same set.)

Basic set theory notions or operations

To avoid Russell's paradox[1], we assume that all considered sets are subsets of a larger set U. Such a set U is often called the universe, e.g., if one considers subsets of the plane, the universe is the plane, if one considers real numbers, the universe is the real line.

1. The union $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$ (or both). The union of sets is also defined for more than two sets.

   (a) Inductively for finitely many sets, $A_1 \cup \cdots \cup A_n = (A_1 \cup \cdots \cup A_{n-1}) \cup A_n$. For example $A \cup B \cup C = \{x \in U \mid x \in A,\ x \in B, \text{ or } x \in C\}$.
   The notation $\bigcup_{i=1}^{n} A_i$ is used as well.

   (b) For countably many sets, $A_1 \cup A_2 \cdots = \{x \in U \mid \text{such that } x \in A_i \text{ for some } i \in \mathbb{N}\}$.
   The notation $\bigcup_{n=1}^{\infty} A_n$ is also used.

   (c) For any number of sets (i.e., finite, countable, uncountable), $\bigcup_{\lambda \in \Lambda} A_\lambda$ is the set $\{x \in U \mid \text{such that } x \in A_\lambda \text{ for some } \lambda \in \Lambda\}$. For example, if Λ is the set of irrational numbers and A_λ is a vertical line in the plane whose first coordinate is λ, then $\bigcup_{\lambda \in \Lambda} A_\lambda$ is the union (in the plane) of all such lines.

2. The intersection $A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$. The intersection is also defined for more than two sets.

   (a) Inductively for finitely many sets, $A_1 \cap \cdots \cap A_n = (A_1 \cap \cdots \cap A_{n-1}) \cap A_n$.
   The notation $\bigcap_{i=1}^{n} A_i$ is also used.

   (b) For countably many sets, $A_1 \cap A_2 \cdots = \{x \in U \mid \text{such that } x \in A_i \text{ for all } i \in \mathbb{N}\}$.
   The notation $\bigcap_{n=1}^{\infty} A_n$ is also used.

   (c) For any number of sets (i.e., finite, countable, uncountable), $\bigcap_{\lambda \in \Lambda} A_\lambda$ is the set $\{x \in U \mid \text{such that } x \in A_\lambda \text{ for all } \lambda \in \Lambda\}$.

3. The set difference $A - B = \{x \in U \mid x \in A \text{ and } x \notin B\}$. Sometimes the set difference is denoted by $A \setminus B$. The set $U - B$ (or $U \setminus B$) is called the complement of B.

4. The Cartesian product of two sets $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.

   (a) For finitely many sets, $A_1 \times \cdots \times A_n = \{(a_1, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$.
   The notation $\Pi_{i=1}^{n} A_i$ is also used.

   (b) For countably many sets, $A_1 \times A_2 \times \cdots = \{(a_1, a_2, \ldots) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots\}$.
   The notation $\Pi_{n=1}^{\infty} A_n$ is also used.

Remark. $\bigcup_{i=1}^{n} A_i$, $\bigcup_{n=1}^{\infty} A_n$, $\bigcup_{\lambda \in \Lambda} A_\lambda$, $\bigcap_{i=1}^{n} A_i$, $\bigcap_{n=1}^{\infty} A_n$, $\bigcap_{\lambda \in \Lambda} A_\lambda$, in display style look like this:

$$\bigcup_{i=1}^{n} A_i, \bigcup_{n=1}^{\infty} A_n, \bigcup_{\lambda \in \Lambda} A_\lambda, \bigcap_{i=1}^{n} A_i, \bigcap_{n=1}^{\infty} A_n, \bigcap_{\lambda \in \Lambda} A_\lambda,$$

[1] Russell's paradox advises that the set $R = \{A \mid A \notin A\}$ is not well-defined. Read about Russell's paradox in wikipedia or another source.

Advanced set theory notions

1. The *cardinality* $|A|$ of a set A is the number of elements of A. Two sets are of the same cardinality if there is a one-to-one matching between the elements of the two sets. We will distinguish sets of three types of cardinality.

   (a) Finite sets, including the empty set.

   (b) Countable sets such as $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$. This cardinality is denoted by $\aleph_0$ (read aleph-zero). (Finite sets are also countable.)

   (c) Sets of cardinality **c** (referred to as the cardinality of the continuum) such as $\mathbb{R}$, the set of irrational numbers, $\mathbb{C}$. These sets are uncountable.

2. The *power set* 2^A of a set A is the set of all subsets of A (including $\emptyset$ and A).

   (a) If A is a finite sets with n elements then 2^A has 2^n elements. For example if $A = \{1, 2\}$, $|A| = 2$, $2^A = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, $|2^A| = 4$.

   (b) $|2^{\mathbb{N}}|=\mathbf{c}$

   (c) $|2^{\mathbb{R}}|$ does not equal **c**. It is greater than **c**. It is also uncountable.