§1.3 Neumann-Poincaré operator K'

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$$(K'\varphi)(x) := \int_\Gamma \frac{\partial \Phi(x,y)}{\partial n_x}\varphi(y)\,ds_y, \quad x\in\Gamma.$$

Example: Γ is a circle with radius 1.

$$\frac{\partial\Phi(x,y)}{\partial n_x} = -\frac{1}{2\pi}\frac{(x-y)\cdot n_x}{|x-y|^2} = -\frac{1}{2\pi}\frac{(x-y)\cdot n_x}{|x-y|^2} = -\frac{1}{4\pi}, \text{ and } K'\varphi = -\frac{1}{4\pi}\int_\Gamma \varphi(y)\,ds_y.$$

K' attains eigenvalues $-\frac{1}{2}$ and 0. The corresponding eigenfunctions are 1 and $e^{in\theta}$ $(n=\pm1,\pm2,\cdots)$.

Denote $\lambda_0=0$, $\lambda_1=\lambda_2=\cdots=\lambda_n=0,\cdots$, $\varphi_0=\frac{1}{\sqrt{2\pi}}$, $\varphi_1^\pm=\frac{1}{\sqrt{2\pi}}e^{\pm i\theta}$, $\varphi_2^\pm=\frac{1}{\sqrt{2\pi}}e^{\pm i2\theta}$, $\cdots$.

We see that K' attains the decomposition

$$K'\varphi = \lambda_0\langle\varphi,\varphi_0\rangle\varphi_0 + \sum_{n\geq1}\lambda_n\left(\langle\varphi,\varphi_n^+\rangle\varphi_n^+ + \langle\varphi,\varphi_n^-\rangle\varphi_n^-\right) \text{ for any } \varphi\in L^2(\Gamma).$$

Namely, K' attains the diagonalization: $K'=\sum_{n=0}^\infty \lambda_n P_n$, where P_0 and P_n $(n\geq1)$ is the projection on $V_0=\mathrm{span}\{\varphi_0\}$ and $V_n=\mathrm{span}\{\varphi_n^\pm\}$ respectively.

Q. For a given closed curve Γ, does K' attain a diagonalization as above?

First, we have the following theorem for the spectrum of a compact operator.

Theorem 1.3.1. Let H be a Hilbert space, and $A: H\to H$ is a compact operator, then (i). $\sigma(A)\setminus\{0\} = \sigma_p(A)\setminus\{0\}$, where $\sigma_p(A)$ is the point spectrum of A.

(ii). $\sigma(A)\setminus\{0\}$ is finite or a sequence tending 0.

Theorem 1.3.2 (Spectrum of K') Let Γ be a smooth curve, then the spectrum of $K': L^2(\Gamma)\to L^2(\Gamma)$ lies in $[-\frac{1}{2},\frac{1}{2})$.

proof. From Theorem 1.3.1, for $\lambda\in\sigma(K')\setminus\{0\}$, λ is an eigenvalue. Consider the homogeneous equation $(K'-\lambda I)\varphi=0$, where (λ,φ) is an eigenpair. Then

$$0=\langle (K'-\lambda I)\varphi, 1\rangle_{L^2(\Gamma)} = \langle\varphi, K[1]\rangle_{L^2(\Gamma)} - \lambda\langle\varphi,1\rangle_{L^2(\Gamma)} = -(\lambda+\tfrac{1}{2})\langle\varphi,1\rangle_{L^2(\Gamma)} = 0.$$

Case (i). $\lambda+\frac{1}{2}=0 \Rightarrow \lambda=-\frac{1}{2}$.

Case (ii). $\langle\varphi,1\rangle_{L^2(\Gamma)}=0$. Next we show that $\lambda\in(-\frac{1}{2},\frac{1}{2})$, and the proof is complete.

Let $u(x)$ be the single layer potential given by $u(x)=\int_\Gamma \Phi(x,y)\varphi(y)\,ds_y$.

$\Delta u(x)=0$ in Ω and $\mathbb{R}^2\setminus\bar\Omega$. In addition,

$|u(x)|=O(\frac{1}{|x|})$ and $|\nabla u(x)|=O(\frac{1}{|x|^2})$ as $|x|\to\infty$. (*)

This can be seen from applying the Green's Theorem for the boundary value problem $\begin{cases}\Delta W=0 \text{ in } \Omega,\\ \frac{\partial W}{\partial n}=\varphi,\end{cases}$ which attains a solution since $\langle\varphi,1\rangle=0$.

We obtain $$u(x)=\int_\Gamma \Phi(x,y)\varphi(y)\,ds_y=\int_\Gamma \Phi(x,y)\frac{\partial W(y)}{\partial n_y}\,ds_y=\int_\Gamma \frac{\partial \Phi(x,y)}{\partial n_y}W(y)\,ds_y.$$

Now let us define two constants A and B:

$$A=\int_\Omega |\nabla u|^2\,dx,\quad B=\int_{\mathbb{R}^2\setminus\bar\Omega}|\nabla u|^2\,dx.$$

Then it follows from (*) that $B<\infty$. Moreover, A and B can not be zero simultaneously. Otherwise, $\varphi\equiv 0$. On the other hand,

$$A=\int_\Omega|\nabla u|^2\,dx=\int_\Gamma \frac{\partial u}{\partial n}\bar u\,ds=\int_\Gamma (K'\varphi+\tfrac{1}{2}\varphi)\bar u\,ds=(\lambda+\tfrac{1}{2})\int_\Gamma \varphi\bar u\,ds,$$

$$B=\int_{\mathbb{R}^2\setminus\bar\Omega}|\nabla u|^2\,dx=-\int_\Gamma \frac{\partial u_+}{\partial n}\bar u\,ds=-\int_\Gamma (K'\varphi-\tfrac{1}{2}\varphi)\bar u\,ds=(\tfrac{1}{2}-\lambda)\int_\Gamma \varphi\bar u\,ds.$$

$\Rightarrow \lambda=\frac{1}{2}\frac{A-B}{A+B}$, and we obtain $|\lambda|\le\frac{1}{2}$. (**)

Now if $\lambda=-\frac{1}{2}$, then $A=0$, and $u\equiv c$ in $\bar\Omega$.

$$B=(\tfrac{1}{2}-\lambda)\int_\Gamma \varphi u\,ds=\int_\Gamma \varphi u\,ds=C\int_\Gamma \varphi\,ds=0,$$

where we have used the fact that $\langle\varphi,1\rangle_{L^2(\Gamma)}=0$. Thus $\varphi\equiv 0$, which is not an eigenfunction, and $\lambda=-\frac{1}{2}$ is not an eigenvalue of K'.

Similarly, it can be shown that $\lambda=\frac{1}{2}$ is neither an eigenvalue of K'.

This combines with (**) proves the assertion in Case (ii).

Exercise.

1. In Theorem 1.2.2 and Theorem 1.2.3 we have shown that

1. In theorem 1.2.2 and theorem 1.2.7, we have shown that

$$V_\pm = K\varphi \pm \tfrac{1}{2}\varphi \quad \text{and} \quad \frac{\partial u_\pm}{\partial n} = K'\varphi \mp \tfrac{1}{2}\varphi$$

for continuous density function $\varphi(x)$. Try to show the equalities also hold for $\varphi \in L^2(\Gamma)$, which is used in the proof of the above theorem. "$f=g$" holds in the sense of "almost everywhere".

Define $L_0^2(\Gamma) := \{\varphi \in L^2(\Gamma) \mid \langle \varphi, 1\rangle = 0\}$. $L_0^2(\Gamma)$ is a closed subspace of $L^2(\Gamma)$.
Note that $\forall \varphi \in L_0^2(\Gamma)$, it follows that $K_0'\varphi \in L_0^2(\Gamma)$.
From the proof of theorem 1.3.2 (case (ii)), we have the following proposition:

Proposition 1.3.3. The spectrum of $K' : L_0^2(\Gamma) \to L_0^2(\Gamma)$ lies in $(-\frac{1}{2}, \frac{1}{2})$.

proposition 1.3.4. For $K' : L^2(\Gamma) \to L^2(\Gamma)$, it holds that $\dim(\frac{1}{2}I + K') = 1$.

proof. For any $\varphi_1, \varphi_2 \in \mathrm{Ker}(\frac{1}{2}I + K')$, we write

$$\varphi_1 = \langle \varphi_1, 1\rangle + \tilde{\varphi}_1, \quad \varphi_2 = \langle \varphi_2, 1\rangle + \tilde{\varphi}_2, \quad \text{where } \langle \tilde{\varphi}_1, 1\rangle = \langle \tilde{\varphi}_2, 1\rangle = 0.$$

Define $\varphi = \frac{\langle \varphi_2, 1\rangle}{\langle \varphi_1, 1\rangle}\varphi_1 - \varphi_2$, then $\varphi \in \mathrm{Ker}(\frac{1}{2}I + K')$. (*)

On the other hand, we see that $\langle \varphi, 1\rangle = 0$. From case (ii) in the proof of Theorem 1.3.2, the eigenvalue associated with the eigenfunction φ lies in $(-\frac{1}{2}, \frac{1}{2})$. (**)

In view of (*) and (**), we deduce that $\varphi \equiv 0$. As such $\varphi_2 = \alpha\varphi_1$, where $\alpha = \frac{\langle \varphi_2, 1\rangle}{\langle \varphi_1, 1\rangle}$. This completes the proof.

Theorem 1.3.5 (Decomposition of compact, self-adjoint operator)

Let H be a separable Hilbert space, $A: H \to H$ be a compact, self-adjoint operator. Then there exist eigenvalues and eigenfunctions

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq \cdots \to 0, \quad \varphi_1, \varphi_2, \cdots, \varphi_n, \cdots,$$

and $A\varphi = \sum_{n=1}^{\infty} \lambda_n \langle \varphi, \varphi_n\rangle \varphi_n \quad \forall \varphi \in H.$

Symmetrization of K'.

Example. Let Γ be a circle with radius 1.

$|x-y| = 2 - 2x\cdot y = 2 - 2(\cos\theta\cdot\cos\theta' + \sin\theta\sin\theta')$, where $x = \cos\theta$, $y = \cos\theta'$,

$|x-y|^2 = 2[1-\cos(\theta-\theta')] = 4\sin^2\frac{\theta-\theta'}{2}$, and $\ln|x-y| = \frac{1}{2}\ln\left(4\sin^2\frac{\theta-\theta'}{2}\right)$.

It follows that

$$S[e^{in\theta}] = -\frac{1}{4\pi}\int_0^{2\pi}\ln\left(4\sin^2\frac{\theta-\theta'}{2}\right)e^{in\theta'}\,d\theta' = \frac{1}{2|n|}e^{in\theta},\quad n\neq 0,$$

where we have used Lemma 1.10.

$$S[1] = -\frac{1}{4\pi}\int_0^{2\pi}\ln\left(4\sin^2\frac{\theta-\theta'}{2}\right)d\theta' = 0.$$

We see that for any $\varphi \in L_0^2(\Gamma) = \operatorname{span}\{e^{\pm i\theta}, e^{\pm i2\theta}, \dots, e^{\pm in\theta}, \dots\}$,

$\langle S\varphi, \varphi\rangle_{L^2(\Gamma)} \geq 0$. In addition, if $\langle S\varphi,\varphi\rangle_{L^2(\Gamma)} = 0$, we have $\varphi = 0$.

This implies that S is "positive definite" on $L_0^2(\Gamma)$. Indeed, as stated below, this holds true in general.

<u>Theorem 1.3.6</u> Let Γ be smooth. It holds that $\langle S\varphi,\varphi\rangle_{L^2(\Gamma)} \geq 0$ $\forall \varphi\in L_0^2(\Gamma)$.

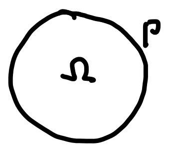
If $\langle S\varphi,\varphi\rangle_{L^2(\Gamma)} = 0$, then $\varphi = 0$ on Γ.

<u>proof</u>. $\forall\varphi\in L_0^2(\Gamma)$, $\langle\varphi, 1\rangle = 0$. Define the single layer potential

$$u(x) = \int_\Gamma \Phi(x,y)\varphi(y)\,ds_y,\quad x\in\mathbb{R}^2\backslash\Gamma.$$

From the proof of theorem 1.3.2, it is known that

$|u(x)| = O(\frac{1}{|x|})$ and $|\nabla u| = O(\frac{1}{|x|^2})$ as $|x|\to\infty$.



$$\int_\Omega |\nabla u|^2\,dx = \int_\Gamma u\frac{\partial u}{\partial n}\,ds \quad\text{and}\quad \int_{\mathbb{R}^2\backslash\bar\Omega}|\nabla u|^2\,dx = -\int_\Gamma u\frac{\partial u_+}{\partial n}\,ds$$

$$\Rightarrow \int_{\mathbb{R}^2}|\nabla u|^2\,dx = \int_\Gamma u\,\overline{\left(\frac{\partial u}{\partial n}-\frac{\partial u_+}{\partial n}\right)}\,ds = \int_\Gamma S\varphi\,\bar\varphi\,ds \geq 0$$

Now if $\langle S\varphi,\varphi\rangle = 0$, it follows that $\nabla u = 0$ in Ω and $\mathbb{R}^2\backslash\bar\Omega$ respectively.

Hence $\varphi = \frac{\partial u}{\partial n} - \frac{\partial u_+}{\partial n} = 0$.

<u>Remark</u>. For a given Curve Γ, it can be shown that S is positive definite on $L^2(\alpha\Gamma)$

for all $\alpha > 0$ except one. See proposition 2.4.4 of [Zoalroskd].

With the above theorem, we introduce the inner product $\langle\cdot,\cdot\rangle_S$ on $L_0^2(\Gamma)$:

$\langle\varphi,\psi\rangle_S := \langle S\varphi,\psi\rangle_{L^2(\Gamma)}$.

Then equipped with the inner product $\langle\cdot,\cdot\rangle_S$, the operator K' is self-adjoint on $L_0^2(\Gamma)$

by using the Calderon formula $SK' = KS$ (Theorem 1.2.9):

$\langle K'\varphi,\psi\rangle = \langle SK'\varphi,\psi\rangle = \langle KS\varphi,\psi\rangle_S = \langle S\varphi, K'\psi\rangle_S = \langle\varphi, K'\psi\rangle$

By completing the space $L_0^2(\Gamma)$ with the norm $\|\cdot\|_S$ induced by the inner product $\langle\cdot,\cdot\rangle_S$, one obtains $H_0^{-1/2}(\Gamma) := \{\varphi \in H^{-1/2}(\Gamma) \mid \langle \varphi, 1\rangle_{L^2(\Gamma)} = 0\}$, where $H^{-1/2}(\Gamma)$ is the dual space of $H^{1/2}(\Gamma)$.

Let us denote $H_0^*(\Gamma)$ by $H_0^{-1/2}(\Gamma)$ equipped with the inner product $\langle\cdot,\cdot\rangle_S$.

One can also extend the definition of K' to $H_0^*(\Gamma)$ and K' is self-adjoint in $H_0^*(\Gamma)$.

Applying Theorem 1.3.5, we obtain the following theorem for the spectral decomposition of K'.

<u>Theorem 1.3.7</u> The operator K' is self-adjoint in $H_0^*(\Gamma)$.

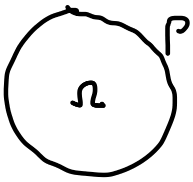
(i) Let $\{(\lambda_j, \varphi_j)\}_{j=1}^{\infty}$ be the eigenpairs of K', then $\lambda_j \in (-\frac{1}{2}, \frac{1}{2})$, and $|\lambda_1| \geq |\lambda_2| \geq \cdots \to 0$.

(ii) For any $\varphi \in H_0^*(\Gamma)$, there holds $K'\varphi = \sum_{j=1}^{\infty} \lambda_j \langle \varphi, \varphi_j\rangle_S \, \varphi_j$.

From the above discussions, we know that S is positive definite on $L_0^2(\Gamma)$, but may not be positive definite on $L^2(\Gamma)$. If $S\varphi_0 = 0$ attains non-trivial solution, then the single layer potential $u(x) := \int_\Gamma \Phi(x,y)\varphi_0(y)\,ds_y$ solves the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases}$$

which attains a unique solution $u \equiv 0$ in $\overline{\Omega}$.



Therefore, $\frac{\partial u_-}{\partial n} = (\frac{1}{2}I + K')\varphi_0 = 0$.

Namely, $\varphi_0 \in \ker(\frac{1}{2}I + K')$. That is $\operatorname{Ker}(S) \subset \operatorname{Ker}(\frac{1}{2}I + K')$.

In view of proposition 1.3.4, $\dim(\operatorname{Ker}(S))$ is at most 1.

Based on this observation, we may introduce a modified operator $\tilde{S}$:

$$\tilde{S}\varphi = \begin{cases} S\varphi & \text{if } \varphi \in L_0^2(\Gamma), \\ 1 & \text{if } \varphi = \varphi_0, \text{ where } (\frac{1}{2}I + K')\varphi_0 = 0,\ \langle \varphi_0, 1\rangle_{L^2(\Gamma)} = 1. \end{cases}$$

Then one can show that $\tilde{S}$ is positive define on $L^2(\Gamma)$, and $\langle\cdot,\cdot\rangle_{\tilde{S}}$, which is defined by $\boxed{\langle \varphi, \psi\rangle_{\tilde{S}} := \langle \tilde{S}\varphi, \psi\rangle_{L^2(\Gamma)}}$, is an inner product in $L^2(\Gamma)$.

In addition, the Calderon formula $K\tilde{S} = \tilde{S}K'$. Therefore, the operator K' can still be symmetrized.