§1.2 Layer potential theory for Laplace equation

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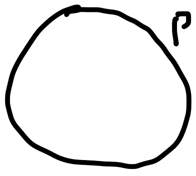
§1.2.2 Boundary integral operators

<u>Definition</u> The single and double layer integral operators are defined by

$$[S\varphi](x) := \int_\Gamma \Phi(x,y)\varphi(y)\,ds_y, \quad x \in \Gamma;$$



$$[K\varphi](x) := \int_\Gamma \frac{\partial \Phi(x,y)}{\partial n_y}\varphi(y)\,ds_y, \quad x\in\Gamma.$$

The normal derivative operators are defined by:

$$[K'\varphi](x) := \int_\Gamma \frac{\partial \Phi(x,y)}{\partial n_x}\varphi(y)\,ds_y, \quad x\in\Gamma,$$

$$[T\varphi](x) := \frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial \Phi(x,y)}{\partial n_y}\varphi(y)\,ds_y, \quad x\in\Gamma.$$

<u>Remark 1.</u> For an integral operator A defined by $A\varphi(x) = \int_\Gamma k(x,y)\varphi(y)\,ds_y$, it is called weakly singular, singular, and hyper-singular if $|k(x,y)| \le \frac{C}{|x-y|^\alpha}$ holds for $0<\alpha<1$, $\alpha=1$, and $\alpha>1$, respectively when $x\neq y$.

S is weakly singular and T is hyper-singular.

<u>Remark 2</u>. In this section, for simplicity, we assume that Γ and φ are sufficiently smooth.

<u>Theorem 1.2.7</u> The following holds for the operators S, K, K' and T:

$$\langle S\varphi,\psi\rangle_{L^2(\Gamma)} = \langle \varphi, S\psi\rangle_{L^2(\Gamma)}, \quad \langle K\varphi,\psi\rangle_{L^2(\Gamma)} = \langle\varphi, K'\psi\rangle_{L^2(\Gamma)},$$

$$\langle T\varphi,\psi\rangle_{L^2(\Gamma)} = \langle\varphi, T\psi\rangle_{L^2(\Gamma)}.$$

<u>proof</u>. From Theorem 1.2.6, we have

$$\langle T\varphi,\psi\rangle = \langle \frac{d}{ds_x} S\frac{d\varphi}{ds_y}, \psi\rangle = -\langle S\frac{d\varphi}{ds_y}, \frac{d}{ds_x}\psi\rangle$$

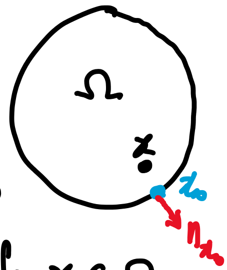
$$= -\langle \frac{d\varphi}{ds_y}, S\frac{d}{ds_x}\psi\rangle = \langle \varphi, \frac{d}{ds_y} S \frac{d}{ds_x}\psi\rangle = \langle\varphi, T\psi\rangle.$$

Lemma 1.2.8 If $u, v \in C^2(\bar\Omega)$, then there holds

$$\int_\Omega \Delta u\, v - u\, \Delta v \, dx = \int_\Gamma \frac{\partial u}{\partial n} v - \frac{\partial v}{\partial n} u \, ds.$$

Theorem 1.2.9 (Calderon formula) The following identities hold for S, K, K' and T:

$$-ST = \tfrac{1}{4} I - K^2, \quad -TS = \tfrac{1}{4} I - (K')^2,$$
$$KS = SK', \qquad TK = K'T.$$



proof: If $\Delta u = 0$ in Ω, then from the Green's formula in Lemma 1.2.8, we have

$$u(x) = \int_\Gamma \Phi(x,y) \frac{\partial u}{\partial n_y} ds_y - \int_\Gamma \frac{\partial \Phi(x,y)}{\partial n_y} u(y)\, ds_y \quad \text{for } x \in \Omega$$

$$\text{and } \nabla u(x) = \nabla_x \int_\Gamma \Phi(x,y) \frac{\partial u}{\partial n_y} ds_y - \nabla_x \int_\Gamma \frac{\partial \Phi(x,y)}{\partial n_y} u(y)\, ds_y \quad \text{for } x \in \Omega.$$

For $\forall x_0 \in \Gamma$, taking the limit $\lim_{h \to 0^+} u(x_0 - h n_{x_0})$ and $\lim_{h\to 0^+} n_{x_0} \cdot \nabla_x u(x_0 - h n_{x_0})$ gives

$$\begin{cases} u(x_0) = S\left[\frac{\partial u}{\partial n}\right] - K[u] + \frac{1}{2} u(x_0), \\ \frac{\partial u(x_0)}{\partial n_{x_0}} = K'\left[\frac{\partial u}{\partial n}\right] + \frac{1}{2} \frac{\partial u}{\partial n} - T[u], \end{cases} \quad (*)$$

which can be expressed as

$$\begin{bmatrix} u \\ \frac{\partial u}{\partial n} \end{bmatrix} = P \begin{bmatrix} u \\ \frac{\partial u}{\partial n} \end{bmatrix}, \text{ where } P = \begin{bmatrix} \frac{1}{2} I - K & S \\ -T & \frac{1}{2} I + K' \end{bmatrix}.$$

P is an operator well defined on $H^{s+1}(\Gamma) \times H^{s}(\Gamma)$, $s \geq \frac{1}{2}$.

Now for any vector function $\begin{bmatrix} v \\ w \end{bmatrix} \in H^{s+1}(\Gamma) \times H^{s}(\Gamma)$, let us define

$$u(x) = \int_\Gamma \Phi(x,y) w(y)\, ds_y - \int_\Gamma \frac{\partial \Phi(x,y)}{\partial n_y} v(y)\, ds_y, \quad x \in \Omega.$$

Then $\Delta u = 0$. In addition, from the same calculation as $(*)$, we have

$$\begin{bmatrix} u \\ \frac{\partial u}{\partial n} \end{bmatrix} = P \begin{bmatrix} v \\ w \end{bmatrix}. \quad (**)$$

Hence, $P[\ldots]$ ... $\{ \ldots \mid u$ defined in Ω s.t. $\Delta u = 0 \}$

Namely, $P\begin{bmatrix} v \\ w \end{bmatrix} \in \left\{ \begin{bmatrix} u, \frac{\partial u}{\partial n} \end{bmatrix}^T : \exists\, u$ defined ... $\right\}$.

In view of (*) & (**), it follows that

$$P^2 \begin{bmatrix} v \\ w \end{bmatrix} = P \begin{bmatrix} u \\ \frac{\partial u}{\partial n} \end{bmatrix} = \begin{bmatrix} u \\ \frac{\partial u}{\partial n} \end{bmatrix} = P \begin{bmatrix} v \\ w \end{bmatrix},$$

and P is a projection operator such that $P^2 = P$.

The proof of the identities follow from $P^2 = P$.

<u>Lemma 1.2.10</u> The following integral formula holds:

$$\frac{1}{2\pi} \int_0^{2\pi} \ln\left(4\sin^2 \frac{t}{2}\right) e^{imt}\, dt = \begin{cases} 0, & m = 0 \\ -\frac{1}{|m|}, & m = \pm 1, \pm 2, \cdots \end{cases}$$

<u>proof</u>. First, it can be shown that

$$1 + 2\sum_{j=1}^{m-1} e^{ijt} + e^{imt} = i\left(1 - e^{imt}\right) \cot\frac{t}{2}, \quad 0 < t < 2\pi.$$

Take the integral of the above yields

$$\int_0^{2\pi} 1 \, dt = -i \int_0^{2\pi} e^{imt} \cot\frac{t}{2}\, dt.$$

$$\Rightarrow \int_0^{2\pi} \cot\frac{t}{2}\, e^{imt}\, dt = 2\pi i, \quad m = 1, 2, \cdots \qquad (*)$$

On the other hand,

$$\frac{d}{dt}\left[\left(e^{imt} - 1\right)\ln\left(\sin^2\frac{t}{2}\right)\right] = im\, e^{imt} \ln\left(4\sin^2\frac{t}{2}\right) + \left(e^{imt} - 1\right)\cot\frac{t}{2}.$$

Take the integral leads to

$$0 = im \int_0^{2\pi} \ln\left(4\sin^2\frac{t}{2}\right) e^{imt}\, dt + \int_0^{2\pi} \cot\frac{t}{2}\, e^{imt}\, dt \qquad (**)$$

Combining (*) & (**) gives $\frac{1}{2\pi}\int_0^{2\pi} \ln\left(4\sin^2\frac{t}{2}\right) e^{imt}\, dt = -\frac{1}{m}$, $m = 1, 2, \cdots$

The formula for $m = -1, -2, \cdots$ follows by setting $\tilde{m} = -m$.

If $m = 0$, note that $\int_0^{2\pi} \ln\left(4\sin^2\frac{t}{2}\right) dt = \int_0^{2\pi} \ln\left(4\cos^2\frac{t}{2}\right) dt$

$$\Rightarrow 2\int_0^{2\pi} \ln\left(4\sin^2\frac{t}{2}\right) dt = \int_0^{2\pi} \ln\left(4\sin^2\frac{t}{2}\right) + \ln\left(4\cos^2\frac{t}{2}\right) dt = \int_0^{2\pi} \ln\left(4\sin^2 t\right) = \frac{1}{2}\int_0^{4\pi} \ln\left(\sin^2\frac{t}{2}\right) dt$$

$\Rightarrow \int_0^{2\pi} \ln\left(4\sin^2\frac{t}{2}\right) dt = 0.$

Theorem 1.2.11 Let Γ be smooth, then
S, K, K' are bounded from $H^p(\Gamma)$ to $H^{p+1}(\Gamma)$, and
T is bounded from $H^{p+1}(\Gamma)$ to $H^p(\Gamma)$ for $p \geq 0$.

proof. (1) Without loss of generality, let us assume that the length $|\Gamma| = 2\pi$.
Γ is parameterized by $\Gamma := \{x(t) \mid t \in [0, 2\pi],\ t \text{ is the arc length}\}$.
Let $\psi(t) = \varphi(x(t))$, then

$$S\psi(t) = -\frac{1}{2\pi}\int_0^{2\pi} \ln|x(t) - x(s)|\, \psi(s)\, ds$$

$$= -\frac{1}{4\pi}\int_0^{2\pi} \ln 4\sin^2\left(\frac{t-s}{2}\right) \psi(s)\, ds + \frac{1}{4\pi}\int_0^{2\pi}\left[\ln 4\sin^2\left(\frac{t-s}{2}\right) - \ln|x(t) - x(s)|^2\right]\psi(s)\, ds$$

$$=: S_1\psi + S_2\psi.$$

If $\varphi \in H^p(\Gamma)$ s.t. $\psi \in H^p[0, 2\pi]$, then $\psi(s) = \sum_{n=-\infty}^{\infty} \beta_n e^{ins}$ and $\|\psi\|_p^2 = \sum_{n=-\infty}^{\infty}(1+n^2)^p|\beta_n|^2 < +\infty$.

From Lemma 1.2.10, $S_1\psi = \sum_{n\neq 0} \frac{1}{2}\frac{\beta_n}{|n|} e^{int}$.

Therefore, $\|S_1\psi\|_{p+1}^2 = \frac{1}{4\pi^2}\sum_{n\neq 0}(1+n^2)^{p+1}\frac{|\beta_n|^2}{n^2} \leq C\|\psi\|_p^2$.

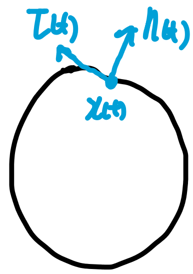
Since S_2 is a smooth operator, we have $S := S_1 + S_2$ bounded from $H^p(\Gamma)$ to $H^{p+1}(\Gamma)$.

(2) T is bounded from $H^{p+1}(\Gamma)$ to $H^p(\Gamma)$ by noting that

$$T = \frac{d}{ds} S \frac{d}{ds} \quad (\text{Theorem } 1.2.6).$$

(3) $$K\varphi(x) := \int_\Gamma \frac{\partial\Phi(x,y)}{\partial n_y}\varphi(y)\, ds_y, \quad \frac{\partial\Phi(x,y)}{\partial n_y} = -\frac{1}{2\pi}\frac{(y-x)\cdot n_y}{|x-y|^2}$$

using the Frenet formula $\frac{d\tau(t)}{dt} = K(t)n(t)$, $\frac{dn(t)}{dt} = -K(t)\tau(t)$, where
K is the curvature of Γ, it follows that



$$y - x = x(s) - x(t) = \tau(t)(s-t) + \frac{1}{2}K(t)n(t)(s-t)^2 + O((s-t)^3).$$

$$n_y = n_x + \frac{dn}{dt}(s-t) + O((s-t)^2) = n(t) - K(t)\tau(t)(s-t) + O((s-t)^2)$$

for $|s-t| \ll 1$.

Thus $|y-x|^2 = (s-t)^2 + O((s-t)^3)$, and

$$(y-x)\cdot n_y = (x(s)-x(t))\cdot n(t)$$
$$= \left[(s-t)\tau + \tfrac{1}{2}\kappa n (s-t)^2 + O((s-t)^3)\right]\cdot\left[n - \kappa\tau(s-t) + O((s-t)^2)\right]$$
$$= -\frac{\kappa}{2}(s-t)^2 + O((s-t)^3).$$

$\Rightarrow$ we have $\dfrac{\partial\Phi(x,y)}{\partial n_y} = -\dfrac{1}{2\pi}\dfrac{(y-x)\cdot n_y}{|x-y|^2} = \dfrac{1}{4\pi}\kappa(t) + O(s-t).$

We see that K is an integral operator with a smooth kernel, and is bounded from $H^p(P)$ to $H^{p+1}(P)$.

<u>proposition 1.2.12</u>. The operators S, K, and K': $H^p(P) \to H^p(P)$ are compact for $p \geq 0$.

Remark: K and K' are called Neumann-Poincare operators.