§1.2 Layer potential theory for Laplace equation

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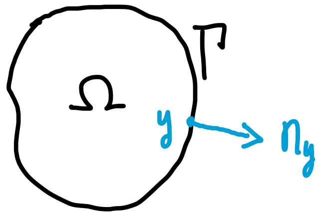
$$\Delta u := \sum_{j=1}^{d} \partial_j^2 u, \quad \Delta u = 0.$$

Fundamental solution:

$$\Phi(x,y) = \begin{cases} -\frac{1}{2\pi}\ln|x-y| & \text{in } \mathbb{R}^2, \\ \frac{1}{4\pi}\cdot\frac{1}{|x-y|} & \text{in } \mathbb{R}^3. \end{cases} \qquad \Delta\Phi(x,y) = -\delta(x-y)$$



Def For $\varphi \in C(\Gamma)$, the single layer potential

$$u(x) := \int_\Gamma \Phi(x,y)\varphi(y)\,ds_y, \quad x \in \mathbb{R}^2\setminus\Gamma,$$

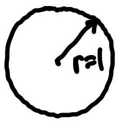
the double layer potential

$$v(x) := \int_\Gamma \frac{\partial\Phi(x,y)}{\partial n_y}\varphi(y)\,ds_y, \quad x \in \mathbb{R}^2\setminus\Gamma. \qquad \boxed{\frac{\partial\Phi(x,y)}{\partial n_y} = \nabla_y\Phi(x,y)\cdot n_y}$$

Both $u(x)$ and $v(x)$ satisfies $\Delta u = 0$ in Ω and $\mathbb{R}^2\setminus\bar\Omega$.

§1.2.1. Behavior of layer potentials when $x \to \Gamma$.

Q1: Is $\int_\Gamma \Phi(x,y)\varphi(y)\,ds_y$ well-defined for $x \in \Gamma$?



$\varphi \equiv 1$ on Γ

$$\int_\Gamma \Phi(x,y)\,ds_y = \int_{|y|=1} -\frac{1}{2\pi}\ln|x-y|\,ds_y$$

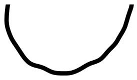
$$x = (\cos\theta, \sin\theta), \quad y = (\cos\theta', \sin\theta')$$

$$|x-y|^2 = \langle x-y, x-y\rangle = |x|^2 + |y|^2 - 2\langle x,y\rangle = 1+1-2(\cos\theta\cdot\cos\theta' + \sin\theta\cdot\sin\theta')$$
$$= 2 - 2\cos(\theta-\theta')$$

$$\int_\Gamma \Phi(x,y)\,ds_y = -\frac{1}{2\pi}\int_0^{2\pi} \ln 2 + \ln[1-\cos(\theta-\theta')]\,d\theta'$$

$\ln[1-\cos(\theta-\theta')] \approx -\frac{1}{2}\ln|\theta-\theta'|$ if $|\theta-\theta'| \ll 1$. Hence the above integral is well-defined.

Q2: Let $x \to x_0 \in \Gamma$, $u(x) \to u(x_0)$?

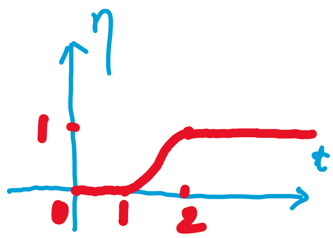


Theorem 1.2.1 The single layer potential $u(x)$ is continuous throughout $\mathbb{R}^2$.

$$u(x) = \int_\Gamma \Phi(x,y)\,\varphi(y)\,ds_y \;,\quad x \in \mathbb{R}^2.$$

proof. Let $\eta(t)$ be a smooth cut-off function, s.t.

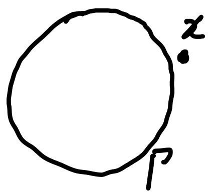
$$\eta(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & t \ge 2 \end{cases} \;.\quad \eta \in C^\infty[0,\infty)$$



$$\eta_m(t) := \eta(mt) \to 1 \,,\quad m \to \infty$$

Let $u_m(x) = \int_\Gamma \eta_m(|x-y|)\cdot\Phi(x,y)\,\varphi(y)\,ds_y$, $x \in \mathbb{R}^2$, continous.

$$|u_m(x) - u(x)| = \left|\int_\Gamma \left(\eta_m(|x-y|) - 1\right)\cdot\Phi(x,y)\,\varphi(y)\,ds_y\right|$$



$$\le \int_{|x-y|<\frac{2}{m}} |\Phi(x,y)|\,ds_y \cdot \|\varphi\|_\infty$$

$$\le C\int_0^{\frac{2}{m}} |\ln\theta|\,d\theta \cdot \|\varphi\|_\infty \quad \text{if } x \text{ is sufficiently close to } \Gamma.$$

$$\le C_m \ln m \cdot \|\varphi\|_\infty \to 0 \quad \text{as } m \to \;.$$

$\Rightarrow$ $u(x)$ is continuous in a region enclosing Γ, and continous in $\mathbb{R}^2$.

Remark. If $|K(x,y)| \le \dfrac{C}{|x-y|^\alpha}$ for $x \ne y$, $0 < \alpha < 1$.

$u(x) := \int_\Gamma K(x,y)\,\varphi(y)\,ds_y$ is continuous throughout $\mathbb{R}^2$.

We call such $K(x,y)$ a <u>weakly singular kernel</u>.

Theorem 1.2.2 The double layer potential $V(x)$ can be extended from $\mathbb{R}^2\backslash\bar{\Omega}$ to $\mathbb{R}^2\backslash\Omega$ or from Ω to $\bar{\Omega}$ with the limits

$$V_\pm(x) = \int_\Gamma \frac{\partial\Phi(x,y)}{\partial n_y}\,\varphi(y)\,ds_y \pm \frac{1}{2}\varphi(x) \quad \text{for } x \in \Gamma,$$

where $V_\pm(x) = \lim\limits_{h \to 0} V(x \pm h\, n_x)$.

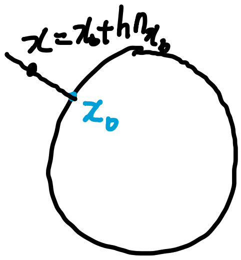
$h\to 0^+$

proof. If $\varphi\equiv 1$, then

$$\int_{\Gamma}\frac{\partial\Phi(x,y)}{\partial n_y}\,ds_y=\begin{cases}-1 & x\in\Omega,\\ -\frac{1}{2} & x\in\Gamma,\\ 0 & x\in\mathbb{R}^2\setminus\overline{\Omega}.\end{cases}$$

The statement holds. If $\varphi\not\equiv 1$, let $x=x_0+h n_{x_0}$, $x_0\in\Gamma$.

$$V(x)=\int_{\Gamma}\frac{\partial\Phi(x,y)}{\partial n_y}\varphi(y)\,ds_y$$



$$=\int_{\Gamma}\frac{\partial\Phi(x,y)}{\partial n_y}\big(\varphi(y)-\varphi(x_0)\big)\,ds_y+\varphi(x_0)\int_{\Gamma}\frac{\partial\Phi(x,y)}{\partial n_y}\,ds_y$$

$$=:V_1(x)+V_2(x)$$

$$\lim_{h\to 0}V_2(x)=\varphi(x_0)\cdot\left[\int_{\Gamma}\frac{\partial\Phi(x_0,y)}{\partial n_y}\,ds_y+\frac{1}{2}\right]$$

$$V_1(x)=\int -\frac{1}{2\pi}\frac{y-x}{|x-y|^2}\cdot n_y\,\big(\varphi(y)-\varphi(x_0)\big)\,ds_y.$$

$$\Rightarrow \lim_{h\to 0}V_1(x_0+h)=\int_{\Gamma}\frac{\partial\Phi(x_0,y)}{\partial n_y}\big(\varphi(y)-\varphi(x_0)\big)\,ds_y.$$

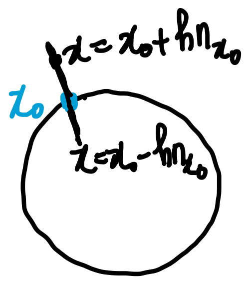
Therefore,

$$\lim_{h\to 0}V(x)=\lim_{h\to 0}V_1(x)+\lim_{h\to 0}V_2(x)$$

$$=\int_{\Gamma}\frac{\partial\Phi(x_0,y)}{\partial n_y}\varphi(y)\,ds_y+\frac{1}{2}\varphi(x_0).$$

Derivative of single and double layer potentials

$$u(x):=\int_{\Gamma}\Phi(x,y)\varphi(y)\,ds_y,\quad x\in\mathbb{R}^2\setminus\Gamma$$



$$\nabla u(x)=\int_{\Gamma}\nabla_x\Phi(x,y)\varphi(y)\,ds_y,\quad x\in\mathbb{R}^2\setminus\Gamma.$$

$$n_{x_0}\cdot\nabla u(x)=\int_{\Gamma}n_{x_0}\cdot\nabla_x\Phi(x,y)\varphi(y)\,ds_y,\quad x_0\in\Gamma,\ x\in\mathbb{R}^2\setminus\Gamma.$$

Theorem 1.2.3 (Derivative of single layer potential I)

Let $\frac{\partial u_{\pm}(x)}{\partial n}:=\lim n_x\cdot\nabla u(x+hn_x)$. If $\varphi\in C(\Gamma)$, then

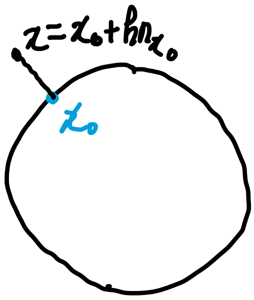
$\partial n_x \qquad h\to 0$

there holds

$$\frac{\partial u_\pm(x)}{\partial n_x} = \int_\Gamma \frac{\partial \Phi(x,y)}{\partial n_x}\varphi(y)\,ds_y \mp \frac{1}{2}\varphi(x) \text{ for } x\in\Gamma.$$

proof. $\nabla_x \Phi(x,y) = -\frac{1}{2\pi}\frac{x-y}{|x-y|^2}$, $\nabla_y \Phi(x,y) = -\frac{1}{2\pi}\frac{y-x}{|x-y|^2}$.

Therefore, for $x = x_0 + h n_{x_0}$, $x_0\in\Gamma$,



$\nabla u(x) = -\int \nabla_y \Phi(x,y)\varphi(y)\,ds_y$, and

$$n_{x_0}\cdot\nabla u(x) + V(x) = \int_\Gamma \nabla_y\Phi(x,y)(n_y - n_{x_0})\varphi(y)\,ds_y.$$

Note that $\int_\Gamma \nabla_y\Phi(x,y)(n_y-n_{x_0})\varphi(y)\,ds_y$

$$= \int -\frac{1}{2\pi}\frac{y-x}{|x-y|^2}\cdot(n_y-n_{x_0})\varphi(y)\,ds_y$$

$$\Rightarrow \lim_{h\to 0}\int_\Gamma -\frac{1}{2\pi}\frac{y-x}{|x-y|^2}\cdot(n_y-n_{x_0})\varphi(y)\,ds_y = \int_\Gamma -\frac{1}{2\pi}\frac{y-x_0}{|x_0-y|}\cdot(n_y-n_{x_0})\varphi(y)\,ds_y.$$

$$\Rightarrow \frac{\partial u_\pm(x_0)}{\partial n_{x_0}} = \lim_{h\to 0} n_{x_0}\cdot\nabla u(x)$$

$$= \lim_{h\to 0}\int_\Gamma \nabla_y\Phi(x,y)(n_y-n_{x_0})\varphi(y)\,ds_y - V_\pm(x_0)$$

$$= \int_\Gamma \nabla_y\Phi(x_0,y)(n_y-n_{x_0})\varphi(y)\,ds_y - \int_\Gamma\frac{\partial\Phi(x_0,y)}{\partial n_y}\varphi(y)\,ds_y \mp \frac{1}{2}\varphi(x_0)$$

$$= \int_\Gamma \frac{\partial\Phi(x_0,y)}{\partial n_{x_0}}\varphi(y)\,ds_y \mp \frac{1}{2}\varphi(x_0).$$

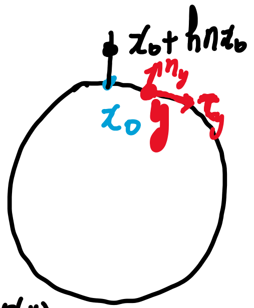
__Theorem 1.2.4__ (Derivative of single layer potential II)

Let $\nabla u_\pm(x) = \lim_{h\to 0}\nabla u(x+hn_x)$. If $\varphi\in C(\Gamma)$, then

$$\nabla u_\pm(x) = \int_\Gamma \nabla_x\Phi(x,y)\varphi(y)\,ds_y \mp \frac{1}{2}\varphi(x)\cdot n_x \text{ for } x\in\Gamma.$$

proof. $\nabla u(x) = \int_\Gamma \nabla_x\Phi(x,y)\varphi(y)\,ds_y$

$$= -\int_\Gamma \nabla_y \Phi(x,y)\, \varphi(y)\, ds_y \quad \text{for } x = x_0 + h n_{x_0},\ x_0 \in \Gamma$$



Write $\nabla_y \Phi(x,y) = \dfrac{\partial \Phi(x,y)}{\partial n_y} n_y + \dfrac{\partial \Phi(x,y)}{\partial \tau_y} \tau_y$,

when τ_y is the tangential vector at y. Then we have

$$\nabla u(x) = -\int_\Gamma \frac{\partial \Phi(x,y)}{\partial n_y} n_y\, \varphi(y)\, ds_y - \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \tau_y} \tau_y\, \varphi(y)\, ds_y .$$

Note that $\displaystyle\int_\Gamma \frac{\partial \Phi(x,y)}{\partial \tau_y} \tau_y\, \varphi(y)\, ds_y = -\int \left(\frac{y-x}{2\pi |x-y|^2} \cdot \underset{(\tau_y - \tau_{x_0}) + \tau_{x_0}}{\tau_y} \right) \tau_y\, \varphi(y)\, ds_y$

we have $\displaystyle\lim_{h\to 0} \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \tau_y} \tau_y\, \varphi(y)\, ds_y = \int_\Gamma \frac{\partial \Phi(x_0,y)}{\partial \tau_y} \tau_y\, \varphi(y)\, ds_y$.

Therefore,

$$\nabla u_\pm(x_0) = -\int_\Gamma \frac{\partial \Phi(x_0,y)}{\partial n_y} n_y\, \varphi(y)\, ds_y \mp \frac{1}{2} n_{x_0} \cdot \varphi(x_0) - \int_\Gamma \frac{\partial \Phi(x_0,y)}{\partial n_y} \tau_y\, \varphi(y)\, ds_y$$

$$= -\int_\Gamma \nabla_y \Phi(x_0,y)\, \varphi(y)\, ds_y \mp \frac{1}{2} n_{x_0} \cdot \varphi(x_0)$$

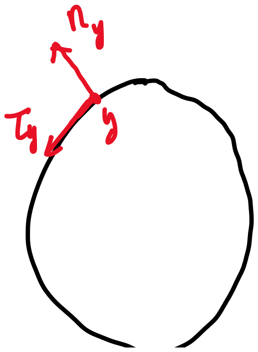
$$= \int_\Gamma \nabla_x \Phi(x_0,y)\, \varphi(y)\, ds_y \mp \frac{1}{2} n_{x_0} \cdot \varphi(x_0) .$$

Lemma 1.2.5 Let $\Phi(x,y) = -\frac{1}{2\pi} \ln|x-y|$, then it holds that

$$\nabla_x \left(\frac{\partial \Phi(x,y)}{\partial n_y} \right) = \left[\nabla_x (\nabla_x \Phi(x,y) \cdot \tau_y \right]^\perp \quad \text{for } x \notin \Gamma \text{ and } y \in \Gamma.$$

In the above, for a vector $\vec{a} = [a_1, a_2]$, we define $\vec{a}^\perp = [a_2, -a_1]$.

τ_y is the tangential vector at y.



proof. $\dfrac{\partial \Phi(x,y)}{\partial n_y} = -\nabla_x \Phi(x,y) \cdot n_y = -\nabla_x \cdot \left(\Phi(x,y) \tau_y^\perp \right)$.

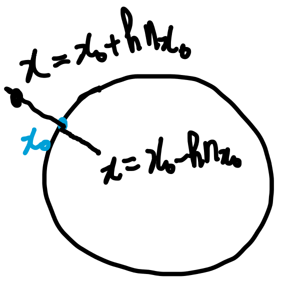
Using the formula $\nabla \times \nabla \times \vec{w} = \nabla(\nabla \cdot \vec{w}) - \Delta \vec{w}$, and

note that $\Delta \Phi(x,y) = 0$ for $x \neq y$, we obtain

Note that ... We obtain

$$\nabla_x \frac{\partial \Phi}{\partial n_y} = -\nabla_x\left(\nabla_x\cdot\left(\Phi(x,y)\,\tau_y^{\perp}\right)\right) = -\nabla_x\times\nabla_x\times\left(\Phi(x,y)\,\tau_y^{\perp}\right)$$

A direct calculation gives $-\nabla_x\times\nabla_x\times\left(\Phi(x,y)\,\tau_y^{\perp}\right) = \left[\nabla_x\left(\nabla_x\Phi(x,y)\cdot\tau_y\right)\right]^{\perp}$.



$$V(x) = \int_{\Gamma} \frac{\partial \Phi(x,y)}{\partial n_y}\,\varphi(y)\,ds_y, \quad x\in\mathbb{R}^2\setminus\Gamma$$

$$\nabla V(x) = \nabla_x \int_{\Gamma} \frac{\partial \Phi(x,y)}{\partial n_y}\,\varphi(y)\,ds_y, \quad x\in\mathbb{R}^2\setminus\Gamma.$$

Define $\dfrac{\partial V_{\pm}(x_0)}{\partial n_{x_0}} := \lim\limits_{h\to 0} n_{x_0}\cdot\nabla_x V(x_0 \pm h n_{x_0}),\quad x_0\in\Gamma.$

Theorem 1.2.6 (Derivative of double layer potential).

If $\varphi\in C^1(\Gamma)$, then the derivative of $V(x)$ can be extended from $\mathbb{R}^2\setminus\overline{\Omega}$ to $\mathbb{R}^2\setminus\Omega$ or from Ω to $\overline{\Omega}$ with the limit

$$\frac{\partial V_{\pm}(x)}{\partial n_x} = \frac{\partial}{\partial n_x}\int_{\Gamma}\frac{\partial \Phi(x,y)}{\partial n_y}\,\varphi(y)\,ds_y = \frac{\partial}{\partial \tau_x}\int_{\Gamma}\Phi(x,y)\,\frac{\partial \varphi(y)}{\partial \tau_y}\,ds_y, \quad x\in\Gamma.$$

Here τ_x and τ_y are tangential vectors of x & y on Γ.

Remark 1 $\dfrac{\partial}{\partial \tau_x}\displaystyle\int_{\Gamma}\Phi(x,y)\frac{\partial\varphi(y)}{\partial\tau_y}\,ds_y = \frac{d}{ds_x}\int_{\Gamma}\Phi(x,y)\frac{d\varphi(y)}{ds_y}\,ds_y$, where $\dfrac{d}{ds}$ is the derivative with respect to arc length.

Remark 2 Note that $\dfrac{\partial^2 \Phi(x,y)}{\partial n_x \partial n_y} = \dfrac{1}{2\pi}\dfrac{n_x \cdot n_y}{|x-y|^2} + \dfrac{1}{\pi}\dfrac{(y-x)\cdot n_y}{|x-y|^3}\,\dfrac{(x-y)\cdot n_x}{|x-y|} \sim O\left(\dfrac{1}{|x-y|^2}\right)$.

The integral $\dfrac{\partial}{\partial n_x}\displaystyle\int_P \dfrac{\partial \Phi(x,y)}{\partial n_y}\varphi(y)$ is understood in the sense of "Hadamard finite part integral".

Hadamard finite part integral

Consider the example $I := \displaystyle\int_{-1}^{1} \dfrac{f(t)}{|t|^{3/2}}\,dt$

Let $I_\varepsilon = \displaystyle\int_{[-1,1]\setminus[-\varepsilon,\varepsilon]} \dfrac{f(t)}{|t|^{3/2}}\,dt$, then I_ε can be splitted as

$$I_\varepsilon = \underbrace{\left(\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1}\right)\frac{1}{|t|^{3/2}}\big(f(t)-f(0)\big)\,dt}_{\text{finite as } \varepsilon\to 0} + f(0)\cdot\underbrace{\left(\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1}\right)\frac{1}{|t|^{3/2}}\,dt}_{=\ \frac{1}{\sqrt{\varepsilon}}\cdot f(0) - f(0)}$$

The Hadamard finite part integral $I_{Hadamard}$

$$I_{Hadamard} := \int_{-1}^{1}\frac{f(t)}{|t|^{3/2}}\,dt = \lim_{\varepsilon\to 0}\left(I_\varepsilon - \frac{1}{\sqrt{\varepsilon}}f(0)\right).$$

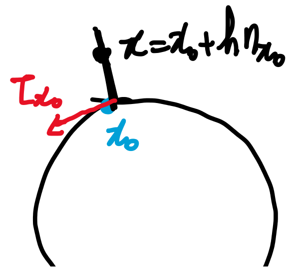
proof of Theorem 1.2.6. In view of Lemma 1.2.5, for $x \notin P$, we have

$$\nabla_x V(x) = \int_P \nabla_x \frac{\partial \Phi(x,y)}{\partial n_y}\varphi(y)\,ds_y = \int_P \left[\nabla_x\left(\nabla_x \Phi(x,y)\cdot \tau_y\right)\right]^{\perp}\varphi(y)\,ds_y$$

$$= \left[\nabla_x \int_P \nabla_x \Phi(x,y)\cdot \tau_y\, \varphi(y)\,ds_y\right]^{\perp} = -\left[\nabla_x \int_P \nabla_y \Phi(x,y)\cdot \tau_y\, \varphi(y)\,ds_y\right]$$

$$= \left[\nabla_x \int_P \Phi(x,y)\frac{\partial \varphi(y)}{\partial \tau_y}\,ds_y\right]^{\perp}.$$

Taking the limit when $x \to P$, it follows that



$$\frac{\partial V_{\pm}(x_0)}{\partial n_{x_0}} = \lim_{h\to 0^+} n_{x_0}\cdot \nabla V(x_0 + h n_{x_0})$$

$$= n_{x_0} \cdot \left[\nabla_{x_0} \int_\Gamma \Phi(x_0, y) \frac{\partial \varphi(y)}{\partial \nu_y} ds_y \mp \frac{1}{2} \frac{\partial \varphi}{\partial \nu_{x_0}} n_{x_0} \right]$$

$$= \frac{\partial}{\partial \nu_x} \int_\Gamma \Phi(x_0, y) \frac{\partial \varphi(y)}{\partial \nu_y} ds_y ,$$

where we have used Theorem 1.2.4.