Overview

Thursday, January 10, 2019 9:39 AM

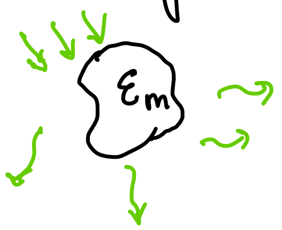
Mathematical studies of plasmon

phonon

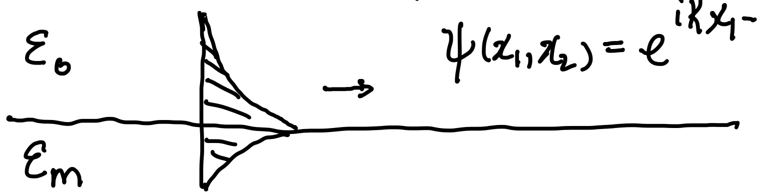
plasmon: collective oscillation of free electrons

Metals: ε_m, $\mathrm{Re}\,\varepsilon_m < 0$,

1. plasmon for nano-particles



2. Surface plasmon polariton



$\psi(x_1, x_2) = e^{ikx_1 - \eta|x_2|}$

$\eta > 0$

3. Extraordinary optical transmission (EOT) through nano-holes

Ebessen's experiment '98,

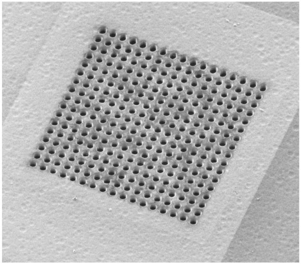
Mathematical tools: layer potential, integral equation, Numerical discretization, asymptotic analysis

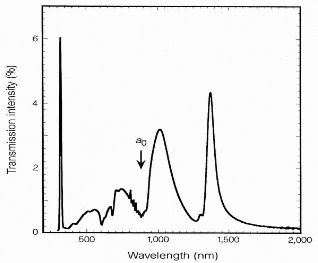
Lycurgus cup

Lycurgus cup in ancient Rome

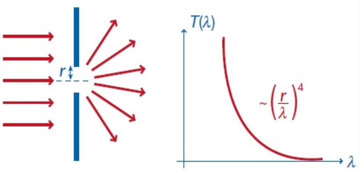
Extraordinary Optical Transmission Through a Small Hole Array

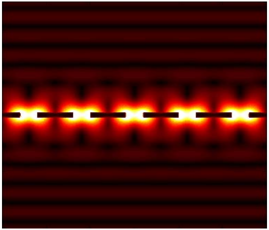
T. W. Ebbessen *et al*, Nature (1998)
Size of each hole: 150 nm, metal thickness: 300 nm.





Classical Bethe theory for diffraction by a small hole





§1.1 Sobolev Spaces

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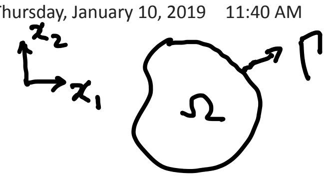

Ω: bnd. smooth in $\mathbb{R}^2$

$H^1(\Omega) := \{ u \mid u \in L^2(\Omega),\ \partial_1 u,\ \partial_2 u \in L^2(\Omega) \}$ is a Hilbert space

Inner product $\langle u, v\rangle_{H^1} := \int_\Omega u(x)\bar{v}(x)\,dx + \int_\Omega \sum_{j=1}^{2} \partial_j u\, \partial_j \bar{v}\, dx$

Norm $\|u\|_1 = \sqrt{\langle u,u\rangle} = \int_\Omega |u|^2 dx + \sum_{j=1}^{2}\int_\Omega |\partial_j u|^2 dx$

$\Gamma: t \to x(t),\ t \in [0, 2\pi]$

The Sobolev Space $H^s[0,2\pi]$, $s \geq 0$

$L^2[0,2\pi]$: $\{e^{int}\}_{n=-\infty}^{\infty}$ is a basis in $L^2[0,2\pi]$

$$u = \sum_{n=-\infty}^{\infty} \hat{u}_n e^{int}, \quad \hat{u}_n = \frac{1}{2\pi}\int_0^{2\pi} u(t) e^{-int}\,dt$$

$\|u\|^2_{L^2[0,2\pi]} = 2\pi \sum_{n=-\infty}^{\infty} |\hat{u}_n|^2$ (parseval's equality)

$H^s[0,2\pi] := \{ u \in L^2[0,2\pi] \mid \sum_{n=-\infty}^{\infty} (1+n^2)^s |\hat{u}_n|^2 < +\infty \}$

Inner product

$$\langle u, v\rangle_{H^s} := \sum_{n=-\infty}^{\infty} (1+n^2)^s\, \hat{u}_n \cdot \overline{\hat{v}_n}$$

Norm: $\|u\|_s = \left[\sum_{n=-\infty}^{\infty} (1+n^2)^s |\hat{u}_n|^2 \right]^{1/2}$

<u>Fact</u>: $u \in C^s_{2\pi}[0,2\pi]$, $s \in \mathbb{N}_+$.

The norm $\|\cdot\|_s$ is equivalent to norm $\|\cdot\|_{s,s}$, where

$$\|u\|^2_{s,s} = \int_0^{2\pi} |u|^2\,dt + \sum_{j=1}^{s} \int_0^{2\pi} \left| \frac{d^{(j)}u}{dt^j} \right|^2 dt.$$

$$\hat{u'}_n = \int_0^{2\pi} u'(t) e^{-int}\,dt = -\int_0^{2\pi} u(t)\, d e^{-int} = in \int_0^{2\pi} u(t) e^{-int}\,dt$$

$$\widehat{u^{(j)}_n} = (in)^j \hat{u}_n \qquad = in\, u_n$$

$$\|u\|_{s,s}^2 = 2\pi \sum_{n=-\infty}^{\infty} |\hat{u}_n|^2 \left[1 + n^2 + \cdots + n^{2s} \right]$$

$$C_1 (1+n^2)^s \le 1 + n^2 + \cdots\cdots n^{2s} \le C_2 (1+n^2)^s$$

$$\Rightarrow C_1 \|u\|_s^2 \le \|u\|_{s,s}^2 \le C_2 \|u\|_s^2 .$$

- Sobolev imbedding

(i) $H^r[0,2\pi] \xhookrightarrow{\text{Compact}} H^s[0,2\pi]$, $r > s$.

(ii) If $s > \frac{1}{2}$, $H^s[0,2\pi] \hookrightarrow C[0,2\pi]$.

Sobolev space $H^s(\Gamma)$

$x(t)$: is of C^p, $p \in \mathbb{N}_+$

$\Gamma = \{ x(t) \mid t \in [0,2\pi] \}$

$H^s(\Gamma) := \{ u \mid u(x(t)) \in H^s[0,2\pi] \}$

$\langle u, v \rangle_{H^s(\Gamma)} := \langle u(x(t)), v(x(t)) \rangle_{H^s[0,2\pi]}$

Fact (i) The definition of $H^s(\Gamma)$ is invariant w.r.t. the parametrization of Γ.

(ii) Trace Theorem: $\|u\|_{H^s(\Gamma)} \le C \|u\|_{H^{s+\frac{1}{2}}(\Omega)}$.