§1.5 Numerical discretization of singular integral operators

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Consider solving the modified integral equation $\left(\frac{1}{2}I + K - i\eta S\right)\varphi = g$ for the exterior Dirichlet problem. We apply the Nyström scheme to discretize the integral operators. The method applies a global approximation of the density function φ, which leads to high-order accuracy. At the same time, the evaluation of singular integral is given analytically.

Let $x(t)$ be a parametrization of the smooth curve Γ. Let $\psi(t) = \varphi(x(t))$, $r(t,\tau) = |x(t) - x(\tau)|$.

$$S\psi(t) = \int_0^{2\pi} \Phi_k(x(t), x(\tau))\, \psi(\tau)\, |x'(\tau)|\, d\tau$$

We define $M(t,\tau) := \Phi_k(x(t), x(\tau))\, |x'(\tau)| = \frac{i}{4} H_0^{(1)}(k r(t,\tau))\, |x'(\tau)|$.

Using the expansion of Bessel functions in Section 1.4.1, the kernel $M(t,\tau)$ can be written as

$$M(t,\tau) = M_1(t,\tau) \ln\left(4\sin^2\frac{t-\tau}{2}\right) + M_2(t,\tau),$$

where $M_1(t,\tau) = -\frac{1}{4\pi} J_0(kr)\, |x'(\tau)|$, and $M_2(t,\tau) = M(t,\tau) - M_1(t,\tau) \ln\left(4\sin^2\frac{t-\tau}{2}\right)$ is smooth.

Thus $$S\psi(t) = \int_0^{2\pi} \ln\left(4\sin^2\frac{t-\tau}{2}\right) M_1(t,\tau)\, \psi(\tau)\, d\tau + \int_0^{2\pi} M_2(t,\tau)\, \psi(\tau)\, d\tau \quad (1.5.1)$$

Similarly, in the parameter space, the double layer operator

$$K\psi(t) = \int_0^{2\pi} -\frac{ik}{4} H_1^{(1)}(k r(t,\tau)) \cdot \frac{x(t) - x(\tau)}{r(t,\tau)} \cdot \frac{x'(\tau)^{\perp}}{|\cancel{x'(\tau)}|}\, \psi(\tau)\, |\cancel{x'(\tau)}|\, d\tau.$$

Define $L(t,\tau) = -\frac{ik}{4} H_1^{(1)}(k r(t,\tau)) \frac{(x(t) - x(\tau)) \cdot x'(\tau)^{\perp}}{r(t,\tau)}$. The kernel $L(t,\tau)$ can be expanded as follows (by keeping the logarithm term):

$$L(t,\tau) = L_1(t,\tau) \ln\left(4\sin^2\frac{t-\tau}{2}\right) + L_2(t,\tau)$$

where

$$L_1(t,\tau) = \frac{k}{4\pi} J_1(kr) \frac{(x(t) - x(\tau)) \cdot x'(\tau)^{\perp}}{r(t,\tau)}, \quad L_2(t,\tau) = L(t,\tau) - L_1(t,\tau) \ln\left(4\sin^2\frac{t-\tau}{2}\right) \text{ is smooth.}$$

Hence $$K\psi(t) = \int_0^{2\pi} \ln\left(4\sin^2\frac{t-\tau}{2}\right) L_1(t,\tau)\, \psi(\tau)\, d\tau + \int_0^{2\pi} L_2(t,\tau)\, \psi(\tau)\, d\tau \quad (1.5.2)$$

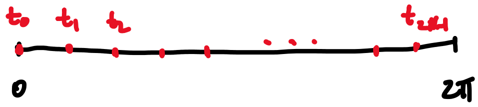
In view of (1.5.1) & (1.5.2), the integral equation $\left(\frac{1}{2}I + K - i\eta S\right)\varphi = g$ takes the following form in the parameter space:

$$\boxed{\frac{1}{2}\psi(t) + \int_0^{2\pi} \ln\left(4\sin^2\frac{t-\tau}{2}\right) K_1(t,\tau)\,\psi(\tau)\,d\tau + \int_0^{2\pi} K_2(t,\tau)\,\psi(\tau)\,d\tau = g(x(t)) \qquad 0\le t\le 2\pi.} \quad (1.5.3)$$

where $K_1(t,\tau) = L_1(t,\tau) - i\eta M_1(t,\tau)$, $K_2(t,\tau) = L_2(t,\tau) - i\eta M_2(t,\tau)$ are smooth.

The Nyström scheme approximates a function by a global trignometric polynomial.

Let $t_n = \frac{n\pi}{N}$, $n=0,1,\cdots,2N-1$ be equidistant points on $[0,2\pi]$ shown below:



We approximate a function $W(t)$ on $[0,2\pi]$ by a trigonometric interpolant

$$W_N(t) = \sum_{m=-N}^{N}{}'' \hat{W}_m e^{imt}, \quad \text{where } \hat{W}_m = \frac{1}{2N}\sum_{n=0}^{2N-1} W(t_n)\, e^{-imt_n}.$$

In the above $\sum''$ denotes halving the first and last terms.

Then it can be shown that $\boxed{\|W-W_N\|_{H^\sigma} \le C\left(\frac{1}{N}\right)^p \cdot \|W\|_{H^{p+\sigma}}}$, $p>0$, $\sigma>0$.

This is given in [Tadmor, 86]. In particular, when W is smooth, we may obtain arbitrary order of accuracy (so-called spectral accuracy)

Using Lemma 1.2.10, the following quadrature formulas hold:

$$\int_0^{2\pi} W_N(\tau)\,d\tau = \frac{\pi}{N}\sum_{n=0}^{2N-1} W(t_n), \qquad (1.5.4)$$

$$\int_0^{2\pi} \ln\left(4\sin^2\frac{t-\tau}{2}\right) W_N(\tau)\,d\tau = \sum_{n=0}^{2N-1} R_n(t)\, W(t_n), \text{ where} \qquad (1.5.5)$$

$$R_n(t) = -\frac{2\pi}{N}\sum_{m=1}^{N-1}\frac{1}{m}\cos\left(m(t-t_n)\right) - \frac{\pi}{N^2}\cos\left(N(t-t_n)\right), \quad n=0,1,\cdots,2N-1.$$

Applying the quadrature formulas (1.5.4) and (1.5.5) to (1.5.2) leads to the following discretization:

$$\frac{1}{2}\psi(t) + \sum_{n=0}^{2N-1} R_n(t)\, K_1(t,\tau_n)\,\psi(\tau_n) + \sum_{n=0}^{2N-1} K_2(t,\tau_n)\,\psi(\tau_n) = g(x(t)), \quad 0\le t\le 2\pi.$$

If one choses the collocation points $t=t_0,\cdots,t_{2N-1}$ for the above equation, then it follows that

$$A\vec{\psi} = \vec{g}.$$

where

$$\begin{bmatrix} R_0(t_0)K_1(t_0,\tau_0) + K_2(t_0,\tau_0) & \cdots & R_{2N-1}(t_0)K_1(t_0,\tau_{2N-1}) + K_2(t_0,\tau_{2N-1}) \end{bmatrix}$$

$$A = \frac{1}{2} I_{(2N-1)\times(2N-1)} + \begin{bmatrix} \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ R_0(t_{2N-1})\, K_1(t_{2N-1},\tau_0) + K_2(t_{2N-1},\tau_0) \ \cdots \ R_{2N-1}(t_{2N-1})\, K_1(t_{2N-1},\tau_{2N-1}) + K_2(t_{2N-1},\tau_{2N-1}) \end{bmatrix},$$

$$\vec{\psi} = [\psi(\tau_0), \psi(\tau_1), \cdots, \psi(\tau_{2N-1})], \quad \vec{g} = [g(x(t_0)), g(x(t_1)), \ldots, g(x(t_{2N-1}))].$$

Solving the linear system $A\vec{\psi} = \vec{g}$ gives the solution at grid points $\{t_n\}_{n=0}^{2N-1}$.

Discretization of hyper-singular operator T

Consider solving the integral equation $T\varphi = g$ on Γ.

From Theorem 1.4.1, we have $T\varphi = \frac{d}{ds_x} S[\frac{d\varphi}{ds_y}] + k^2 S[\varphi\, n_x \cdot n_y]$.

The discretization of the single layer operator $S[\varphi\, n_x\cdot n_y]$ follows from the discussions above. Using $\frac{d}{dt} = \frac{ds}{dt}\cdot\frac{d}{ds} = |x'(t)|\frac{d}{ds}$, in the parameterized form,

$$\frac{d}{ds_x} S\left[\frac{d\varphi}{ds_y}\right] = \frac{1}{|x'(t)|}\frac{d}{dt}\int_0^{2\pi} \frac{i}{4} H_0^{(1)}(k\, r(t,\tau))\, \frac{d\psi(\tau)}{d\tau}\, d\tau, \quad \text{where } r = |x(t) - x(\tau)|,\ \psi(\tau) = \varphi(x(\tau)).$$

Decompose the fundament solution as $\frac{i}{4} H_0^{(1)}(kr) = -\frac{1}{4\pi}\ln\left(4\sin^2\frac{t-\tau}{2}\right) + N(t,\tau)$,

where $N(t,\tau) = -\frac{1}{4\pi}(J_0(kr) - 1)\ln(4\sin^2\frac{t-\tau}{2})$ + Smooth terms.

Then

$$\frac{d}{ds_x} S\left[\frac{d\varphi}{ds_y}\right] = -\frac{1}{|x'(t)|}\cdot\frac{1}{4\pi}\int_0^{2\pi}\frac{d}{dt}\ln\left(\sin^2\frac{t-\tau}{2}\right)\frac{d\psi(\tau)}{d\tau}\,d\tau + \frac{1}{|x'(t)|}\int_0^{2\pi}\frac{\partial N(t,\tau)}{\partial t}\frac{d\psi(\tau)}{d\tau}\,d\tau$$

$$= \frac{1}{|x'(t)|}\cdot\frac{1}{4\pi}\int_0^{2\pi}\cot\left(\frac{\tau-t}{2}\right)\frac{d\psi(\tau)}{d\tau}\,d\tau + \frac{1}{|x'(t)|}\int_0^{2\pi}\frac{\partial N(t,\tau)}{\partial t}\frac{d\psi(\tau)}{d\tau}\,d\tau.$$

$$=: I_1 + I_2.$$

By using a global interpolant for $\psi(t)$ then $\psi(t) \approx \psi_N(t) = \sum_{m=-N}^{N}{}'' \hat{\psi}_m e^{im\tau}$, $\hat{\psi}_m = \frac{1}{2N}\sum_{n=0}^{2N-1}\psi(t_n)\, e^{-imt_n}$.

Note that $\frac{1}{2\pi}\int_0^{2\pi}\cot(\frac{\tau-t}{2})\,(e^{im\tau})'\,d\tau = \frac{im}{2\pi}\int_0^{2\pi}\cot(\frac{\tau-t}{2})e^{im\tau}\,d\tau = -|m|\,e^{imt}$. (Lemma 1.2.10)

One obtains the quadrature formula

$$\frac{1}{2\pi}\int_0^{2\pi}\cot\left(\frac{\tau-t}{2}\right)\frac{d\psi_N(\tau)}{d\tau}\,d\tau = \sum_{n=0}^{2N-1} T_n(t)\,\psi(t_n), \quad \text{where}$$

$$T_n(t) = -\frac{1}{N}\sum_{m=1}^{N-1} m\cos(m(t-t_0)) - \frac{1}{2}\cos(N(t-t_0)).$$

Hence the discretization of I_1 is obtained. Now for I_2, integration by parts leads to

$$I_2 = -\frac{1}{|z'(t)|}\int_0^{2\pi} \frac{\partial^2 N(t,\tau)}{\partial t\,\partial \tau}\varphi(\tau)\,d\tau$$

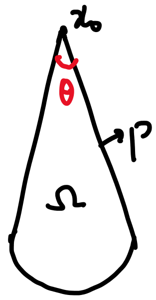
From the expression of $N(t,\tau)$, we see that

$$\frac{\partial^2 N(t,\tau)}{\partial t\,\partial \tau} = N_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + N_2(t,\tau),$$ where N_1 and N_2 are smooth.

Thus $$I_2 = -\frac{1}{|z'(t)|}\int_0^{2\pi} N_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)\varphi(\tau) - \frac{1}{|z'(t)|}\int_0^{2\pi} N_2(t,\tau)\varphi(\tau)\,d\tau.$$

An application of the quadrature formula (1.5.4) and (1.5.5) gives the desired discretization.

Discretization of integral operators for domains with corners.



Consider solving the integral equation $(\frac{1}{2}I + K)\varphi = g$.

(1) The operator K is not compact. $\frac{1}{r}$ singularity for the kernel near x_0.

(2) The solution φ is not smooth globally.

Using the relation $\int_\Gamma \frac{\partial \Phi(x,y)}{\partial n_y}\,ds_y = -\frac{1}{2}$, $x \in \Gamma \setminus \{x_0\}$, where $\Phi(x,y) = -\frac{1}{2\pi}\ln|x-y|$,

the integral equation can be modified as

$$\frac{1}{2}\varphi(x) + \int_\Gamma \frac{\partial \Phi(x,y)}{\partial n_y}\varphi(y)\,ds_y - \left(\frac{1}{2} + \int_\Gamma \frac{\partial \Phi(x,y)}{\partial n_y}\right)\cdot \varphi(x_0)\cdot W_{x_0,r}(x) = g,$$

where $W_{x_0,r}(x)$ is a smooth cut-off function such that $W_{x_0,r}(x) = \begin{cases} 1, & |x-x_0| \le \frac{r}{2}, \\ 0, & |x-x_0| \ge r. \end{cases}$

Then the above integral equation can be written as

$$\frac{1}{2}\left(\varphi(x) - \varphi(x_0)W_{x_0,r}(x)\right) + \int_\Gamma\left(\frac{\partial \Phi(x,y)}{\partial n_y} - \frac{\partial \Phi(x,y)}{\partial n_y}\right)\varphi(y)\,ds_y + (1 - W_{x_0,r})\int_\Gamma \frac{\partial \Phi(x,y)}{\partial n_y}\varphi(y)\,ds_y$$

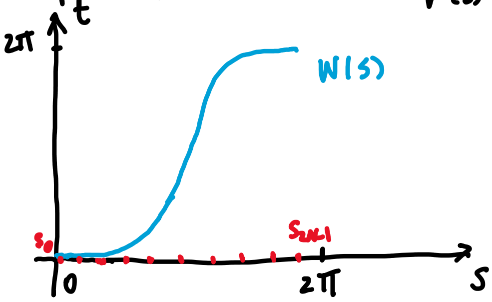
$$+ W_{x_0,r}\cdot\int_\Gamma \frac{\partial \Phi(x,y)}{\partial n_y}(\varphi(y) - \varphi(x_0))\,ds_y = g$$

The first two integral operators attain weakly singular / smooth kernels, and are compact on $C(\Gamma)$. On the other hand, by choosing sufficiently small r, it can be shown that $\left\| W_{x_0,r}\int_\Gamma \frac{\partial \Phi(x,y)}{\partial n_y}(\varphi(y) - \varphi(x_0))\right\|_{C(\Gamma)} < \delta$

for a prescribed small constant δ. Therefore, $\frac{1}{2}I+k$ is invertible on $C(\Gamma)$ by a combination of the Fredholm alternative and the Neumann Series.

Let $x(t)$ be a parametrization of Γ s.t $x_0 = x(0) = x(2\pi)$.
To resolve problem (2), we consider the approximation of the integral in the parametrized form: $I = \int_0^{2\pi} f(t)\,dt$,
where $f(t)$ attains singularity at $t=0, 2\pi$. Typically, there holds $\varphi \sim O(r^{\beta(\theta)})$ near x_0 for the solution of the integral equation, where $r = |x - x_0|$, $0<\beta(\theta)<1$, and $\beta(\theta)$ decreases with the angle θ. Hence we assume that f attains the same singularity near the end points $t=0, 2\pi$.

Introduce a parameterization $t = W(s)$, $0 \le s \le 2\pi$, where $W^{(j)}(0) = W^{(j)}(2\pi) = 0$, $j = 0, 1, \cdots, p-1$.
One example is given by $W(s) = 2\pi \dfrac{V^p(s)}{V^p(s) + V^p(2\pi - s)}$, where $V(s) = \left(\frac{1}{p} - \frac{1}{2}\right)\left(\frac{\pi - s}{\pi}\right)^3 + \frac{1}{p}\frac{s-\pi}{\pi} + \frac{1}{2}$.



Then $I = \int_0^{2\pi} f(W(s))\,W'(s)\,ds$, and the integrand is "smoothed out" at the corner.
We apply a global trigonometric interpolant for the integrand, then the formula (1.5.4) leads to an approximation of the integral:

$$I \approx I_N = \frac{\pi}{N} \sum_{n=0}^{2N-1} f(W(s_n))\,W'(s_n).$$

The accuracy of the approximation is given by following statement:

proposition 1.5.1 If $\int_0^{2\pi} \frac{1}{t^\alpha} f'(t)\,dt < +\infty$ for some $0<\alpha<1$, and $\alpha p \ge 1$,
then $|I - I_N| \le \frac{C}{N}$.

Remark: Through the parameterization $t = W(s)$, for a uniformly space grid points $\{s_n\}_{n=0}^{2N-1}$, the grid points $\{t_n\}_{n=0}^{2N-1}$ are dense near $t=0$ and 2π. In addition, if β is smaller such that for φ is more singular, from the above

proposition, p has to be larger to gurantee the same order of accuracy.
Namely, more refined grid points have to be imposed near the end points.