§1.4 Layer potential and boundary integral equation for Helmholtz equation

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§1.4.1 Helmholtz equation

$\Delta u + k^2 u = 0$ in $\mathbb{R}^d$, $d=2,3$, $k=\frac{\omega}{c}$ is called wavenumber,

where ω is the frequency, and c is the wave speed.

The Helmholtz equation arises naturally from the modeling of time-harmonic acoustic or electromagnetic wave.

Wave equation: $\frac{1}{c^2}\frac{\partial^2 W(x,t)}{\partial t^2} - \Delta W(x,t) = 0$ in $\mathbb{R}^d \times (0,T)$

Maxwell's equations:
$$\begin{cases} \varepsilon_0 \frac{\partial \vec{E}}{\partial t} - \nabla \times \vec{H} = 0, \\ \mu_0 \frac{\partial \vec{H}}{\partial t} - \nabla \times \vec{E} = 0, \\ \nabla \cdot \vec{E} = 0, \\ \nabla \cdot \vec{H} = 0. \end{cases} \quad \text{in } \mathbb{R}^3 \times (0,T).$$

ε_0 & μ_0: electric permittivity and magnetic permeability in a homogeneous medium.

$\vec{E}$ & $\vec{H}$: electric field and magnetic field

If one set $W(x,t) = u(x) e^{-i\omega t}$, then u solves $\Delta u + k^2 u = 0$, $k = \frac{\omega}{c}$.

Similarly, if $\vec{E}(x,t) = \vec{U}(x) e^{-i\omega t}$, then $\Delta \vec{U} + k^2 \vec{U} = 0$, $k = \frac{\omega}{c}$, $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$.

Sommerfeld radiation condition:

$$\frac{\partial u}{\partial r} \pm iku = o\left(\frac{1}{r}\right) \text{ as } r \to \infty,\ r = |x|,\ d=3.$$

$$\frac{\partial u}{\partial r} \pm iku = o\left(\frac{1}{\sqrt{r}}\right) \text{ as } r \to \infty,\ r = |x|,\ d=2.$$

When "$-$" is taken, it is called outgoing radiation condition, Otherwise, it is called incoming radiation condition.

Fundamental solution of Helmholtz equation $\Phi_k(x,y)$

Let $\Phi_k(x,y) = f(kr)$, $r = |x-y|$.

In $\mathbb{R}^2$, f satisfies the Bessel equation $z^2 f''(z) + z f'(z) + z^2 f(z) = 0$, where $z = kr$.

There exist two linearly independent Bessel functions (of order zero)

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{z}{2}\right)^{2n}, \quad Y_0(z) = \frac{2}{\pi}\left(\ln\frac{z}{2} + C\right) J_0(z) - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{z}{2}\right)^{2n} \cdot 2\left(\sum_{m=1}^{n}\frac{1}{m}\right),$$

where C is the Euler-Mascheroni constant given by $C = \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n} - \ln N\right)$.

The following asymptotics holds for J_0 and Y_0:

As $z \to 0$, $J_0(z) \to 1$ and $Y_0(z) \sim \frac{2}{\pi}\ln z$. (*)

As $z \to +\infty$, $J_0(z) \sim \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{4}\right)$ and $Y_0(z) \sim \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{\pi}{4}\right)$.

We also define the Hankel's functions

$H_0^{(1)}(z) = J_0(z) + iY_0(z)$ and $H_0^{(2)}(z) = J_0(z) - iY_0(z)$

Then from the asymptotic behavior of J_0 and Y_0, we see that $H_0^{(1)}(kr)$ and $H_0^{(2)}(kr)$ satisfies the outgoing and incoming radiation, respectively. (**)

From (*) and (**), one deduces that the fundamental solution in $\mathbb{R}^2$ is given by

$\Phi_k(x,y) = \frac{i}{4} H_0^{(1)}(kr)$, $r = |x-y|$. It satisfies
$$\begin{cases} \Delta \Phi_k(x,y) = -\delta(x-y) & \text{in } \mathbb{R}^2 \\ \frac{\partial \Phi_k}{\partial r} - ik\Phi_k = o\left(\frac{1}{\sqrt{r}}\right) & \text{as } r \to \infty. \end{cases}$$

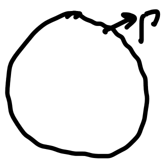
In $\mathbb{R}^3$, the radial solution $f(r)$ of the Helmholtz equation satisfies
$$\frac{1}{r^2}\left(r^2 f'(r)\right)' + k^2 f(r) = 0,$$
which attains two linearly independent solutions $\frac{e^{ikr}}{r}$ and $\frac{e^{-ikr}}{r}$.

Thus the outgoing fundamental solution $\Phi_k(x,y) = \frac{e^{ikr}}{4\pi r}$, $r = |x-y|$.

§ 1.4.2 Layer potentials and boundary integral operators

single layer potential $u(x) := \int_\Gamma \Phi_k(x,y)\varphi(y)\,ds_y$, $x \in \mathbb{R}^2 \setminus \Gamma$,

double layer potential $v(x) := \int_\Gamma \frac{\partial \Phi_k(x,y)}{\partial n_y}\varphi(y)\,ds_y$, $x \in \mathbb{R}^2 \setminus \Gamma$.



$\Phi_k(x,y) = \frac{i}{4} H_0^{(1)}(k|x-y|)$, $H_0^{(1)}(z) = J_0(z) + iY_0(z)$.

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{z}{2}\right)^{2n}, \quad Y_0(z) = \frac{2}{\pi}\left(\ln\frac{z}{2} + C\right)J_0(z) - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{z}{2}\right)^{2n} \cdot 2\left(\sum_{m=1}^{n}\frac{1}{m}\right)$$

If $z \ll 1$, $J_0(z) = 1 + O(z^2)$, $Y_0(z) = \frac{2}{\pi}\left(\ln\frac{z}{2} + C\right) + O(z^2\ln z)$, we obtain
$$H_0(z) = 1 + i\,\frac{2}{\pi}\left(\ln\frac{z}{2} + C\right) + O(z^2\ln z) \quad \text{for } z \ll 1.$$

Thus $\Phi_k(x,y) = -\frac{1}{2\pi}\ln|x-y| - \frac{1}{2\pi}\left(\ln\frac{k}{2} + C\right) + \frac{i}{4} + O\left(|x-y|^2\ln\frac{1}{|x-y|}\right)$ if $|x-y| \ll 1$.

By decomposing $u(x)$ as $u(x) = u_0(x) + u_1(x)$, where
$$u_0(x) = \int_\Gamma \Phi(x,y)\varphi(y)\,ds_y, \quad \Phi(x,y) = -\frac{1}{2\pi}\ln|x-y|$$
$$u_1(x) = \int_\Gamma \left(\Phi_k(x,y) - \Phi(x,y)\right)\varphi(y)\,ds_y$$

From theorem 1.2.1 and the asymptotic expansion of $\Phi_k(x,y)$ above, we see that both $U_0(x)$ and $U_1(x)$, are continuous in $\mathbb{R}^2$, and $U(x)$ is continuous throughout $\mathbb{R}^2$.

$$\left(H_0^{(1)}(z)\right)' = -H_1^{(1)}(z) = -\left(J_1(z) + iY_1(z)\right) = -\left(-\frac{2i}{\pi}\frac{1}{z} + O\left(z\ln\frac{1}{z}\right)\right) = \frac{2i}{\pi}\frac{1}{z} + O\left(z\ln\frac{1}{z}\right),\ z \ll 1.$$

$$\nabla_y \Phi_k(x,y) = \frac{i}{4}\left(H_0^{(1)}(k|x-y|)\right)' \cdot k \cdot \frac{y-x}{|x-y|} = -\frac{1}{2\pi}\frac{y-x}{|x-y|^2} + O\left(|x-y|\ln\frac{1}{|x-y|}\right)$$

Therefore, by decomposing $V(x) = V_0(x) + V_1(x)$, where

$$V_0(x) = \int_\Gamma \frac{\partial \Phi(x,y)}{\partial n_y}\varphi(y)\,ds_y,\quad V_1(x) = \int_\Gamma \left(\frac{\partial \Phi_k(x,y)}{\partial n_y} - \frac{\partial \Phi(x,y)}{\partial n_y}\right)\varphi(y)\,ds_y,$$

then $V_1(x)$ is continuous in $\mathbb{R}^2$. Applying Theorem 1.2.2 for $V_0(x)$, we obtain

$$V_\pm(x) = \int_\Gamma \frac{\partial \Phi(x,y)}{\partial n_y}\varphi(y)\,ds_y \pm \frac{1}{2}\varphi(x).$$

Theorem 1.4.1 Let $\varphi \in C(\Gamma)$, then the single layer potential $U(x)$ is continuous in $\mathbb{R}^2$. The double layer potential $V(x)$ can be extended from $\mathbb{R}^2\setminus\bar\Omega$ to $\mathbb{R}^2\setminus\Omega$ or from Ω to $\bar\Omega$ with the limit

$$V_\pm(x) = \int_\Gamma \frac{\partial \Phi_k(x,y)}{\partial n_y}\varphi(y)\,ds_y \pm \frac{1}{2}\varphi(x),\quad x \in \Gamma.$$

In addition, the following holds for the derivative of $U(x)$ and $V(x)$:

$$\frac{\partial U_\pm(x)}{\partial n_x} = \int_\Gamma \frac{\partial \Phi_k(x,y)}{\partial n_x}\varphi(y)\,ds_y \mp \frac{1}{2}\varphi(x),\quad x \in \Gamma.$$

$$\frac{\partial V_\pm(x)}{\partial n_x} = \int_\Gamma \frac{\partial^2 \Phi_k(x,y)}{\partial n_x \partial n_y}\varphi(y)\,ds_y = \frac{d}{ds_x} S\left(\frac{d\varphi}{ds_y}\right) + k^2 S\left(\varphi\, n_x \cdot n_y\right),\ x \in \Gamma,$$

where S is the single layer potential defined below, and the density function $\varphi \in C^1(\Gamma)$ in the last equality.

We define the integral operators:

$$[S\varphi](x) = \int_\Gamma \Phi_k(x,y)\varphi(y)\,ds_y,\quad [K\varphi](x) = \int_\Gamma \frac{\partial \Phi_k(x,y)}{\partial n_y}\varphi(y)\,ds_y,\quad x \in \Gamma;$$

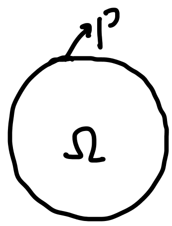
$$[K'\varphi](x) = \int_\Gamma \frac{\partial \Phi_k(x,y)}{\partial n_x}\varphi(y)\,ds_y,\quad [T\varphi](x) = \int_\Gamma \frac{\partial^2 \Phi_k(x,y)}{\partial n_x \partial n_y}\varphi(y)\,ds_y,\quad x \in \Gamma.$$

Theorem 1.4.2 Let Γ be smooth, then S, K, K' are bounded from $H^p(\Gamma)$ to $H^{p+1}(\Gamma)$, and T is bounded from $H^{p+1}(\Gamma)$ to $H^p(\Gamma)$, $p \geq 0$.

§ 1.4.3 Integral equations for boundary value problems

Consider the exterior and interior Dirichlet problems

$$(\mathrm{I})\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^2\setminus\bar{\Omega}, \\ u = g & \text{on } \Gamma, \\ \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0. \end{cases} \qquad (\mathrm{I}')\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega, \\ u = g & \text{on } \Gamma. \end{cases}$$



The exterior problem (I) attains a unique solution for all real k.
For the interior problem (I'), it attains a unique solution if k is not an eigenvalue of the homogenous problem.

(i) Integral equation of the first kind

Consider solving the exterior problem (I), we express the solution $u(x)$ as the single layer potential with density function $\varphi(x)$:

$$u(x) = \int_\Gamma \Phi(x,y)\,\varphi(y)\,ds_y.$$

Then by taking the limit to the boundary Γ, we obtain an integral equation

$$S\varphi = g \quad \text{on } \Gamma.$$

<u>Theorem 1.4.3</u> The integral equation $S\varphi = g$ attains a unique solution provided k is not an interior Dirichlet eigenvalue.

<u>proof</u>. (Existence) If k is not an interior Dirichlet eigenvalue, then (I') attains a unique solution $u^i(x)$. Let $u^e(x)$ be the solution of the exterior problem (I). Applying the Green's second identity in Ω and $\mathbb{R}^2\setminus\bar{\Omega}$ leads to

$$u^i(x) = \int_\Gamma \Phi_k(x,y)\frac{\partial u^i(y)}{\partial n} - \frac{\partial \Phi_k(x,y)}{\partial n_y}u^i(y)\,ds_y, \quad x\in\Omega,$$

$$0 = \int_\Gamma \Phi_k(x,y)\frac{\partial u^e(y)}{\partial n} - \frac{\partial \Phi_k(x,y)}{\partial n_y}u^e(y)\,ds_y, \quad x\in\Omega.$$

Therefore, $u^i(x) = \displaystyle\int_\Gamma \Phi_k(x,y)\left(\frac{\partial u^i(y)}{\partial n} - \frac{\partial u^e(y)}{\partial n}\right)ds_y.$

Taking the limit to the boundary Γ gives $S\left[\frac{\partial u^i}{\partial n} - \frac{\partial u^e}{\partial n}\right] = g.$

(Uniqueness) If k is not an interior Dirichlet eigenvalue, we have unique solutions

$u^- \equiv 0$ in Ω and $u^+ \equiv 0$ in $\mathbb{R}^2 \setminus \Omega$ for the homogeneous interior and exterior problems ($g=0$).

Let φ_0 be a solution of $S\varphi_0 = 0$, then the corresponding single layer potential $u(x) = \int_\Gamma \Phi_k(x,y)\varphi_0(y)\,ds_y$ is zero in $\mathbb{R}^2$. From the jump relation $\varphi_0 = \frac{\partial u_-}{\partial n} - \frac{\partial u_+}{\partial n}$, we obtain $\varphi_0 = 0$.

(ii) Integral equation of the second kind

We express the solution as the double layer potential

$$u(x) = \int_\Gamma \frac{\partial \Phi_k(x,y)}{\partial n_y}\varphi(y)\,ds_y, \quad x \notin \Gamma.$$

By taking the limit to the boundary Γ, we obtain the following integral equations for the exterior and interior problem respectively:

$$\left(\tfrac{1}{2}I + K\right)\varphi = g \quad \text{and} \quad \left(-\tfrac{1}{2}I + K\right)\varphi = g.$$

We apply the "Fredholm alternative" to study the above integral equation.

Fredholm alternative: Let A be a compact operator on the Hilbert space H.
Then either $\varphi - A\varphi = f$ attains a unique solution for any $f \in H$,
or else $\varphi - A\varphi = 0$ has non-trivial solutions.

Theorem 1.4.4 The integral equations $\left(\frac{1}{2}I + K\right)\varphi = g$ and $\left(-\frac{1}{2}I + K\right)\varphi = g$ attain a unique solution if k is not an eigenvalue of the interior Neumann and Dirichlet eigenvalue respectively.

The assertion of the theorem follows from the Fredholm alternative and Lemmas 1.4.5 and 1.4.9.

Lemma 1.4.5 The operator $K: H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is compact. There holds

$$\mathrm{Ker}\left(\tfrac{1}{2}I + K\right) = V, \text{ where } V := \left\{ u|_\Gamma \in H^{\frac{1}{2}}(\Gamma) \,\middle|\, \Delta u + k^2 u = 0 \text{ in } \Omega, \tfrac{\partial u}{\partial n} = 0 \text{ on } \Gamma \right\}.$$

proof. The compactness of K follows from Theorem 1.4.2 and the Sobolev compact imbedding.

If $\varphi_0 \in \mathrm{Ker}\left(\frac{1}{2}I + K\right)$, we define $v(x) = \int \frac{\partial \Phi_k(x,y)}{\partial n_y}\varphi_0(y)\,ds_y$ for $x \notin \Gamma$.

then $V(x)$ solves the exterior problem (I) with $g=0$. The uniquness of the solution for the exterior problem implies that $V\equiv 0$ in $\mathbb{R}^2\setminus\Omega$, as such $\frac{\partial V_+}{\partial n}=0$ on Γ.

In view of Theorem 1.4.1, we obtain $\frac{\partial V_-}{\partial n}=\frac{\partial V_+}{\partial n}=0$. Thus

$$\begin{cases} \Delta V+k^2V=0 \text{ in } \Omega, \\ \frac{\partial V_-}{\partial n}=0 \text{ on } \Gamma. \end{cases}$$

Therefore, $\varphi_0=V_+-V_-=-V_-\in V$.

If $\varphi_0\in V$, then there exists $V(x)$ such that $\begin{cases} \Delta V+k^2V=0 \text{ in } \Omega, \\ \frac{\partial V}{\partial n}=0, \ V=\varphi_0 \text{ on } \Gamma. \end{cases}$

Apply the Green's second identity for $x\in\mathbb{R}^2\setminus\bar\Omega$ and use the boundary conditions for $V(x)$, we obtain

$$\int_\Gamma \frac{\partial\Phi_k(x,y)}{\partial n_y}\varphi_0(y)\,ds_y=0.$$

Taking the limit of the above to the boundary Γ leads to $(\frac{1}{2}I+K)\varphi_0=0$, and $\varphi_0\in \mathrm{Ker}(\frac{1}{2}I+K)$.

(iii) Modified integral equation

We express the solution as a combination of single and double layer potential:

$$u(x)=\int_\Gamma\left(\frac{\partial\Phi_k(x,y)}{\partial n_y}-i\eta\,\Phi_k(x,y)\right)\varphi(y)\,ds_y,\quad x\notin\Gamma,\quad \eta\neq 0.$$

For the exterior problem (I), this leads to the following integral equation:

$$\left(\tfrac{1}{2}I+K-i\eta S\right)\varphi=g \text{ on } \Gamma.$$

<u>Theorem 1.4.6</u> The integral equation $(\frac{1}{2}I+K-i\eta S)\varphi=g$ is uniquely solvable for all real k.

<u>proof</u>. Consider the homogeneous equation $(\frac{1}{2}I+K-i\eta S)\varphi_0=0$.

Let $u(x)=\int_\Gamma\left(\frac{\partial\Phi_k(x,y)}{\partial n_y}-i\eta\,\Phi_k(x,y)\right)\varphi_0(y)\,ds_y,\quad x\notin\Gamma$.

Then $u(x)$ solves the exterior problem (I) with $g=0$. From the uniquesness of the solution, we have $u(x)\equiv 0$ in $\mathbb{R}^2\setminus\Omega$, and $\frac{\partial u_+}{\partial n}=0$. In view of Theorem 1.4.1,

$\varphi_0=u_+-u_-=-u_-$, $\quad i\eta\varphi_0=\frac{\partial u_+}{\partial n}-\frac{\partial u_-}{\partial n}=-\frac{\partial u_-}{\partial n}$.

Namely, u satisfies $\begin{cases} \Delta u+k^2u=0 \text{ in } \Omega, \\ u=-u_- \quad \frac{\partial u_-}{\partial n} \ldots \end{cases}$

$\ldots\ T_0,\ \frac{\partial u}{\partial n} = -i\eta\varphi_0$ on Γ.

Applying the Green's first identity and using the boundary conditions gives

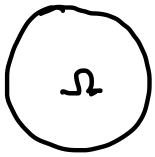
$$\int_\Omega |\nabla u|^2 - k|u|^2\,dx = -i\eta\int_\Gamma |\varphi_0|^2\,ds.$$

Since $\eta \neq 0$, by taking the imaginary part of the above, we obtain $\varphi_0 = 0$ on Γ.

Now the Fredholm alternative implies that the inhomogeneous equation attains a unique solution.

Consider the exterior and interior Neumann problems

$$(\mathrm{II})\begin{cases}\Delta u + k^2 u = 0 & \text{in } \mathbb{R}^2\setminus\bar\Omega,\\ \frac{\partial u}{\partial n} = h & \text{on } \Gamma,\\ \lim_{r\to\infty}\sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0.\end{cases} \qquad (\mathrm{II}')\begin{cases}\Delta u + k^2 u = 0 & \text{in } \Omega,\\ \frac{\partial u}{\partial n} = h & \text{on } \Gamma.\end{cases}$$



The exterior problem (II) attains a unique solution, while the interior problem (II') attains a unique solution if k is not an eigenvalue. Similar to the Dirichlet problems we may have integral equation of the first and second kind.

(i) Integral equation of the first kind.

Express the solution of the boundary value problems as the double layer potential

$$u(x) = \int_\Gamma \frac{\partial\Phi_k(x,y)}{\partial n_y}\varphi(y)\,ds_y,\quad x\notin\Gamma.$$

Then the Neumann problems leads to the integral equation $T\varphi = h$ on Γ.

<u>Theorem 1.4.7</u> The integral equation $T\varphi = h$ attains a unique solution if k is not an eigenvalue for the interior Neumann problem.

The proof is similar to that of Theorem 1.4.3.

(ii) Integral equation of the second kind.

Express the solution as the single layer potential

$$u(x) = \int_\Gamma \Phi_k(x,y)\varphi(y)\,ds_y \quad x\notin\Gamma.$$

We obtain the integral equation $(-\frac{1}{2}I + K')\varphi = h$ and $(\frac{1}{2}I + K')\varphi = h$ for the exterior and interior problem respectively.

<u>Theorem 1.4.8</u> The integral equation $(-\frac{1}{2}I + K')\varphi = h$ and $(\frac{1}{2}I + K')\varphi = h$ attains a unique solution if k is not an eigenvalue for the interior Dirichlet

and Neumann problem respectively.

The proof follows from the Fredholm alternative and Lemmas 1.4.9 and 1.4.5.

Lemma 1.4.9 The operator K' is compact on $H^{\frac{1}{2}}(\Gamma)$. There holds

$$\mathrm{Ker}\left(-\tfrac{1}{2}I+K'\right)=W, \text{ where } W=\left\{\frac{\partial u}{\partial n}\Big|_{\Gamma}\in H^{\frac{1}{2}}(\Gamma)\;\Big|\;\Delta u+k^2u=0 \text{ in } \Omega, u=0 \text{ on } \Gamma\right\}.$$

The proof is similar to that of Lemma 1.4.5.

(iii) Modified integral equation

Express the solution as a combination of single and double layer potential:

$$u(x)=\int_{\Gamma}\left(\Phi_k(x,y)+i\eta\frac{\partial\Phi_k(x,y)}{\partial n_y}\right)\varphi(y)\,ds_y,\quad x\notin\Gamma,\quad \eta\neq 0.$$

This leads to the integral equation $\left(-\frac{1}{2}I+K'+i\eta T\right)\varphi=h$ for the exterior problem (I).

The integral equation attains a unique solution for all real k.

Exercises. Fill in the proof of Theorem 1.4.7 and Lemma 1.4.9.