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THE EXPONENTIAL ACCURACY OF FOURIER AND CHEBYSHEV DIFFERENCING METHODS*

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Abstract. It is shown that when differencing analytic functions using the pseudospectral Fourier or Chebyshev methods, the error committed decays to zero at an *exponential* rate.

Key words. Fourier approximation, Chebyshev approximation, pseudospectral differencing, exponential decay

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1. Introduction. The pseudospectral differencing methods involve the exact differentiation of interpolants which are based on different sets of selected points. Each method is usually named after the base functions used to expand such interpolants.

We discuss the pseudospectral Fourier and Chebyshev differencing methods—the two most extensively used among all of the above, see for example the survey of Gottlieb, Hussaini and Orszag [5] and the references therein. This stems from the possibility of implementing the FFT in these cases. One can efficiently travel between the "physical" and "phase" spaces, making the (global) pseudospectral calculations in these two cases almost as economical as the (local) finite difference ones. The definitive advantage of the former lies, however, in their remarkable accuracy properties, which is the topic of this paper.

As is well known, the pseudospectral differencing of (sufficiently) smooth functions, enjoys "infinite" order of accuracy. That is, measured w.r.t. the inverse number of selected points, the error committed is bounded by *any fixed polynomial* order (see for example Kreiss and Oliger [8] for the Fourier case, and a different detailed study of Canuto and Quarteroni [1], which includes, among others, the Chebyshev case).

Here we show, that if the function under consideration is further assumed to be analytic, then the asymptotic decay rate of the error with either the Fourier or Chebyshev differencing is, in fact, *exponential*. This should be compared with the *polynomial* decay rate obtained by finite difference/finite element differencing methods.

In § 2 we begin discussing the Fourier differencing of smooth functions. Following [8], we first derive the aliasing relation, which implies "infinite" order of accuracy in this case. In § 3, we show the exponential decay rate of the error, with Fourier differencing of analytic functions. The Chebyshev differencing method is likewise treated in § 4. After putting the aliasing relation in an identical form to the one obtained in the Fourier case, the various error estimates follow along the same lines.

Similar to our treatment of the stability question in [15, Part II], we emphasize here the central role played by the aliasing relations, from which we derive all the results below. Thanks to these aliasing relations, the error decay behavior is "essentially" due to the corresponding decay of either the Fourier or Chebyshev coefficients; an exponential decay of the latter is widely known in the analytic case. Also, by considering the Fourier/Chebyshev coefficients, the above derivation may still offer

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an exponential decay rate of fractional order in *nonanalytic, smooth* cases (for example, standard cut-off functions).

In closing, we would like to point out that the above results are intimately related to Bernstein's theorem, regarding the exponential convergence of best polynomial approximations. Specifically, given an analytic function, Bernstein's proof verifies the exponential convergence of its truncated Chebyshev series expansion, see for example [11, § 6]. Using the Gauss-Chebyshev rule to compute that expansion's coefficients, we are then led to the Chebyshev interpolant; the further error inferred by such discretization (which is *exactly* an aliasing error), is known to be also exponentially small, see for example [3, p. 329]. In other words, we conclude that the above Chebyshev interpolant—so-called near minimax polynomial—approximates a given analytic function within an exponentially decaying error. In fact, the results below indicate that given an analytic function, both the Fourier and Chebyshev interpolants approximate the function *and its derivatives*, within an exponential accuracy. Indeed, these results manifest themselves in the global error behaviour of pseudospectrally solved PDE's, see for example [5], [6], [13].

2. Fourier differencing of smooth functions. Let $w(x)$ be a 2π-periodic function, whose values, $w_\nu = w(x_\nu)$, are assumed known at the $2N$ equidistant grid points $x_\nu = \nu h$, $h = \pi/N$, $\nu = 0, 1, \cdots, 2N-1$. The (pseudospectral) Fourier differencing of such a function, refers to differentiation of the *trigonometric* interpolant of these grid values. One constructs the trigonometric interpolant[1]

$$\tilde{w}(x) = \tilde{w}(x; N) = \sum_{p=-N}^{N}{}'' \hat{w}_p e^{ipx}, \qquad \hat{w}_p = \frac{1}{2N} \cdot \sum_{\nu=0}^{2N-1} w_\nu e^{-ip\nu h}, \tag{2.1}$$

and use its derivative

$$\frac{d\tilde{w}}{dx}(x_\nu) = \sum_{p=-N}^{N}{}'' ip\hat{w}_p e^{ipx_\nu}$$

to approximate the "true" value, $dw/dx\ (x = x_\nu)$.

In order to examine the error we commit by such an approximation, it is convenient to work with Sobolev space W^s, defined for integral orders s,

$$W^s \equiv W_2^s = \left\{ w(x) \,\middle|\, \|w\|_{W^s}^2 = \sum_{k=0}^{s} \left\| \frac{d^{(k)}w}{dx^k} \right\|_{L^2[0,2\pi]}^2 < \infty \right\}, \tag{2.2}$$

and extended by interpolation for fractional orders. Thanks to Plancherel's formula, W^s is isometrically isomorphic to H^s. Assuming $w(x)$ admits a formal Fourier expansion

$$w(x) \sim \sum_{p=-\infty}^{\infty} \hat{w}(p)\, e^{ipx}, \qquad \hat{w}(p) = \frac{1}{2\pi} \cdot \int_0^{2\pi} w(\xi)\, e^{-ip\xi}\, d\xi; \tag{2.3a}$$

then we can equally work with H^s, s real, which consists of those functions $w(x)$, having a finite Sobolev norm of order s,

$$H^s = \left\{ w(x) \,\middle|\, \|w\|_{H^s}^2 = \sum_{p=-\infty}^{\infty} (1+|p|)^{2s} |\hat{w}(p)|^2 < \infty \right\}. \tag{2.3b}$$

The following lemma relating the Fourier coefficients of $w(x)$,

$$\hat{w}(p) = \frac{1}{2\pi} \cdot s \int_0^{2\pi} w(\xi)\, e^{-ip\xi}\, d\xi, \qquad -\infty < p < \infty, \tag{2.4}$$

[1] (Double) primed summation indicates halving first (and last) terms.

with those of its trigonometric interpolant, $\tilde{w}(x; N)$,

$$\hat{w}_p = \frac{1}{2N} \cdot \sum_{\nu=0}^{2N-1} w(x_\nu)\, e^{-ip\nu h}, \qquad -N \leqq p \leqq N, \tag{2.5}$$

is at the heart of our discussion (see for example, Kreiss and Oliger [8]).

LEMMA 2.1 (Aliasing). *Assume $w(x)$ is in H^s, $s > \frac{1}{2}$. Then the following equality holds*:

$$\hat{w}_p = \sum_{k=-\infty}^{\infty} \hat{w}(p+2kN), \qquad -N \leqq p \leqq N. \tag{2.6}$$

Verification of Lemma 2.1 consists of inserting the Fourier expansion (2.3a) evaluated at $x = x_\nu$ into (2.5), interchanging summations and obtaining (2.6).

Equipped with the aliasing lemma, we now may turn to estimate the error between $w(x)$ and its equidistant interpolant $\tilde{w}(x)$: rewriting

$$w(x) = \left[\sum_{|p| \leqq N}'' + \sum_{|p| \geqq N}'' \right] \hat{w}(p)\, e^{ipx}, \tag{2.7}$$

and, with the help of (2.6),

$$\tilde{w}(x) = \sum_{|p| \leqq N}'' \hat{w}(p)\, e^{ipx} + \sum_{|p| \leqq N}'' \left[\sum_{k \neq 0} \hat{w}(p+2kN) \right] e^{ipx}, \tag{2.8}$$

the difference $w(x) - \tilde{w}(x)$ is readily verified to equal

$$w(x) - \tilde{w}(x) = -\sum_{|p| \leqq N}'' \left[\sum_{k \neq 0} \hat{w}(p+2kN) \right] e^{ipx} + \sum_{|p| \geqq N}'' \hat{w}(p)\, e^{ipx}. \tag{2.9}$$

The first summation on the right represents *aliasing* of the higher modes with the lower ones, $|p| \leqq N$, while the second summation consists of the *truncated* higher mode, $|p| \geqq N$. A quantitative study of both terms gives us Lemma 2.2 (compare, for example, Kreiss and Oliger [9], Pasciak [12]).

LEMMA 2.2 (Error estimate). *Assume $w(x)$ is in H^s, $s > \frac{1}{2}$. Then for any real σ, $0 \leqq \sigma \leqq s$, we have*

$$\|w(x) - \tilde{w}(x; N)\|_{H^\sigma} \leqq \left(1 + 2 \cdot \sum_{k=1}^{\infty} (2k-1)^{-2s} \right)^{1/2} \cdot \|w\|_{H^s} \cdot \left(\frac{1}{N} \right)^{s-\sigma}. \tag{2.10}$$

Proof. Beginning with (2.9), then by definition

$$\begin{aligned} \|w(x) - \tilde{w}(x; N)\|_{H^\sigma}^2 &= \sum_{|p| \leqq N}'' (1+|p|)^{2\sigma} \left| \sum_{k \neq 0} \hat{w}(p+2kN) \right|^2 \\ &\quad + \sum_{|p| \geqq N}'' (1+|p|)^{2\sigma} |\hat{w}(p)|^2. \end{aligned} \tag{2.11}$$

The Cauchy-Schwarz inequality implies

$$\left| \sum_{k \neq 0} \hat{w}(p+2kN) \right|^2 \leqq \sum_{k \neq 0} (1+|p+2kN|)^{2s} \cdot |\hat{w}(p+2kN)|^2 \cdot \sum_{k \neq 0} (1+|p+2kN|)^{-2s},$$

with the second summation not exceeding a value of

$$\sum_{k \neq 0} (1+|p+2kN|)^{-2s} \leqq 2N^{-2s} \cdot \sum_{k=1}^{\infty} (2k-1)^{-2s}, \qquad |p| \leqq N.$$

Inserted into (2.11), we find that the aliasing part of the error given in the first term on the right is bounded by

$$\begin{aligned}&2N^{-2s}\cdot\sum_{k=1}^{\infty}(2k-1)^{-2s}\cdot\sum_{|p|\leqq N}N^{2\sigma}\sum_{k\neq 0}(1+|p+2kN|)^{2s}|\hat{w}(p+2kN)|^2\\&\leqq 2\cdot\sum_{k=1}^{\infty}(2k-1)^{-2s}\cdot\left(\frac{1}{N}\right)^{2(s-\sigma)}\cdot\|w\|_s^2.\end{aligned}$$

The truncation error, given in the second term on the right of (2.11), is equally found to be bounded by

$$\sum_{|p|\geqq N}'' N^{2(\sigma-s)}\cdot(1+|p|)^{2s}|\hat{w}(p)|^2\leqq\left(\frac{1}{N}\right)^{2(s-\sigma)}\cdot\|w\|_s^2.$$

Added together, the last two estimates yield (2.10).

Remark 1. Observe that requiring $w(x)$ to have more than "one-half" bounded derivative enables us to control the *aliasing part* of the error. Apart from that restriction, there is an error decay in any Sobolev norm weaker than that of $w(x)$, which is *equally* due to aliasing and truncation errors.

Remark 2. The aliasing relation (2.5) for the 0th mode, $p=0$, implies that the trapezoidal rule is highly accurate for the integration of smooth periodic functions (Davis and Rabinowitz [3]). Indeed, the error committed in this case is solely due to aliasing

$$\frac{1}{2N}\cdot\sum_{\nu=0}^{2N}{}'' w(x_\nu)-\frac{1}{2\pi}\cdot\int_0^{2\pi}w(\xi)\,d\xi=\sum_{k\neq 0}\hat{w}(2kN).$$

This allows us to replace the H^σ-norm, measuring the error on the left of (2.10), with its more applicable discrete counterpart (Gottlieb et al. [5])

$$|||w(x)-\tilde{w}(x;N)|||_{H^\sigma}^2=\sum_{k=0}^{\sigma}\frac{1}{2N}\cdot\sum_{\nu=0}^{2N}{}''\left[\frac{d^{(k)}w}{dx^k}(x_\nu)-\frac{d^{(k)}\tilde{w}}{dx^k}(x_\nu;N)\right]^2,\qquad \sigma\text{ integral}.$$

Returning to our original question, we find—choosing $\sigma=1$ in Lemma 2.1—that the error in Fourier differencing does not exceed

$$\left\|\frac{dw}{dx}(x)-\frac{d\tilde{w}}{dx}(x;N)\right\|\leqq\text{Const}\cdot\|w\|_{H^s}\cdot\left(\frac{1}{N}\right)^{s-1},\tag{2.12}$$

for arbitrary real s, $s>1$. The norm on the left refers, of course, to the $H^0=L^2$ norm of the error, with a uniform Constant $=2$ on the right. It can be replaced, in fact, by any other reasonable (possibly discrete) norm; for example, Sobolev's inequality implies for the somewhat more applicative maximum norm

$$\underset{0\leqq\nu\leqq 2N-1}{\text{Max}}\left|\frac{dw}{dx}(x_\nu)-\frac{d\tilde{w}}{dx}(x_\nu;N)\right|\leqq\text{Const}\cdot\|w\|_{H^s}\cdot\left(\frac{1}{N}\right)^{s-3/2},\qquad s>\frac{3}{2}.$$

Consider now a sufficiently smooth 2π-periodic function $w(x)$. Differencing such a function by local methods, such as finite difference or finite element methods, leads to an error bound of the type (2.12) with a finite, *fixed*[2] degree, polynomial decay. The latter is usually identified with the *accuracy order* of the differencing method. With this terminology in mind, the (global) Fourier differencing method is thus shown to

[2] That is, independent of N.

be "infinitely" order accurate. The discretization error decays faster than *any fixed* degree polynomial rate, for example, [1], [2], [4]–[7], [14], [15]. It is worth emphasizing that phrasing the error estimate (2.12) as "infinite" order of accuracy is limited on both accounts:

1. Consider a sufficiently smooth function $w(x)$ in H^s, $s \gg 1$. The error's *order of magnitude* for a given Fourier differencing of such functions may be difficult to calculate. An a priori knowledge regarding the *size* of the factors $\|w\|_{H^k}$, $k \leqq s$, is required in this case.

2. Assume $w(x)$ is a C^∞-function. One cannot detect the exact asymptotic decay *rate*, according to the error estimate (2.12). Because of its factor dependence on the power s—when s increases so does $\|w\|_{H^s}$—one may not conclude, for example, an exponential convergence *rate* simply by placing *arbitrarily* large powers s, since the optimal s depends of course (usually in an unknown manner) on N.

3. Fourier differencing of analytic functions. In this section, we show that the Fourier differencing of 2π-periodic analytic functions admits an exponentially decaying error. Furthermore, in some cases, the error's order of magnitude may be calculated as well.

To this end, assume

$$-\eta_0 < \operatorname{Im} z < \eta_0 \tag{3.1a}$$

to be the strip of analyticity where $w(z)$ admits the absolutely convergent expansion

$$w(z) = \sum_{p=-\infty}^{\infty} \hat{w}(p)\, e^{ipz}, \qquad |\operatorname{Im} z| \leqq \eta < \eta_0. \tag{3.1b}$$

Denoting

$$M(\eta) = \operatorname*{Max}_{|\operatorname{Im} z| \leqq \eta} |w(z)|, \tag{3.2}$$

we may now state

THEOREM 3.1. *Assume $w(x)$ is 2π-periodic analytic, with analyticity strip of width $2\eta_0$. Then for any η, $0 < \eta < \eta_0$, we have*

$$\left\| \frac{dw}{dx}(x) - \frac{d\tilde{w}}{dx}(x; N) \right\| \leqq 4M(\eta) \left(\frac{\operatorname{ctgh}(N\eta)}{e^{2\eta} - 1} \right)^{1/2} \cdot N e^{-N\eta}. \tag{3.3}$$

Proof. Making the change of variables, $\zeta = e^{iz}$, then $v(\zeta) = w(z = -i \log \zeta)$ admits the power series expansion

$$v(\zeta) = w(z = -i \log \zeta) = \sum_{p=-\infty}^{\infty} \hat{w}(p) \zeta^p. \tag{3.4}$$

By the periodic analyticity of $w(z)$ in the strip $|\operatorname{Im} z| < \eta_0$, $v(\zeta)$ is found to be single-valued analytic in the corresponding annulus $e^{-\eta_0} < \zeta < e^{\eta_0}$, whose Laurent expansion is given in (3.4)

$$\hat{w}(q) = \frac{1}{2\pi i} \cdot \int_{|\zeta| = r} \frac{v(\zeta)}{\zeta^{q+1}}\, d\zeta, \qquad e^{-\eta_0} < r < e^{\eta_0}. \tag{3.5}$$

To estimate the error of Fourier differencing in this case, we employ (2.11) with $\sigma = 1$, obtaining

$$\|w(x) - \tilde{w}(x; N)\|_{H^1}^2 \leqq N^2 \cdot \sum_{|p| \leqq N}'' \left| \sum_{k \neq 0} \hat{w}(p + 2kN) \right|^2 + \sum_{|p| \geqq N}'' (1 + |p|)^2 |\hat{w}(p)|^2. \tag{3.6}$$

Using (3.5), we sum the aliased amplitudes

$$\left[\sum_{k<0}+\sum_{k>0}\right]\hat{w}(p+2kN)=\frac{1}{2\pi i}\left[\int_{|\zeta|=r}\frac{v(\zeta)\,d\zeta}{\zeta^{p+1}(\zeta^{2N}-1)}+\int_{|\zeta|=r^{-1}}\frac{v(\zeta)\,d\zeta}{\zeta^{p+1}(\zeta^{-2N}-1)}\right],\qquad r=e^{\eta}>1,$$

so that the first term on the right of (3.6) does not exceed a value of

$$4N^2\frac{M^2(\eta)}{(e^{2N\eta}-1)^2}\cdot\sum_{|p|\leqq N}{}'' e^{2\eta p}\leqq 4M^2(\eta)\frac{\operatorname{ctgh}(N\eta)}{e^{2\eta}-1}N^2e^{-2N\eta}. \tag{3.7a}$$

The truncation contribution to the error in the second term on the right of (3.6), does not exceed[3]

$$4M^2(\eta)\left[\sum_{p\geqq N}{}'(1+p^2)\,e^{-2\eta p}+\sum_{p\leqq -N}{}'(1+p^2)\,e^{2\eta p}\right]\leqq 8\frac{M^2(\eta)}{e^{2\eta}-1}N^2e^{-2N\eta}. \tag{3.7b}$$

Adding the last two bounds yields (3.3).

Remark 3. According to the above derivation, the exponential decay of the overall error is due to equal size contributions of the aliasing and truncation parts, both admitting a loss of a factor of N. One can do better, however, by taking into account higher derivatives bounds

$$M_k(\eta)=e^{2k\eta}\cdot\sum_{j=0}^{k}\operatorname*{Max}_{|\zeta|=e^{\eta}}|v^{(j)}(\zeta)|.$$

Indeed, by invoking the relation

$$q\hat{w}(q)=\frac{1}{2\pi i}\cdot\int_{|\zeta|=r}\frac{dv}{d\zeta}(\zeta)\zeta^{-q}\,d\zeta,$$

the truncation contribution in (3.6) is, in fact, found to be bounded by

$$2M_1^2(\eta)\left[\sum_{p\geqq N}{}' e^{-2\eta p}+\sum_{p\leqq -N}{}' e^{2\eta p}\right]\leqq 4\frac{M_1^2(\eta)}{e^{2\eta}-1}e^{-2N\eta}.$$

Compared with the truncation estimate in (3.7b), we see that the loss of the N-factor is regained here. The aliasing error can be upper bounded similarly.

Remark 4. Estimate (3.3) shows that the error with Fourier differencing of an analytic function $w(x)$, decays exponentially w.r.t. its asymptotic dependence on N. Furthermore, equipped with a bound on $w(x)$ when moved into the complex plane, one can estimate the *size* of the error in this case, using the somewhat more aesthetic upper bound

$$\left\|\frac{dw}{dx}(x)-\frac{d\tilde{w}}{dx}(x;N)\right\|\leqq\frac{4M(\eta)}{\sinh(\eta)}Ne^{-N\eta}. \tag{3.8}$$

Remark 5. The exponential convergence follows for derivatives higher than one. With the usual loss of a factor of N for each derivative, we obtain

$$\|w(x)-\tilde{w}(x;N)\|_{H^\sigma}\leqq\text{Const}_\sigma\cdot\frac{M(\eta)}{\sinh(\eta)}N^\sigma e^{-N\eta}. \tag{3.9}$$

The preferable discrete estimates follow along the lines of an earlier remark, or alternatively, using Sobolev inequality to implement L^∞ error estimates. Moreover, the

[3] We assume N is sufficiently large, $N>(e^{2\eta}-1)^{-1}$.

loss of the polynomial factor N^σ in these cases can be regained, compensated instead by using bounds which involve $M_\sigma(\eta)$, as previously argued in Remark 3.

4. Chebyshev differencing—the nonperiodic case. In the nonperiodic case, the Chebyshev differencing is usually advocated, see for example [1], [2], [4]-[6], [10], [13], [14]. Let $w(x)$ be defined for $-1 \leqq x \leqq 1$, and assume its values $w_\nu = w(x_\nu)$ are known at the $N+1$ gridpoints $x_\nu = \cos(\nu h)$, $h = \pi/N$, $\nu = 0, 1, \cdots, N$. The (pseudo-spectral) Chebyshev differencing of such a function refers to differentiation of the *polynomial* interpolant of these gridvalues. One constructs the polynomial interpolant

$$\tilde{w}_T(x) = \tilde{w}_T(x, N) = \sum_{p=0}^{N}{}'' \hat{w}_p T_p(x), \qquad \hat{w}_p = \frac{2}{N} \cdot \sum_{\nu=0}^{N}{}'' w_\nu T_p(x_\nu), \tag{4.1}$$

in terms of Chebyshev polynomials $T_p(x) = \cos[p(\cos^{-1} x)]$, and uses its derivative

$$\frac{d\tilde{w}}{dx}(x_\nu) = \sum_{p=0}^{N}{}'' \hat{w}_p \frac{dT_p}{dx}(x = x_\nu)$$

to approximate the "true" value, $dw/dx\ (x = x_\nu)$. The latter summation can be translated into standard cosine FFT-like summation using a *single* two-step recursion formula, see [4]-[6]. Thus Chebyshev differencing admits a fast efficient implementation.

To measure the error in this case, one usually employs the appropriately weighted Chebyshev norm

$$\|w\|_T^2 = \int_{-1}^{1} \frac{w^2(x)}{(1-x^2)^{1/2}}\, dx$$

and the corresponding weighted spaces under the W_T^s norm, s integral,

$$W_T^s = \left\{ w(x) \,\middle|\, \|w\|_{W_T^s}^2 = \sum_{k=0}^{s} \left\| \frac{d^{(k)} w}{dx^k} \right\|_T^2 < \infty \right\}; \tag{4.2}$$

Chebyshev spaces W_T^s of fractional order s are suitably interpreted by interpolation.

We have found it more convenient, however, to work below within the spaces H_T^s, s real. Assuming $w(x)$ admits a formal Chebyshev expansion

$$w(x) \sim \sum_{p=0}^{\infty}{}' \hat{w}(p) T_p(x), \qquad \hat{w}(p) = \frac{2}{\pi} \cdot \int_{-1}^{1} \frac{w(\xi) T_p(\xi)}{(1-\xi^2)^{1/2}}\, d\xi, \tag{4.3}$$

then, in complete analogy with (2.3b), we introduce

$$H_T^s = \left\{ w(x) \,\middle|\, \|w\|_{H_T^s}^2 = \sum_{p=0}^{\infty} (1+p)^{2s} |\hat{w}(p)|^2 < \infty \right\}. \tag{4.4}$$

Unlike the Fourier case (endowed with the usual Euclidean weighting), W_T^s and H_T^s are not equivalent unless $s = 0$, in which case they are in fact isometrically isomorphic by the Chebyshev transform

$$\|w\|_{H_T^0}^2 = \frac{2}{\pi} \|w\|_{W_T^0}^2. \tag{4.5}$$

Making use of the inverse inequalities of Canuto and Quarteroni [1], will enable us, later on, to recover the H_T^s-estimates derived below, within the more standard W_T^s-spaces. We begin with the aliasing relation, which in this case reads (see for example Gottlieb [4], Reyna [14]).

LEMMA 4.1 (Aliasing). *Assume $w(x)$ is in H_T^s, $s>\frac{1}{2}$. Then the following equality holds*

$$\hat{w}_p=\hat{w}(p)+\sum_{k=1}^{\infty}[\hat{w}(-p+2kN)+\hat{w}(p+2kN)], \qquad 0\leqq p\leqq N. \tag{4.6}$$

Verification of Lemma 4.1 consists of inserting the Chebyshev expansion (4.3) evaluated at $x=x_\nu$ into (4.1), yielding

$$\hat{w}_p=\frac{2}{N}\sum_{\nu=0}^{N}{}''\left[\sum_{q=0}^{\infty}{}'\hat{w}(q)T_q(x_\nu)\right]T_p(x_\nu)=\frac{2}{N}\sum_{q=0}^{\infty}{}'\hat{w}(q)\left[\sum_{\nu=0}^{N}{}''T_q(x_\nu)T_p(x_\nu)\right];$$

to calculate the inner summation we employ the identity $2T_q(x)T_q(x)=T_{p+q}(x)+T_{|p-q|}(x)$, ending up with

$$\hat{w}_p=\sum_{q=0}^{\infty}{}'\hat{w}(q)\left[\delta_{qp}+\delta_{q0}\cdot\delta_{p0}+\sum_{k=1}^{\infty}\delta_{q,2kN\pm p}\right]$$

and (4.6) follows.

Let us define $T_{-p}(x)=T_p(x)$ so that $\hat{w}(-p)=\hat{w}(p)$. The Chebyshev expansion (4.3) takes now the Fourier-like symmetric form

$$w(x)\sim\tfrac{1}{2}\cdot\sum_{p=-\infty}^{\infty}\hat{w}(p)T_p(x) \tag{4.7}$$

with an aliasing formula *identical* to the one we had before in Lemma 2.1

$$\hat{w}_p=\sum_{k=-\infty}^{\infty}\hat{w}(p+2kN). \tag{4.8}$$

Hence, we can equally conclude the corresponding error estimate, which we quote from Lemma 2.2.

LEMMA 4.2 (Error estimate). *Assume $w(x)$ is in H_T^s, $s>\frac{1}{2}$. Then for any real σ, $0\leqq\sigma\leqq s$, we have*

$$\|w(x)-\tilde{w}_T(x;N)\|_{H_T^\sigma}\leqq 2\left(1+2\cdot\sum_{k=1}^{\infty}(2k-1)^{-2s}\right)^{1/2}\cdot\|w\|_{H_T^s}\cdot\left(\frac{1}{N}\right)^{s-\sigma}. \tag{4.9}$$

Setting $\sigma=0$ in (4.9) gives us, in view of (4.5)

$$\|w(x)-\tilde{w}_T(x;N)\|_{W_T^0}\leqq\left[2\pi\left(1+2\cdot\sum_{k=1}^{\infty}(2k-1)^{-2s}\right)\right]^{1/2}\cdot\|w\|_{H_T^s}\cdot\left(\frac{1}{N}\right)^{s}. \tag{4.10}$$

Using the inverse inequality [1, Lemma 2.1], one can "raise" the Sobolev norm on the left of (4.10), obtaining (for details see Canuto and Quarteroni [1, Thm. 3.1], Maday and Quarteroni [10]).

COROLLARY 4.3 (Error estimate). *Assume $w(x)$ is in W_T^s, $s>\frac{1}{2}$. Then for any real σ, $0\leqq 2\sigma\leqq s$, we have*

$$\|w(x)-\tilde{w}_T(x;N)\|_{W_T^\sigma}\leqq \mathrm{Const}_s\cdot\|w\|_{W_T^s}\cdot\left(\frac{1}{N}\right)^{s-2\sigma}. \tag{4.11}$$

Thus, each derivative infers a loss of N^2 factor in this case, rather than the usual factor N associated with the Fourier differencing.

Remark 6. According to Y. Maday (private communication), the factor dependence on the right of (4.11) is factorial, $\mathrm{Const}_s\sim s!$.

We turn now to consider the case where $w(x)$ is analytic in the interval $[-1, 1]$. To this end, we employ Bernstein's regularity ellipse, E_r, with foci ± 1 and with sum of its semiaxis equals r, see for example [11, § 6]. Denoting

$$M^T(\eta) = \operatorname*{Max}_{z \in E_r} |w(z)|, \qquad r = e^\eta, \tag{4.12}$$

we may now state

THEOREM 4.4. *Assume* $w(x)$ *is analytic in* $[-1, 1]$, *having a regularity ellipse whose sum of its semiaxis equals* $r_0 = e^{\eta_0} > 1$. *Then for any* $\eta, 0 < \eta < \eta_0$, *we have*

$$\|w(x) - \tilde{w}_T(x; N)\|_{H_T^1} \leqq 8 M^T(\eta) \left(\frac{\operatorname{ctgh}(N\eta)}{e^{2\eta} - 1} \right)^{1/2} \cdot N e^{-N\eta}. \tag{4.13}$$

Proof. The transformation, $(\zeta + \zeta^{-1})/2 = z$, takes the regularity ellipse E_{r_0} in the z-plane, into the annulus $r_0^{-1} < |\zeta| < r_0$ in the ζ-plane. Hence, $v(\zeta) = 2w(z = (\zeta + \zeta^{-1})/2)$ admits the power series expansion

$$v(\zeta) = 2w\left(\frac{\zeta + \zeta^{-1}}{2} \right) = \sum_{p=-\infty}^{\infty} \hat{w}(p) \zeta^p, \qquad r_0^{-1} < |\zeta| < r_0 = e^{\eta_0}. \tag{4.14}$$

Indeed, upon setting $\zeta = e^{i\theta}$ and recalling that $\hat{w}(-p) = \hat{w}(p)$, the above expansion clearly describes the real interval $[-1, 1]$,

$$w(z = \cos \theta) = \sum_{p=0}^{\infty}{}' \hat{w}(p) \cos(p\theta).$$

For the Laurent expansion given in (4.14), we then find

$$\hat{w}(q) = \frac{1}{2\pi i} \cdot \int_{|\zeta| = r} \frac{v(\zeta)}{\zeta^{q+1}} d\zeta, \qquad e^{-\eta_0} < r < e^{\eta_0}. \tag{4.15}$$

Comparing (4.15) and (3.5), we end up with the same Cauchy integral formulae for the amplitudes in both the Fourier and Chebyshev expansions; coupled with the identical aliasing relations, (4.13) follows along the lines of Theorem 3.1.

Remark 7. As before, the factor $(\operatorname{ctgh}(N\eta)/e^{2\eta} - 1)^{1/2}$ on the right of (4.13), can be replaced by the more aesthetic bound of $1/\sinh(\eta)$, yielding

$$\|w(x) - \tilde{w}_T(x; N)\|_{H_T^1} \leqq 8 \frac{M^T(\eta)}{\sinh(\eta)} N e^{-N\eta}. \tag{4.16}$$

Next, an exponential error estimate in terms of the Sobolev norm W_T^1 can be derived. With the loss of an additional factor of N in the spirit of an earlier remark, we then find

COROLLARY 4.5. *Assume* $w(x)$ *is analytic in* $[-1, 1]$. *Then we have*

$$\|w(x) - \tilde{w}_T(x; N)\|_{W_T^\sigma} \leqq \text{Const}_\sigma \cdot \frac{M^T(\eta)}{\sinh(\eta)} N^{2\sigma} e^{-N\eta}, \qquad 0 < \eta < \eta_0. \tag{4.17}$$

Making use of the Sobolev inequality, for example, [10], implies in particular a discrete maximum estimate of the form

COROLLARY 4.6. *Assume* $w(x)$ *is analytic in* $[-1, 1]$. *Then we have*

$$\operatorname*{Max}_{0 \leqq \nu \leqq N} \left| \frac{dw}{dx}(x_\nu) - \frac{d\tilde{w}}{dx} T(x_\nu; N) \right| \leqq \text{Const} \cdot \frac{M^T(\eta)}{\sinh(\eta)} N^{5/2} e^{-N\eta}, \qquad 0 < \eta < \eta_0. \tag{4.18}$$

We conclude by noting that the growth of the polynomial factors in the last error bounds can be decreased, increasing the derivative bound $M^T(\eta)$, accordingly.

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