

Eigenvalues and Eigenfunctions of Double Layer Potentials

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Abstract

Eigenvalues and eigenfunctions of two- and three-dimensional double layer potentials are considered. Let Ω be a C^2 bounded region in \mathbf{R}^n ($n = 2, 3$). The double layer potential $K : L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$ is defined by

$$(K\psi)(x) \equiv \int_{\partial\Omega} \psi(y) \cdot \nu_y E(x, y) ds_y,$$

where

$$E(x, y) = \begin{cases} \frac{1}{\pi} \log \frac{1}{|x-y|} & \text{if } n = 2, \\ \frac{1}{2\pi} \frac{1}{|x-y|} & \text{if } n = 3, \end{cases}$$

ds_y is the line or surface element and ν_y means the outer normal derivative on $\partial\Omega$. It is known that K is a compact operator on $L^2(\partial\Omega)$ and consists of at most a countable number of eigenvalues, with 0 the only possible limit point. The aim of this paper is to establish some relationships between eigenvalues, eigenfunctions and the geometry of $\partial\Omega$.

1 Introduction and Results

Let Ω be a C^2 bounded region in \mathbf{R}^n ($n = 2, 3$). Consider the double layer potential $K : L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$:

$$(K\psi)(x) \equiv \int_{\partial\Omega} \psi(y) \cdot \nu_y E(x, y) ds_y,$$

where

$$E(x, y) = \begin{cases} \frac{1}{\pi} \log \frac{1}{|x-y|} & \text{if } n = 2, \\ \frac{1}{2\pi} \frac{1}{|x-y|} & \text{if } n = 3, \end{cases}$$

ds_y is the line or surface element and ν_y means the outer normal derivative on $\partial\Omega$. We know that K is a compact operator on $L^2(\partial\Omega)$ and consists of at most a countable number of eigenvalues, with 0 the only possible limit point. It is also known that the eigenvalues of the double layer potential integral operator lie in the interval $[-1, 1)$ and the eigenvalue -1 corresponds to constant eigenfunctions (See [P1] and see also [Ta] for some recent progress).

We set the ordered eigenvalues and eigenfunctions counting multiplicities by

$$\sigma_p(K) = \{ \lambda_j \mid |\lambda_0| > |\lambda_1| \geq |\lambda_2| \geq \cdots \},$$

where

$$Ke_{\lambda_j}(x) = \lambda_j e_{\lambda_j}(x).$$

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Recall also that every compact operator K on Hilbert space takes the following canonical form

$$K\psi = \sum_{j=1}^{\infty} \alpha_j \langle \psi, v_j \rangle u_j$$

for some orthonormal basis $\{u_j\}$ and $\{v_j\}$, where α_j are singular values of K (i.e. the eigenvalues of $(K^*K)^{1/2}$) and $\langle \cdot, \cdot \rangle$ means the $L^2(\partial\Omega)$ inner product. The singular values are non-negative and we denote the ordered singular values by

$$\sigma_{sing}(K) = \{ \alpha_j \mid \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \dots \}.$$

With this in mind, our main concerns are two natural questions:

- (i) What can we say about the geometry of $\partial\Omega$ given the eigen or singular values?
- (ii) What can we say about the eigenvalues, singular values and eigenfunctions given the geometry?

What we will attempt to prove in this paper are some selected aspects of those two questions: isoperimetric eigen and singular value problems, decay rates of eigen and singular values, and nodal sets of eigenfunctions. Note that these questions are taken from the questions of the spectral geometry for elliptic operators. As will be mentioned in the last of this section, there are many studies in this direction. In other words our aim is to develop the spectral geometry of double layer potentials.

For this purpose, in §2 we start studying two questions for two dimensional double layer potentials:

- (Q1) What types of eigen and singular values give the isoperimetric property of $\partial\Omega$?
- (Q2) What types of sequences can occur as eigen and singular values?

An answer of (Q1) is given by:

Theorem 2.7. Let $n = 2$ and Ω be a simply connected region with C^2 boundary. Then

$$\sigma_{sing}(K) \setminus \{0\} = \{1\} \text{ is necessary and sufficient for } \partial\Omega = S^1.$$

It follows that $(K\psi)(x) = 0$ for all $\psi(x) \in L_0^2(\partial\Omega)$, then $\partial\Omega = S^1$ (See Corollary 2.8). There are some proofs of this theorem (See e.g. [Li]). In §2.1, a short alternative proof is given by using Hilbert-Schmidt norm of K . For $\partial\Omega = S^1$, K has an eigenvalue -1 of multiplicity 1 and an eigenvalue 0 of infinite countable multiplicities (See [Ah1] and §2.1). So the condition $\sigma_{sing}(K) \setminus \{0\} = \{1\}$ can be replaced by $\sigma_{sing}(K) = \{1, 0\}$. Moreover the theory of quasi-conformal mapping states that $\sigma_p(K) = \{-1, 0\}$ is also necessary and sufficient for $\partial\Omega = S^1$. We mention about this result in the last of §2.1.

To answer (Q2), we consider Schatten norm of K and estimate a decay rate ^[1] of eigen and singular values by the regularity of $\partial\Omega$:

Theorem 2.12. Let $n = 2$ and Ω be a C^k ($k \geq 2$) region. For any $\alpha > -2k + 3$,

$$\alpha_j = o(j^{\alpha/2}) \text{ and } \lambda_j = o(j^{\alpha/2}) \text{ as } j \rightarrow \infty$$

where o means the small order.

It follows that if Ω be a C^∞ region, we have $\lambda_j = o(j^{-\infty})$. For an ellipse $\partial\Omega$, for instance, direct calculations give $\lambda_j = O(e^{-cj})$ (See [Ah1], [KPS, §8.3] and example 2.2). It should be emphasized that the ellipses are analytic curves and so the eigenvalues are presumed to have the stronger decay properties than the case of smooth curves. These viewpoints shed some new lights on eigenvalue asymptotics.

Few studies have focused on the eigenfunctions. In §3, we show a question for two dimensional double layer potentials:

- (Q3) What can we say about the nodal sets of eigenfunctions?

Here we establish the holomorphic extension of e_{λ_j} for analytic curves and give the growth of zeroes of analytic eigenfunctions:

Theorem 3.10. Let $n = 2$ and Ω be a real analytic region and $\{e_{\lambda_j}(x)\} \subset C^\omega(\partial\Omega)$ be real analytic eigenfunctions. There exists $C > 0$, depending only on Ω , such that the zeroes $N(e_{\lambda_j}(x))$ satisfy

$$\#N(e_{\lambda_j}(x)) \leq C |\log |\lambda_j||.$$

^[1]When considering the asymptotics of eigenvalues and eigenfunctions, we henceafter assume that the eigenvalues counting multiplicities are infinite.

From Theorem 3.10 one can expect that the positive eigenfunctions correspond to eigenvalue -1 , so the positive eigenfunction is constant. This fact holds true even for a much more general case (See Theorem 3.1).

Apart from $n = 2$, for the case of $n = 3$, the analogy of the above theorems is difficult to handle. So we shall discuss only some remarks and conjectures in §4. The behavior of $\sigma_p(K)$, for instance, changes to be drastic:

Remark 4.1. Let $n = 3$ and Ω be a smooth region. For $\alpha > -\frac{1}{2}$, we have

$$\lambda_j = o(j^\alpha) \quad \text{as } j \rightarrow \infty.$$

For $n = 3$, no satisfactory answer of isoperimetric eigenvalue problems has been found yet. Instead, in the context of studying the eigenvalue problems we propose reasonable conjectures:

Conjecture 1. Let $n = 3$ and $\underline{\lambda} \equiv \min \sigma_p(K) \setminus \{-1\}$. We have

$$\sup_{\partial\Omega} \underline{\lambda} = -\frac{1}{3}$$

where the supremum is taken over all C^∞ simply connected closed surfaces. The supremum is achieved if and only if $\partial\Omega = S^2$.

Conjecture 2. Let $n = 3$. For $p > 1$, we have

$$\inf_{\partial\Omega} \text{tr}\{(K^*K)^p\} = \left(1 - \frac{1}{2^{2p-1}}\right) \zeta(2p-1)$$

where the infimum is taken over all C^∞ simply connected closed surfaces and $\zeta(s)$ denotes the Riemann zeta function. The infimum is achieved if and only if $\partial\Omega = S^2$.

We confirm the validity of these conjectures. When C^∞ closed surfaces are replaced by ellipsoids, these conjectures will be proved (See Theorem 4.3).

We end the introduction by comparing with the above results and the spectral geometry of Laplacian on manifolds. In the case of Laplacian, the isoperimetric properties of manifolds are characterized by the first eigenvalue or second eigenvalue or eigenvalue asymptotics, etc. (See e.g. [Be] and references therein). Theorem 2.7, Conjecture 1 and Conjecture 2 correspond to these results. Theorem 2.12 can be viewed as eigenvalue asymptotics called Weyl's law. For Laplacian, Weyl's law includes the information about the dimension and volume of manifolds, and etc. (See e.g. [CH] and [ANPS] and references therein). Theorems about zeroes of Laplace eigenfunctions are known as Courant's nodal line theorem and Donnelly-Fefferman's results and etc. (See e.g. [CH], [DF] and [Ze]). Roughly speaking, they estimate the Hausdorff dimension and measure of nodal sets by the eigenvalues. Indeed we prove Theorem 3.10 by using the modified Donnelly-Fefferman value distribution theory.

2 Eigenvalues and singular values of two dimensional double layer potentials

In §2 we shall restrict ourselves to two dimensional double layer potentials. Such a situation allows us to treat Hilbert-Schmidt norm and Schatten norm of K . Using these norms, we obtain isoperimetric properties of singular values in §2.1 and decay estimates in §2.2.

2.1 The trace of K^*K and its application to isoperimetric problems

We consider the boundary integral equation:

$$(K\psi)(x) \equiv \frac{1}{\pi} \int_{\partial\Omega} \psi(y) \cdot \nu_y \log \frac{1}{|x-y|} ds_y, \quad (1)$$

where Ω is a C^2 bounded region in \mathbf{R}^2 and ν_y means the outer normal derivative on $\partial\Omega$. $\partial\Omega \in C^2 \subset C^{1,\alpha}$ is a Lyapunov curve and K as well as K^* are compact operators on $L^2(\partial\Omega)$. Moreover the spectra in $L^2(\partial\Omega)$ and

in $C^0(\partial\Omega)$ are identical (e.g. [Mik, Theorem 7.3.2]). A standard result in two-dimensional potential theory (See [Tr, p.78-80] and see also Lemma 2.12) states that for closed C^2 curves $\partial\Omega$

$$\lim_{\substack{x \rightarrow y \\ x \in \partial\Omega}} \nu_y \log \frac{1}{|x-y|} = -\frac{1}{2}\kappa(y),$$

where $\kappa(y)$ denotes the curvature of $\partial\Omega$. Consequently, unlike the singular nature of the double layer potentials in \mathbf{R}^3 , the double layer kernel in \mathbf{R}^2 is continuous for all points x and y on $\partial\Omega$, including when $x = y$. It is also known the eigenvalues of the integral operator K , defined in equation (1), lie in the interval $[-1, 1)$ and are symmetric with respect to the origin (e.g. [BM], [Sh]). The only exception is the eigenvalue -1 corresponding to constant eigenfunctions. Summarizing these results, we have the ‘‘formal’’ trivial trace formula ^[2] for K :

$$\text{tr}(K) \equiv \sum_{\substack{\lambda_i: \text{eigenvalue of } K \\ |\lambda_0| > |\lambda_1| = |\lambda_2| \geq |\lambda_3| = |\lambda_4| \geq \dots}} \lambda_i = \int_{\partial\Omega} -\frac{1}{2\pi}\kappa(y)ds_y = -1.$$

Here K is not selfadjoint or even normal, but K^*K is a selfadjoint trace class operators. Thus the trace of K^*K :

$$\text{tr}(K^*K)$$

is also considered. Consequently we obtain some asymptotic properties of the singular values of K . In §2, we start out by rapidly going over basic examples of $\text{tr}(K^*K)$.

Example 2.1 (The circle (See [Ah1])). Let $\partial\Omega$ be a circle of radius R . We find

$$\begin{aligned} \sigma_p(K) &= \{-1, 0\}, \\ \text{tr}(K^*K) &= 1, \end{aligned}$$

where $\sigma_p(K)$ means the set of eigenvalues of K .

In the case of ellipse, we have $\text{tr}(K^*K) > 1$.

Example 2.2 (The ellipse (See [Ah1], [KPS, §8.3])). For $R > 0$ and $c > 0$, we define the ellipse by $\partial\Omega = \{(x, y) \mid x = \frac{1}{2}c \cosh R \cos \theta, y = \frac{1}{2}c \sinh R \sin \theta\}$. Then

$$\begin{aligned} \sigma_p(K) &= \{-1, \pm e^{-2mR} \mid m \in \mathbf{N}\}, \\ \text{tr}(K^*K) &> 1. \end{aligned}$$

Seeing this, we want to characterize the region of which $\text{tr}(K^*K) = 1$.

Lemma 2.3. K^*K is a trace class operator on $L^2(\partial\Omega)$, i.e., K is a Hilbert-Schmidt class operator and

$$\text{tr}(K^*K) = \frac{1}{\pi^2} \int_{\partial\Omega_y} \int_{\partial\Omega_x} |\nu_y \log |x-y||^2 ds_x ds_y.$$

Proof.

$$(KK^*\psi)(x) = \frac{1}{\pi^2} \int_{\partial\Omega_z} \int_{\partial\Omega_y} (\nu_z \log |x-z|)(\nu_z \log |y-z|) \psi(y) ds_y ds_z.$$

The kernel $K(x, y)$ of KK^* is continuous symmetric non-negative definite on $\partial\Omega_x \times \partial\Omega_y$. By Mercer’s theorem (See e.g. [CH, p.138], [FM, Theorem 1.1] and [Kö]), there is an orthonormal set $\{u_i\}_i$ of $L^2(\partial\Omega)$ consisting of eigenfunctions of KK^* such that corresponding eigenvalues $\{\mu_i\}_i$ are nonnegative. The eigenfunctions corresponding to non-zero eigenvalues are continuous on $\partial\Omega$ and $K(x, y)$ has the representation

$$K(x, y) = \sum_{i=1}^{\infty} \mu_i u_i(x) u_i(y),$$

where the convergence is absolute and uniform. This leads to

$$\text{tr}(K^*K) = \text{tr}(KK^*) = \frac{1}{\pi^2} \int_{\partial\Omega_y} \int_{\partial\Omega_x} |\nu_y \log |x-y||^2 ds_x ds_y.$$

□

^[2] K is not always the usual trace class operator. The above ‘‘formal’’ trace formula is defined by only a conditional summation. Note that if $\partial\Omega$ is C^3 curve, K is the usual trace class operator (See the proof of Theorem 2.12 and Remark 2.16).

Recall that every compact operator K on Hilbert space takes the following canonical form

$$K\psi = \sum_{j=1}^{\infty} \alpha_j \langle \psi, v_j \rangle u_j$$

for some orthonormal basis $\{u_j\}$ and $\{v_j\}$, where α_i are singular values of K (i.e. the eigenvalues of $(K^*K)^{1/2}$) and $\langle \cdot, \cdot \rangle$ means the L^2 inner product. Also, the usual operator norm is $\|K\| = \sup_j(\alpha_j)$. $\text{tr}(K^*K) = \sum_{j=1}^{\infty} |\alpha_j|^2$ and by using Weyl's inequality (See. e.g. [Si], [Te]) :

$$\sum_{j=1}^{\infty} |\alpha_j|^2 \geq \sum_{\lambda_j \in \sigma_p(K)} |\lambda_j|^2,$$

we obtain:

Lemma 2.4.

$$\text{tr}(K^*K) \geq \|K\|^2, \quad \text{tr}(K^*K) \geq \sum_{\lambda_j \in \sigma_p(K)} |\lambda_j|^2 \geq 1.$$

Remark 2.5. From Lemma 2.4 the ordered eigenvalues satisfy

$$\sum_{\lambda_j \in \sigma_p(K)} |\lambda_j|^2 < \infty.$$

So for all $\epsilon > 0$ there exists N such that

$$(n - N)|\lambda_n|^2 \leq \sum_{N+1}^n |\lambda_j|^2 < \epsilon$$

and hence

$$n|\lambda_n|^2 < 2\epsilon \quad \text{for all } n > 2N.$$

Accordingly $\lambda_j = o(j^{-1/2})$. This is not the best possible estimate (See example 2.1, 2.2 and see also §2.2).

In the following of this subsection, we apply the trace for the analysis of singular values. The minimizer of $\text{tr}(K^*K)$ is attained by $\partial\Omega = S^1$.

Theorem 2.6. Let Ω be a simply connected region with C^2 boundary.

$$\text{tr}(K^*K) = 1 \text{ is necessary and sufficient for } \partial\Omega = S^1.$$

Proof. We note that $\int_{\partial\Omega_x} \nu_y \log|x-y| ds_y = \pi$. Letting $C \equiv \pi \cdot (\text{length of } \partial\Omega)^{-1}$,

$$\begin{aligned} \text{tr}(K^*K) &= \frac{1}{\pi^2} \int_{\partial\Omega_y} \int_{\partial\Omega_x} |\nu_y \log|x-y||^2 ds_x ds_y \\ &= \frac{1}{\pi^2} \int_{\partial\Omega_y} \int_{\partial\Omega_x} |\nu_y \log|x-y| - C|^2 ds_x ds_y + \frac{2C}{\pi^2} \int_{\partial\Omega_y} \int_{\partial\Omega_x} \nu_y \log|x-y| ds_x ds_y - \frac{C^2}{\pi^2} \int_{\partial\Omega_y} \int_{\partial\Omega_x} ds_x ds_y \\ &= \frac{1}{\pi^2} \int_{\partial\Omega_y} \int_{\partial\Omega_x} |\nu_y \log|x-y| - C|^2 ds_x ds_y + 1. \end{aligned}$$

It follows that $\text{tr}(K^*K) = 1 \Rightarrow \nu_y \log|x-y| = C$ for all $(x, y) \in \partial\Omega_x \times \partial\Omega_y$.

By the continuity of $\nu_y \log|x-y|$,

$$\frac{1}{2}\kappa(x) = C \quad (\text{constant})$$

as desired. □

Suppose $\text{tr}(K^*K) = 1$. Then only one singular value takes 1, otherwise $\alpha_j = 0$. Thus we obtain:

Theorem 2.7. Let Ω be a simply connected region with C^2 boundary.

$$\sigma_{sing}(K) \setminus \{0\} = \{1\} \text{ is necessary and sufficient for } \partial\Omega = S^1$$

where $\sigma_{sing}(K)$ denotes the set of singular values.

From the canonical form of K , the ball symmetry property of double layer potentials is also obtained:

Corollary 2.8. (See [Li, Theorem 1.3]) Let Ω be a simply connected region with C^2 boundary and $L_0^2(\partial\Omega) = \{\psi \in L^2(\partial\Omega) \mid \int_{\partial\Omega} \psi \, ds = 0\}$. If $(K\psi)(x) = 0$ for all $\psi(x) \in L_0^2(\partial\Omega)$, then $\partial\Omega = S^1$.

M. Lim proved that if K is self-adjoint, then $\partial\Omega$ is circle. Lim's work is essentially based on the "moving hyperplane" method of Alexandroff and Serrin, but we are not aware of any studies in this direction (See [Re], [Se]). It is also known [S] that the disk is the only planar domain for which K has finite rank.

Remark 2.9. For higher dimensions, K^*K is not always a trace class operator (See §4).

In the following we introduce well-known classical results on $\sigma_p(K)$ (See e.g. [Scho1]): Even when $\sigma_{sing}(K)$ is replaced by $\sigma_p(K)$, Theorem 2.7 holds true. The points $\lambda \in \sigma_p(K) \setminus \{-1\}$ are known as the Fredholm eigenvalues of $\partial\Omega$. The largest eigenvalue $\bar{\lambda}$ is often of interest. By the symmetry of eigenvalues, we have $\bar{\lambda} = -\lambda \equiv -\inf \sigma_p(K) \setminus \{-1\}$.

Let $\partial\Omega$ be on the Riemann sphere $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. Then $\partial\Omega$ divides $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ into complementary simply connected domains Ω and $\bar{\Omega}^c$. Let \mathcal{H} be the family of all functions u continuous in $\hat{\mathbf{C}}$ and harmonic in $\Omega \cup \bar{\Omega}^c$, with $0 < \mathcal{D}_\Omega(u) + \mathcal{D}_{\bar{\Omega}^c}(u) < \infty$. Here $\mathcal{D}_A(u)$ denotes the Dirichlet integral on A :

$$\mathcal{D}_A(u) = \int \int_A u_x^2 + u_y^2 \, dx dy.$$

Ahlfors [Ahl] showed the relationships between the Fredholm eigenvalues, the Dirichlet integral and quasiconformal mappings. Especially the value $\bar{\lambda}$ can be represented in terms of the Dirichlet integral:

$$\bar{\lambda} = \sup_{u \in \mathcal{H}} \frac{|\mathcal{D}_\Omega(u) - \mathcal{D}_{\bar{\Omega}^c}(u)|}{\mathcal{D}_\Omega(u) + \mathcal{D}_{\bar{\Omega}^c}(u)}.$$

Since conformal mappings preserve harmonic functions and Dirichlet integrals, $\bar{\lambda}$ is invariant under linear fractional transformations. Let $f : \Omega \cup \bar{\Omega}^c \rightarrow \Omega \cup \bar{\Omega}^c$ of $\hat{\mathbf{C}}$ be a orientation preserving homeomorphism whose distributional partial derivatives are in L_{loc}^2 . If f preserves the curve $\partial\Omega$, the reflection coefficient of f is defined by

$$q_{\partial\Omega} = \inf \|\partial_{\bar{z}} f / \partial_z f\|_\infty$$

where the infimum is taken over all quasireflections across $\partial\Omega$ provided these exist and is attained by some quasireflection f_0 . The number M satisfying

$$q_{\partial\Omega} = \frac{M+1}{M-1}$$

is called the quasiconformal constant. The M -quasiconformal mapping is an orientation-preserving diffeomorphism whose derivative maps infinitesimal circles to infinitesimal ellipses with eccentricity at most M . A basic ingredient for estimating $\bar{\lambda}$ is known as Ahlfors inequality [Ahl]:

$$\bar{\lambda} \geq \frac{1}{q_{\partial\Omega}}.$$

If $\bar{\lambda} = 0$, then $q_{\partial\Omega} = \infty$ and $M = 1$. So f_0 is 1-conformal, hence conformal. The conformal mapping of $\Omega \cup \bar{\Omega}^c$ onto $\Omega \cup \bar{\Omega}^c$ can be extended to a 1-conformal mapping of $\hat{\mathbf{C}}$ onto $\hat{\mathbf{C}}$. The only such mappings are linear fractional transformations, and so, since $\partial\Omega$ is mapped onto $\partial\Omega$, it must itself be S^1 . Thus $q_{\partial\Omega} = \infty$, $\bar{\lambda} = 0$ and $\sigma_p(K) \setminus \{-1\} = \{0\}$ only for the circle.

Remark 2.10. Many authors study in this direction. We mention only some fascinating results.

1. If $\partial\Omega$ is convex, then $\bar{\lambda} \geq \{1 - (|\partial\Omega|/2\pi R)\}^{-1}$ where R is the supremum of radii of all circles which intersect $\partial\Omega$ at least 3 points. (In case $\partial\Omega$ is smooth, R is the maximum radius of curvature). This is due to C. Neumann (e.g. [Scho2], [Wa]).

2. Recently Krushkal proved the celebrated inequality (See [Kr1, p.358] and reference in [Kr2]):

$$\frac{3}{2\sqrt{2}} \frac{1}{q_{\partial\Omega}} \geq \bar{\lambda} \geq \frac{1}{q_{\partial\Omega}}.$$

3. For higher dimensions, Fredholm eigenvalues are also characterized by Dirichlet integrals (e.g. [S], [KPS]).

Remark 2.11. Taking the limit $R \rightarrow \infty$ in example 2.2, we have $\sup_{\partial\Omega} \bar{\lambda} = 1$ where the supremum is taken over all C^∞ domain Ω .

2.2 Asymptotic properties of $\sigma_p(K)$

In the preceding subsection, we considered Hilbert-Schmidt norm of K . More generally K is in Schatten classes of $r > \frac{2}{2k-3}$ for C^k ($k \geq 2$) closed curve $\partial\Omega$. (For details on the notion of the Schatten classes, see e.g. [Mc]).

Let λ_n be eigenvalues of K satisfying

$$|\lambda_0| > |\lambda_1| = |\lambda_2| \geq |\lambda_3| = |\lambda_4| \geq \dots$$

In the case of ellipse, we find

$$\lambda_j = O(e^{-cj})$$

where $\lambda_j = O(e^{-cj})$ means that there exists a constant $C > 0$ such that $\lambda_j \leq Ce^{-cj}$ for large $j \in \mathbf{N}$. For general C^k closed curves $\partial\Omega$, we obtain:

Theorem 2.12. Let $n = 2$ and Ω be a C^k ($k \geq 2$) bounded region. For any $\alpha > -2k + 3$,

$$\alpha_j = o(j^{\alpha/2}) \text{ and } \lambda_j = o(j^{\alpha/2}) \text{ as } j \rightarrow \infty.$$

Thus the boundary regularity is essential to the decay rate of eigenvalues. To prove Theorem 2.12, we first prepare a fundamental lemma. For the sake of the readers' convenience, we also give the proof to the following.

Lemma 2.13. If $k \geq 2$, then $E \in C^{k-2}(\partial\Omega \times \partial\Omega)$. Especially we have

$$\lim_{\substack{x \rightarrow y \\ x \in \partial\Omega}} \nu_y \log \frac{1}{|x - y|} = -\frac{1}{2} \kappa(y),$$

where $\kappa(y)$ denotes the curvature of $\partial\Omega$.

Proof. For every point P on $\partial\Omega$ there exists a small neighborhood $B_\epsilon(P)$ such that the part of $B_\epsilon(P) \cap \partial\Omega$ for some orientation of the axes of coordinate system (ξ, η) , admits a representation (See Fig.1)

$$\partial\Omega \cap B_\epsilon(P) = \{(\xi, \eta) \mid \eta = F(\xi), |\xi| < \epsilon'\}$$

where $F(\xi) \in C^k$.

For $x = (\xi_1, \eta_1)$ and $y = (\xi_2, \eta_2)$, ν_y and $\log|x - y|$ is given by

$$\nu_y = \left(\frac{F'(\xi_2)}{\{1 + (F'(\xi_2))^2\}^{1/2}} \frac{\partial}{\partial \xi_2}, \frac{-1}{\{1 + (F'(\xi_2))^2\}^{1/2}} \frac{\partial}{\partial \eta_2} \right),$$

$$\log|x - y| = \frac{1}{2} \log\{(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2\}.$$

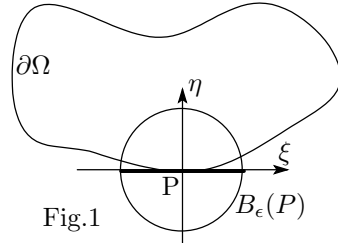


Fig.1

Now

$$\begin{aligned} \nu_y \log|x - y| &= \frac{(\xi_2 - \xi_1)F'(\xi_2) - (\eta_2 - \eta_1)}{\{(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2\}^{1/2} \{1 + (F'(\xi_2))^2\}^{1/2}} \\ &= \frac{(\xi_2 - \xi_1)F'(\xi_2) - (F(\xi_2) - F(\xi_1))}{\{(\xi_1 - \xi_2)^2 + (F(\xi_1) - F(\xi_2))^2\}^{1/2} \{1 + (F'(\xi_2))^2\}^{1/2}}. \end{aligned}$$

Since

$$F(\xi_1) - F(\xi_2) - (\xi_1 - \xi_2)F'(\xi_2) = (\xi_1 - \xi_2)^2 \int_0^1 tF''(\xi_2 + (\xi_1 - \xi_2)t) dt$$

and

$$F(\xi_1) - F(\xi_2) = (\xi_1 - \xi_2) \int_0^1 F'(\xi_2 + (\xi_1 - \xi_2)t) dt,$$

we obtain

$$\begin{aligned} \nu_y \log |x - y| &= \frac{(\xi_1 - \xi_2)^2 \int_0^1 t F''(\xi_2 + (\xi_1 - \xi_2)t) dt}{[(\xi_1 - \xi_2)^2 + (\xi_1 - \xi_2)^2 \{ \int_0^1 F'(\xi_2 + (\xi_1 - \xi_2)t) dt \}^2] \{1 + (F'(\xi_2))^2\}^{1/2}} \\ &= \frac{\int_0^1 t F''(\xi_2 + (\xi_1 - \xi_2)t) dt}{[1 + \{ \int_0^1 F'(\xi_2 + (\xi_1 - \xi_2)t) dt \}^2] \{1 + (F'(\xi_2))^2\}^{1/2}}. \end{aligned}$$

The positive denominator is of class C^{k-1} and the numerator is of class C^{k-2} , including when $x = y$. Moreover

$$\lim_{\substack{x \rightarrow y \\ x \in \partial\Omega}} \nu_y \log \frac{1}{|x - y|} = -\frac{\frac{1}{2} F''(\xi_2)}{\{1 + (F'(\xi_2))^2\}^{3/2}} = -\frac{1}{2} \kappa(y).$$

□

For $p < 2$, the Schatten class $S_p(L^2)$ cannot be characterized as in the case $p = 2$ by a property analogous to the square integrability of integral kernels. To obtain criteria for operators to belong to Schatten classes for $p < 2$, we use the result of J. Delgado and M. Ruzhansky:

Theorem 2.14 ([DR] Theorem 3.6). Let M be a closed smooth manifold of dimension n and let $\mu_1, \mu_2 \geq 0$. Let $K \in L^2(M \times M)$ be such that $E(x, y) \in H_{x,y}^{\mu_1, \mu_2}(M \times M)$. Then the integral operator K on $L^2(M)$, defined by

$$(Kf)(x) = \int_M E(x, y) f(y) dy,$$

is in the Schatten classes $S_r(L^2(M))$ for $r > \frac{2n}{n+2(\mu_1+\mu_2)}$.

Proof of Theorem 2.12. Taking a C^∞ atlas on $M = \partial\Omega$ like Lemma 2.13, we see

$$E(x, y) \in C_{x,y}^{k-2}(M \times M).$$

Let $n = \dim \partial\Omega = 1$ and $\mu_1 + \mu_2 = k - 2$. From Theorem 2.14, we have

$$K \in S_r(L^2(M)) \quad \text{for all } r > \frac{2}{2k-3}.$$

Using Weyl's inequality again (See e.g. [Si], [Te]),

$$\left\{ \sum_{j=1}^{\infty} |\alpha_j|^r \right\}^{1/r} \geq \left\{ \sum_{\lambda_j \in \sigma_p(K)} |\lambda_j|^r \right\}^{1/r}.$$

The L.H.S. is the Schatten norm of K which is finite. □

Corollary 2.15. Let $n = 2$ and Ω be a C^∞ region.

$$\alpha_j = o(j^{-\infty}) \text{ and } \lambda_j = o(j^{-\infty}) \text{ as } j \rightarrow \infty.$$

Remark 2.16. If Ω is a C^6 region, then $E(x, y) \in C_{x,y}^{2,2}(M \times M)$. From [DR, Corollary 4.4], K is a trace class operator and its trace is given by

$$\sum_{\lambda_i \in \sigma_p(K)} \lambda_i \equiv \text{tr}(K) = \int_{\partial\Omega} -\frac{1}{2\pi} \kappa(y) ds_y = -1.$$

-1 is an eigenvalue of K , so the sum of Fredholm eigenvalues is 0.

The $L^p \rightarrow L^q$ estimate of eigenfunctions is one of the main interests in spectral geometry. From Lemma 2.13, we find a fundamental estimate of eigenfunction:

Remark 2.17. Let $n = 2$ and Ω be a C^2 region. There exists a constant C , depending only on Ω , such that

$$\|e_{\lambda_j}\|_{L^\infty(\partial\Omega)} \leq C \lambda_j^{-1} \|e_{\lambda_j}\|_{L^1(\partial\Omega)}.$$

Presumably this is the best $L^1 \rightarrow L^\infty$ estimate of eigenfunctions.

3 Nodal sets of eigenfunctions

Few studies have focused on the eigenfunctions. In this section, we introduce some fundamental estimates for nodal sets of eigenfunctions of two-dimensional double layer potentials.

3.1 Basic properties of nodal sets

The nodal set $N(e_\lambda(x))$ of eigenfunction $e_\lambda(x)$ is defined by:

$$N(e_\lambda(x)) \equiv \{ x \in \partial\Omega \mid e_\lambda(x) = 0 \}.$$

We note that the nodal set of non-constant eigenfunction is not empty:

Theorem 3.1. Let Ω be a bounded C^2 region in \mathbf{R}^n and $0 < \phi(x) \in C(\partial\Omega)$ be an eigenfunction of K . Then $\phi(x) = \text{const}$.

This theorem holds true even for $n \geq 3$. To prove Theorem 3.1, we closely follow [KPS] and introduce the properties of symmetrizable operators. The proposition below is aimed at and will be directly applicable to double layer potentials K . We know that K is in some Schatten classes (See §2 and §4 for the case of $n = 3$). Moreover the eigenvalues of symmetrizable Schatten class operators are given by Min-Max methods (See e.g. [KPS, §3 and Proposition 3]):

Proposition 3.2 (Min-Max principle for double layer potentials). Let $\lambda_1^+ \geq \lambda_2^+ \geq \dots \geq 0 \geq \dots \geq \lambda_1^- > \lambda_0^- = -1$ be the eigenvalues of K repeated according to their multiplicity, and let ϕ_k^+ , ϕ_k^- be the corresponding eigenfunctions.

Then,

$$\lambda_k^+ = \max_{f \perp \{\phi_1^+, \dots, \phi_{k-1}^+\}} \frac{\langle SKf, f \rangle}{\langle Sf, f \rangle},$$

and similarly

$$\lambda_k^- = \min_{f \perp \{\phi_0^-, \dots, \phi_{k-1}^-\}} \frac{\langle SKf, f \rangle}{\langle Sf, f \rangle}.$$

Here we may employ the single layer potential S defined by

$$(S\psi)(x) \equiv \int_{\partial\Omega} E(x, y)\psi(y) dS_y$$

and $f \perp g$ means $\langle f, Sg \rangle = 0$. Especially if $\lambda \neq -1$, $e_\lambda(x) \in \{\phi(x) \in L^2(\partial\Omega) \mid \langle \phi, S1 \rangle = 0\}$.

Proof of Theorem 3.1. From Min-Max principle for double layer potentials, non constant eigenfunctions $\{e_\lambda(x)\}$ satisfy $e_\lambda(x) \in \{\phi(x) \in L^2(\partial\Omega) \mid \langle \phi, S1 \rangle = 0\}$. Remarking that $f(x) = S1(x) > 0$ for $n \geq 3$ and

$$\int_{\partial\Omega} f(x)\phi(x) dS_x = 0,$$

there exists subset $N^- \subset \partial\Omega$ such that $\phi(x) < 0$ on N^- .

For $n = 2$, eigenfunctions and eigenvalues are equivalent under the self-similar transformations. Indeed, letting $x_\epsilon = \epsilon x$, $y_\epsilon = \epsilon y$, $\Omega_\epsilon = \{x_\epsilon \mid x \in \Omega\}$ and $\psi(x_\epsilon) \equiv \psi(x)$, we have

$$(K_\epsilon\psi)(x_\epsilon) \equiv \int_{\partial\Omega_\epsilon} \psi(y_\epsilon) \cdot \nu_{y_\epsilon} E(x_\epsilon, y_\epsilon) ds_{y_\epsilon} = \int_{\partial\Omega} \psi(y) \cdot \nu_y E(x, y) ds_y = (K\psi)(x).$$

Since $S1(x) > 0$ for the shrinking region Ω_ϵ . Again using the min-max principle, there exists subset $N^- \subset \partial\Omega$ such that $\phi(x) < 0$ on N^- . \square

For convex region, we can give another short proof of Theorem 3.1 without Proposition 3.2.

Remark 3.3. Let Ω be a convex region in \mathbf{R}^n and $\phi(x) > 0$ be an eigenfunction of K . Then $\phi(x) = \text{const}$.

Proof. From a convex separation theorem,

$$\nu_y E(x, y) = C \frac{x - y}{|x - y|^{n-1}} \cdot \mathbf{n}_y \leq 0 \quad (\forall x, y \in \partial\Omega).$$

Remarking that $(K1)(x) \equiv \int_{\partial\Omega} \nu_y E(x, y) ds_y = -1$ and using the first mean value theorem for integration, for all $x \in \partial\Omega$ there exists $x' \in \partial\Omega$ satisfying

$$(K\phi)(x) = -\phi(x').$$

For a non-constant eigenfunction $\phi(x) > 0$, we know $(K\phi)(x) = \lambda\phi(x)$ with $|\lambda| < 1$. Thus

$$\inf_{x \in \partial\Omega} |(K\phi)(x)| = \inf_{x \in \partial\Omega} |\lambda\phi(x)| < \inf_{x' \in \partial\Omega} |\phi(x')| = \inf_{x \in \partial\Omega} |(K\phi)(x)|.$$

This is a contradiction. \square

In the following example, the nodal set of second eigenfunction of double layer potential divides $\partial\Omega$ into many pieces.

Example 3.4. Let $\partial\Omega = S^1$. For an arbitrary non-empty closed set $A \subsetneq \partial\Omega$, there exists eigenfunction $e_0(x) \neq 0$ such that

$$A \subset N(e_0(x)).$$

Proof. From Corollary 2.8, we just choose the non-constant function $e_0(x) \in C(\partial\Omega) \cap L_0^2(\partial\Omega)$ to satisfy $e_0(x) = 0$ on A . \square

We recall Courant's nodal line theorem (CNLT). CNLT states that if the eigenvalues λ_n of Laplacian are ordered increasingly, then each eigenfunctions $u_n(x)$ corresponding to λ_n , divides the region by its nodal set, into at most n subdomains. Unlike the CNLT, we find that the nodal set of double layer eigenfunction e_n is characterized by not n but λ_n .

3.2 Two dimensional analytic boundary

In this subsection, we only consider the analytic domains $\Omega \subset \mathbf{R}^2$ and real analytic eigenfunctions $\{e_\lambda(x)\} \subset C^\omega(\partial\Omega)$. This assumption is reasonable since the continuous eigenfunction $e_\lambda(x)$ is also analytic for $\lambda \neq 0$ (See Remark 3.8).

We prove the boundary zeroes $N(e_\lambda(x))$ satisfy

$$\sharp N(e_\lambda(x)) < C |\log |\lambda||$$

where $Ke_\lambda(x) = \lambda e_\lambda(x)$.

3.2.1 Holomorphic extentions of eigenfunctions

The following notations and results are heavily borrowed from Garabedian (See [Ga]), Millar (See [Mi1], [Mi2], [Mi3]) and Toth-Zelditch (See [TZ]): We denote points \mathbf{R}^2 and also in \mathbb{C}^2 by (x, y) . We further write $z = x + iy$, $z^* = x - iy$. Note that z, z^* are independent holomorphic coordinates on \mathbb{C}^2 and are characteristic coordinates for the Laplacian $\frac{1}{4}\Delta$, in that Laplacian analytically extends to $\frac{\partial^2}{\partial z \partial z^*}$. When dealing with the kernel functions of two variables, we use (ξ, η) in the same way as (x, y) for the second variable.

When the boundary is real analytic, the complexification $\partial\Omega \subset \mathbb{C}$ is the image of analytic continuation of a real analytic parametrization. For simplicity and without loss of generality, we will assume that the length of $\partial\Omega = 2\pi$. We denote a real parametrization by arc-length by $Q : \mathbf{S}^1 \rightarrow \partial\Omega \subset \mathbb{C}$, and also write the parametrization as a periodic function

$$q(t) = Q(e^{it}) : [0, 2\pi] \rightarrow \partial\Omega$$

on $[0, 2\pi]$. We then put the complex conjugate by $q(s) = q_1(s) + iq_2(s)$, $\bar{q}(s) = q_1(s) - iq_2(s)$ for $s \in [0, 2\pi]$.

We complexify $\partial\Omega$ by holomorphically extending the parametrization to $Q^{\mathbb{C}}$ on the annulus

$$A(\epsilon) \equiv \{ \tau \in \mathbb{C} : e^{-\epsilon} < |\tau| < e^\epsilon \}$$

for $\epsilon > 0$ small enough. Note that the complex conjugate parametrization \bar{Q} extends holomorphically to $A(\epsilon)$ as $Q^{*\mathbb{C}}$. The $q(t)$ parametrization analytically continues to a periodic function $q^{\mathbb{C}}(t)$ on $[0, 2\pi] + i[-\epsilon, \epsilon]$. The complexification $\partial\Omega_{\mathbb{C}}(\epsilon)$ of $\partial\Omega$ is denoted by

$$\partial\Omega_{\mathbb{C}}(\epsilon) \equiv Q^{\mathbb{C}}(A(\epsilon)) \subset \mathbb{C}.$$

Next, we put $r^2((x, y); (\xi, \eta)) = (\xi - x)^2 + (\eta - y)^2$. For $s \in \mathbf{R}$ and $t \in \mathbb{C}$, we have $q(s) = \xi(s) + i\eta(s)$, $q^{\mathbb{C}}(t) = x(t) + iy(t)$, $q^{\mathbb{C}*}(t) = x(t) - iy(t)$ and we write $r^2(q(s); q^{\mathbb{C}}(t))$. Thus

$$r^2((x, y); (\xi, \eta)) = (q(s) - q^{\mathbb{C}}(t))(\bar{q}(s) - q^{\mathbb{C}*}(t)) \in \mathbb{C}.$$

To clarify the notation, we consider two examples:

Example 3.5 (The circle). Let $\partial\Omega = S^1$. Then, $q(s) = e^{is}$, $t = \theta + i\xi$, $q^{\mathbb{C}}(t) = e^{i(\theta+i\xi)}$, $q^{\mathbb{C}*}(t) = e^{-i(\theta+i\xi)}$, $\bar{q}^{\mathbb{C}*}(t) = e^{i(\theta-i\xi)}$, and

$$r^2(s, t) = (e^{i(\theta+i\xi)} - e^{is})(e^{-i(\theta+i\xi)} - e^{-is}) = 4 \sin^2 \frac{\theta - s + i\xi}{2}.$$

Thus, $\log r^2 = \log(4 \sin^2 \frac{\theta - s + i\xi}{2})$.

Example 3.6 (The ellipse). Let $\partial\Omega = \{(x, y) \mid \frac{x^2}{9} + y^2 = 1\}$. Then, $q(s) = 2e^{is} + e^{-is}$, $t = \theta + i\xi$, $q^{\mathbb{C}}(t) = 2e^{i(\theta+i\xi)} + e^{-i(\theta+i\xi)}$, $q^{\mathbb{C}*}(t) = 2e^{-i(\theta+i\xi)} + e^{i(\theta+i\xi)}$, $\bar{q}^{\mathbb{C}*}(t) = 2e^{i(\theta-i\xi)} + e^{-i(\theta-i\xi)}$, and

$$r^2(s, t) = (2e^{i(\theta+i\xi)} + e^{-i(\theta+i\xi)} - e^{is})(2e^{-i(\theta+i\xi)} + e^{i(\theta+i\xi)} - e^{-is}).$$

We denote by $\frac{\partial}{\partial n}$ the not-necessarity-unit normal derivative in the direction $iq'(s)$. Thus, in terms of the notation $\frac{\partial}{\partial \nu}$ above, $\frac{\partial}{\partial n} = |q'(s)| \frac{\partial}{\partial \nu}$. When we are using an arc-length parametrization, $\frac{\partial}{\partial n} = \frac{\partial}{\partial \nu}$. One has

$$\frac{d}{ds} \log r = \frac{1}{2} \left[\frac{q'(s)}{q(s) - q^{\mathbb{C}}(t)} + \frac{\bar{q}'(s)}{\bar{q}(s) - q^{\mathbb{C}*}(t)} \right], \quad \frac{\partial}{\partial n} \log r = \frac{-i}{2} \left[\frac{q'(s)}{q(s) - q^{\mathbb{C}}(t)} - \frac{\bar{q}'(s)}{\bar{q}(s) - q^{\mathbb{C}*}(t)} \right].$$

3.2.2 Analytic continuation of eigenfunctions through layer potential representation

Since $r^2(s, t) = 0$ when $s = t$, the logarithmic factor in K now gives rise to a multi-valued integrand. Nevertheless any derivative of $\log r^2$ is unambiguously defined and the analytic continuation of complex representation was given by Millar (See [Mi1, p.508 (7.2)]):

Proposition 3.7. The integral $Ke_{\lambda}(q(s)) = \frac{1}{\pi} \int_0^{2\pi} e_{\lambda}(q(s)) \frac{\partial}{\partial \nu} \log r(s, t) ds = \frac{1}{\pi} \int_0^{2\pi} e_{\lambda}(q(s)) \frac{1}{r} \frac{\partial r}{\partial \nu}(s, t) ds$ is real analytic on the parameter interval S^1 parametrizing $\partial\Omega$ and holomorphically extended to an annulus $A(\epsilon)$ by the formula

$$Ke_{\lambda}(q^{\mathbb{C}}(t)) = \frac{1}{2\pi i} \int_0^{2\pi} e_{\lambda}(q(s)) \left(\frac{q'(s)}{q(s) - q^{\mathbb{C}}(t)} - \frac{\bar{q}'(s)}{\bar{q}(s) - q^{\mathbb{C}*}(t)} \right) ds.$$

Proof. We first remark that $\frac{\partial}{\partial \nu} = |q'(s)|^{-1} \frac{\partial}{\partial n}$, so the integral representation is invariant under reparametrization. Any derivative of $\log r^2$ is unambiguously defined and we already have

$$\frac{1}{r} \frac{\partial r}{\partial n} = \frac{\partial \log r}{\partial n} = \frac{1}{2i} \left[\frac{q'(s)}{q(s) - q^{\mathbb{C}}(t)} - \frac{\bar{q}'(s)}{\bar{q}(s) - q^{\mathbb{C}*}(t)} \right].$$

In the real domain $q^{\mathbb{C}*}(t) = \bar{q}^{\mathbb{C}}(t)$, so

$$\frac{1}{r} \frac{\partial r}{\partial n} = \text{Im} \frac{q'(s)}{q(s) - q(t)}.$$

Here $\text{Im } z$ denotes the imaginary part of z . We recall that in terms of the real parametrization, $\frac{1}{r} \frac{\partial r}{\partial \nu}$ is real and continuous (See Lemma 2.13).

In complex notation, the same statement follows from the fact that

$$\lim_{t \rightarrow s} \frac{q(s) - q^{\mathbb{C}}(t)}{s - t} = q'(s) \quad \Rightarrow \quad \frac{q'(s)}{q(s) - q^{\mathbb{C}}(t)} = \frac{1}{s - t} + O(1), \quad (s \rightarrow t),$$

where $\frac{1}{s-t}$ is real when $s, t \in \mathbf{R}$. Hence $\text{Im} \frac{q'(s)}{q(s)-q^{\mathbb{C}}(t)}$ is continuous for $s, t \in [0, 2\pi]$ and since $q(s), q(t)$ are real analytic, the map

$$s \rightarrow \left[\frac{q'(s)}{q(s)-q^{\mathbb{C}}(t)} - \frac{\bar{q}'(s)}{\bar{q}(s)-q^{\mathbb{C}*}(t)} \right]$$

is a continuous map from $s \in [0, 2\pi]$ to the space of holomorphic functions of t . So the integral admits an holomorphic extention. \square

Remark 3.8. We notice that the continuous eigenfunction satisfies

$$e_\lambda(q(s)) = \frac{1}{\lambda} K e_\lambda(q(s)) \quad \text{for } s \in \mathbf{S}^1.$$

From Proposition 3.7, if $\lambda \neq 0$, the continuous eigenfunction is also analytic.

3.2.3 Growth of zeroes and Growth of $e_\lambda^{\mathbb{C}}(q^{\mathbb{C}}(t))$

The main purpose of this subsection is to give an upper bound for the number of complex zeroes of $e_\lambda^{\mathbb{C}}$ in $\partial\Omega_{\mathbb{C}}(\epsilon)$ in terms of the growth of $|e_\lambda^{\mathbb{C}}(q^{\mathbb{C}}(t))|$. For the eigenvalue λ and for a region $D \subset \partial\Omega_{\mathbb{C}}(\epsilon)$ we denote by

$$n(\lambda, D) = \#\{q^{\mathbb{C}}(t) \in D : e_\lambda^{\mathbb{C}}(q^{\mathbb{C}}(t)) = 0\}.$$

To the reader's convenience, we recall that the classical distribution theory of holomorphic functions is concerned with the relation between the growth of the number of zeroes of a holomorphic function f and the growth of $\max_{|z|=r} \log |f(z)|$ on discs of increasing radius. The following estimate, suggested by Lemma 6.1 of Donnelly-Fefferman (See [DF]), gives an upper bound on the number of zeroes in terms of the growth of the family:

Proposition 3.9. Normalize e_λ so that $\|e_\lambda\|_{L^2(\partial\Omega)} = 2\pi$. Then there exists a constant $C(\epsilon) > 0$ such that for any $\epsilon > 0$,

$$n(\lambda, \partial\Omega_{\mathbb{C}}(\epsilon/2)) \leq C(\epsilon) \max_{q^{\mathbb{C}}(t) \in \partial\Omega_{\mathbb{C}}(\epsilon)} \left| \log |e_\lambda^{\mathbb{C}}(q^{\mathbb{C}}(t))| \right|.$$

Proof. Let G_ϵ denote the Dirichlet Green's function of $\frac{2}{\pi} \frac{\partial^2}{\partial z \partial \bar{z}}$ in the 'annulus' $\partial\Omega_{\mathbb{C}}(\epsilon)$. Also, let $\{a_k\}_{k=1}^{n(\lambda, \partial\Omega_{\mathbb{C}}(\epsilon/2))}$ denote the zeroes of $e_\lambda^{\mathbb{C}}$ in the sub-annulus $\partial\Omega_{\mathbb{C}}(\epsilon/2)$. Let $f_\lambda = \frac{e_\lambda^{\mathbb{C}}}{\|e_\lambda^{\mathbb{C}}\|_{\partial\Omega_{\mathbb{C}}(\epsilon)}}$ where $\|u\|_{\partial\Omega_{\mathbb{C}}(\epsilon)} = \max_{\zeta \in \partial\Omega_{\mathbb{C}}(\epsilon)} |u(\zeta)|$. Then $\log |f_\lambda(q^{\mathbb{C}}(t))|$ can be separated into two terms:

$$\begin{aligned} \log |f_\lambda(q^{\mathbb{C}}(t))| &= \int_{\partial\Omega_{\mathbb{C}}(\epsilon/2)} G_\epsilon(q^{\mathbb{C}}(t), w) \frac{i}{\pi} \partial \bar{\partial} \log |e_\lambda^{\mathbb{C}}(w)| + F_\lambda(q^{\mathbb{C}}(t)) \\ &= \sum_{a_k \in \partial\Omega_{\mathbb{C}}(\epsilon/2) : e_\lambda^{\mathbb{C}}(a_k)=0} G_\epsilon(q^{\mathbb{C}}(t), a_k) + F_\lambda(q^{\mathbb{C}}(t)), \end{aligned}$$

since $\frac{i}{\pi} \partial \bar{\partial} \log |e_\lambda^{\mathbb{C}}(w)| = \sum_{a_k \in \partial\Omega_{\mathbb{C}}(\epsilon/2) : e_\lambda^{\mathbb{C}}(a_k)=0} \delta_{a_k}$ which is called Poincaré-Lelong formula of holomorphic functions (See e.g. [De, p.9 (3.6)]). Moreover the function F_λ is subharmonic on $\partial\Omega_{\mathbb{C}}(\epsilon)$ in the sense of distribution:

$$\frac{i}{\pi} \partial \bar{\partial} F_\lambda = \frac{i}{\pi} \partial \bar{\partial} \log |f_\lambda(q^{\mathbb{C}}(t))| - \sum_{a_k \in \partial\Omega_{\mathbb{C}}(\epsilon/2) : e_\lambda^{\mathbb{C}}(a_k)=0} \frac{i}{\pi} \partial \bar{\partial} G_\epsilon(q^{\mathbb{C}}(t), a_k) = \sum_{a_k \in \partial\Omega_{\mathbb{C}}(\epsilon) \setminus \partial\Omega_{\mathbb{C}}(\epsilon/2) : e_\lambda^{\mathbb{C}}(a_k)=0} \delta_{a_k} > 0.$$

So, by the maximum principle for subharmonic functions, we obtain

$$\max_{\partial\Omega_{\mathbb{C}}(\epsilon)} F_\lambda(q^{\mathbb{C}}(t)) \leq \max_{\partial(\partial\Omega_{\mathbb{C}}(\epsilon))} F_\lambda(q^{\mathbb{C}}(t)) = \max_{\partial(\partial\Omega_{\mathbb{C}}(\epsilon))} \log |f_\lambda(q^{\mathbb{C}}(t))| = 0.$$

It follows that

$$\log |f_\lambda(q^{\mathbb{C}}(t))| \leq \sum_{a_k \in \partial\Omega_{\mathbb{C}}(\epsilon/2) : e_\lambda^{\mathbb{C}}(a_k)=0} G_\epsilon(q^{\mathbb{C}}(t), a_k),$$

hence that

$$\max_{q^{\mathbb{C}}(t) \in \partial\Omega_{\mathbb{C}}(\epsilon)} \log |f_\lambda(q^{\mathbb{C}}(t))| \leq \left(\max_{z, w \in \partial\Omega_{\mathbb{C}}(\epsilon/2)} G_\epsilon(z, w) \right) n(\lambda, \partial\Omega_{\mathbb{C}}(\epsilon/2)).$$

Now $G_\epsilon(z, w) \leq \max_{w \in \partial(\partial\Omega_C(\epsilon))} G_\epsilon(z, w) = 0$ and so $G_\epsilon(z, w) < 0$ for $z, w \in \partial\Omega_C(\epsilon/2)$. It follows that there exists a constant $\nu(\epsilon) < 0$ so that $\max_{z, w \in \partial\Omega_C(\epsilon/2)} G_\epsilon(z, w) \leq \nu(\epsilon)$. Hence,

$$\max_{q^C(t) \in \partial\Omega_C(\epsilon/2)} \log |f_\lambda(q^C(t))| \leq \nu(\epsilon)n(\lambda, \partial\Omega_C(\epsilon/2)).$$

Since both sides are negative, we obtain

$$\begin{aligned} n(\lambda, \partial\Omega_C(\epsilon/2)) &\leq \frac{1}{|\nu(\epsilon)|} \left| \max_{q^C(t) \in \partial\Omega_C(\epsilon/2)} \log |f_\lambda(q^C(t))| \right| \\ &\leq \frac{1}{|\nu(\epsilon)|} \left(\max_{q^C(t) \in \partial\Omega_C(\epsilon)} \log |e_\lambda^C(q^C(t))| - \max_{q^C(t) \in \partial\Omega_C(\epsilon/2)} \log |e_\lambda^C(q^C(t))| \right) \\ &\leq \frac{1}{|\nu(\epsilon)|} \max_{q^C(t) \in \partial\Omega_C(\epsilon)} \log |e_\lambda^C(q^C(t))|, \end{aligned}$$

where in the last inequality we use that $\max_{q^C(t) \in \partial\Omega_C(\epsilon/2)} \log |e_\lambda^C(q^C(t))| \geq 0$, which holds since $|e_\lambda^C| \geq 1$ at some point in $\partial\Omega_C(\epsilon/2)$. Indeed, by our normalization, $\|e_\lambda\|_{L^2(\partial\Omega)} = 2\pi$, and so there must already exist a points on $\partial\Omega$ with $|e_\lambda| > 1$. Putting $C(\epsilon) = \frac{1}{|\nu(\epsilon)|}$ we have the desired result. \square

We obtain the main theorem:

Theorem 3.10. Let $\Omega \subset \mathbf{R}^2$ be a real analytic domain and $|\lambda| \neq 0$. For real analytic eigenfunctions $e_\lambda(x)$ we have

$$\sharp N(e_\lambda(x)) < C |\log |\lambda||.$$

Proof. For real $t \in S^1 = [0, 2\pi]/\sim$,

$$e_\lambda(q(t)) = \frac{1}{\lambda} K e_\lambda = \frac{1}{\lambda} \int_0^{2\pi} e_\lambda(q(s)) \frac{1}{r} \frac{\partial r}{\partial \nu}(s, t) ds.$$

The holomorphic extention of $e_\lambda(q(s)) \in C^\omega(S^1)$ to $C^\omega(A(\epsilon))$ is unique and hence from Proposition 3.7,

$$e_\lambda^C(q^C(t)) = \frac{1}{2\pi i \lambda} \int_0^{2\pi} e_\lambda(q(s)) \left(\frac{q'(s)}{q(s) - q^C(t)} - \frac{\bar{q}'(s)}{\bar{q}(s) - q^{C*}(t)} \right) ds.$$

Remarking that the function (\dots) is continuous and bounded on $A(\epsilon)$ from the proof of Proposition 3.7. So using Cauchy-Schwarz inequality, there exists $C_{A(\epsilon)} > 0$ such that

$$\begin{aligned} |e_\lambda^C(q^C(t))| &\leq \left| \frac{1}{2\pi i \lambda} \right| \cdot \|e_\lambda(q(s))\|_{L^2(\partial\Omega)} \int_0^{2\pi} \left| \frac{q'(s)}{q(s) - q^C(t)} - \frac{\bar{q}'(s)}{\bar{q}(s) - q^{C*}(t)} \right|^2 ds \\ &\leq \left| \frac{1}{\lambda} \right| \cdot C_{A(\epsilon)} \cdot \|e_\lambda(q(s))\|_{L^2(\partial\Omega)}. \end{aligned}$$

Letting $\|e_\lambda(q(s))\|_{L^2(\partial\Omega)} = 2\pi$ and by Proposition 3.9,

$$\begin{aligned} n(\lambda, \partial\Omega_C(\epsilon/2)) &\leq C(\epsilon) \max_{q^C(t) \in \partial\Omega_C(\epsilon)} \left| \log |e_\lambda^C(q^C(t))| \right| \\ &\leq C(\epsilon) \left| \log \left[\left| \frac{1}{\lambda} \right| \cdot C_{A(\epsilon)} \cdot \|e_\lambda(q(s))\|_{L^2(\partial\Omega)} \right] \right| \\ &\leq \tilde{C}(\epsilon) |\log |\lambda|| \end{aligned}$$

as desired. \square

4 Double layer potentials in \mathbf{R}^3

Plemelj [Pl] derived a fundamental result on the double layer potential in \mathbf{R}^3 which states that the eigenvalues of K satisfy the following inequality

$$-1 \leq \lambda_j < 1$$

For the case of a sphere, however, it is known that the eigenvalue of K are negative, and by a straightforward calculation it can be shown that the eigenvalues are given by

$$\lambda_j = -\frac{1}{2j+1}, \quad (j = 0, 1, 2, \dots)$$

with multiplicity $2j+1$. So $\sigma_p(K)$ for $n=3$ is very different from it for $n=2$ (See example 2.1, example 2.2 and Theorem 2.7). Furthermore, Ahner and Arenstorf [AA] have shown that when $\partial\Omega$ is a prolate spheroid, the corresponding eigenvalues are also negative. Consequently, for this geometry, the spectrum of K also lies in the closed interval $[-1, 0]$.

Apart from these calculations, for the case of a special oblate spheroid, Ahner [Ah2, p.333] finds the positive eigenvalue $\lambda = 0.0598615 \dots < 1$. This is an example of positive eigenvalues. Unfortunately, this supremum of eigenvalues becomes a formidable task for general region.

Nevertheless, for $\bar{\lambda} = \sup\{\lambda_j \mid \lambda_j \in \sigma_p(K)\}$, we know the supremum of the boundary variation ([ADR, Lemma 3.2, Theorem 3.4])

$$\sup_{\partial\Omega} \bar{\lambda} = 1$$

where the supremum is taken over all C^∞ domain Ω . Letting $\underline{\lambda} = \inf\{\lambda_j \mid \lambda_j \in \sigma_p(K) \setminus \{-1\}\}$, we also know ([ADR, Lemma 3.2], [KPS, Theorem 5])

$$\inf_{\partial\Omega} \underline{\lambda} = -1.$$

Here we introduce a result about $\underline{\lambda}$: Steinbach and Wendland prove that

$$(1 - \sqrt{1 - c_0}) \|w\|_{S^{-1}} \leq \|(I \pm K)w\|_{S^{-1}} \leq (1 + \sqrt{1 - c_0}) \|w\|_{S^{-1}}$$

where $c_0 = \inf_{w \in H_*^{1/2}} \frac{\langle Kw, w \rangle}{\langle S^{-1}w, w \rangle}$ and $\|w\|_{S^{-1}} = \sqrt{\langle S^{-1}w, w \rangle}_{L^2(\partial\Omega)}$ for $w \in H^{1/2}(\partial\Omega)$. They show $c_0 \leq 1$. (These constants are slightly different from those in the original papers. For more information see [SW, Theorem 3.2].) Especially for the negative eigenvalue $\underline{\lambda}$

$$\underline{\lambda} \geq -\sqrt{1 - c_0}.$$

Thus the shape dependent constant c_0 controls the eigenvalue $\underline{\lambda}$. Note that Pechstein recently gives the lower bound of c_0 by using the isoperimetric constant $\gamma(\Omega)$ and Sobolev extension constants (See [Pe, Corollary 6.14], [KRW]). In the case of $\partial\Omega = S^2$, $c_0 = \frac{8}{9}$ and $\underline{\lambda} = -\sqrt{1 - c_0} = -\frac{1}{3}$.

4.1 Asymptotic properties of $\sigma(K)$ for $n=3$

For the case of $n=3$, D. Khavinson, M. Putinar and H. S. Shapiro briefly mentioned only a result: K is in the Schatten class $\mathcal{S}_p(L^2(\partial\Omega))$, $p > 2$ (See [KPS, p.150]). We shall explain it in more detail for smooth $\partial\Omega$. Following [Ke, p.303], the nature of the diagonal singularity of the kernel $\nu_y E(x, y)$ shows that

$$E_2(x, y) = \int_{\Omega_z} \nu_z E(z, x) \cdot \nu_z E(z, y) dS_z = A(x, y) + B(x, y) \log(|x - y|)$$

where $A(x, y), B(x, y) \in C^\infty(\partial\Omega \times \partial\Omega)$. Since $E_2(x, y) \in H_{x,y}^{\mu_1, \mu_2}$ with $\mu_1 + \mu_2 < 1$, applying Theorem 2.14 to E_2

$$K^*K \in \mathcal{S}_r(L^2(\partial\Omega)) \quad \text{for } r > \frac{4}{2+2} = 1.$$

This means that K is in the Schatten class $\mathcal{S}_p(L^2(\partial\Omega))$, $p > 2$.

We note that the regularity of $A(x, y)$ and $B(x, y)$ is essential to the above result. Immediately a decay rate of $\sigma_p(K)$ is obtained:

Remark 4.1. Let $n=3$ and Ω be a smooth region. For $\alpha > -\frac{1}{2}$,

$$\lambda_j = o(j^\alpha) \quad \text{as } j \rightarrow \infty.$$

In the case of a sphere, this is the best possible estimate.

4.2 Isoperimetric properties of K

We want to characterize the isoperimetric properties by $\sigma_p(K)$. For the case of $n = 3$, however, the explicit formula have not been obtained yet. In this subsection, some expected properties and conjectures are introduced. Seeing the case of $n = 2$, $-\underline{\lambda}$ and Schatten norm are expected to minimize by $\partial\Omega = S^2$. So we expect the following conjectures:

Conjecture 1. Let $n = 3$ and $\underline{\lambda} \equiv \min \sigma_p(K) \setminus \{-1\}$. We have

$$\sup_{\partial\Omega} \underline{\lambda} = -\frac{1}{3},$$

where the supremum is taken over all C^∞ simply connected closed surfaces. The supremum is achieved if and only if $\partial\Omega = S^2$.

Note that for the case of $\partial\Omega = S^2$, $\underline{\lambda} = -\frac{1}{3}$ is obtained by direct calculations.

Conjecture 2. Let $n = 3$. For $p > 1$, we have

$$\inf_{\partial\Omega} \text{tr}\{(K^*K)^p\} = \left(1 - \frac{1}{2^{2p-1}}\right) \zeta(2p-1)$$

where the infimum is taken over all C^∞ simply connected closed surfaces and $\zeta(x)$ denotes the Riemann zeta function. The infimum is achieved if and only if $\partial\Omega = S^2$.

To confirm the validity of conjectures, henceforce, we consider the case of ellipsoids. For the case of ellipsoids $\{(x, y, z) \in \mathbf{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$ Ritter [Ri1][Ri2] has shown that $\sigma_p(K)$ is completely solved by ellipsoidal harmonics (Lamé polynomials); note that there are exactly $2l + 1$ linearly independent Lamé polynomials of order $l \geq 0$ (See [H]). Also Martensen [Ma, Theorem 1] proved :

Proposition 4.2. For any $2l + 1$ linearly independent Lamé polynomials of order $l \geq 0$, considered as eigenfunctions of K , the sum of corresponding eigenvalues is equal -1 .

We denote these eigenvalues by $\lambda_{k,l}$ ($k = 1, 2, \dots, 2l + 1$) and so

$$\sum_{k=1}^{2l+1} \lambda_{k,l} = -1.$$

Furthermore, deformation of the sphere into a triaxial ellipsoid yields to bifurcation $-\frac{1}{2l+1}$ into $2l + 1$ different eigenvalues of order l , say $\lambda_{k,l}$, $k = 1, 2, \dots, 2l + 1$, each with multiplicity one (See [Ri2]).

Consequently proofs of conjectures for ellipsoids are given:

Theorem 4.3. Let $n = 3$ and $p > 1$. For the case of ellipsoids $\partial\Omega$, we have

$$\sup_{\partial\Omega} \underline{\lambda} = -\frac{1}{3} \quad \text{and} \quad \inf_{\partial\Omega} \text{tr}\{(K^*K)^p\} = \left(1 - \frac{1}{2^{2p-1}}\right) \zeta(2p-1).$$

The supremum and infimum are achieved if and only if $\partial\Omega = S^2$.

Proof. For $l = 1$,

$$\lambda_{1,1} + \lambda_{2,1} + \lambda_{3,1} = -1.$$

Thus $\underline{\lambda} \leq \min(\lambda_{1,1}, \lambda_{2,1}, \lambda_{3,1}) \leq -\frac{1}{3}$. Equality holds if and only if $\lambda_{1,1} = \lambda_{2,1} = \lambda_{3,1} = -\frac{1}{3}$. So we have the first equation.

To prove the second equation, we note that from Hölder's inequality

$$1 = |(1, 1, \dots, 1) \cdot (\lambda_{1,l}, \lambda_{2,l}, \dots, \lambda_{2l+1,l})| \leq (2l+1)^{(2p-1)/2p} \left(\sum_{k=0}^{2l+1} |\lambda_{k,l}|^{2p} \right)^{1/2p}.$$

This leads to $s_l = \sum_{k=0}^{2l+1} |\lambda_{k,l}|^{2p} \geq \left(\frac{1}{2l+1}\right)^{2p-1}$. Remarking that $(K^*K)^p$ is in trace class and using Weyl's inequality,

$$\begin{aligned} \operatorname{tr}\{(K^*K)^p\} &\geq \sum_{l=0}^{\infty} \sum_{k=0}^{2l+1} |\lambda_{k,l}|^{2p} \\ &= \sum_{l=0}^{\infty} s_l \\ &\geq \sum_{l=0}^{\infty} \left(\frac{1}{2l+1}\right)^{2p-1} = \left(1 - \frac{1}{2^{2p-1}}\right) \zeta(2p-1) \end{aligned}$$

as desired. \square

For the general smooth surfaces, we mention equivalent statements of conjectures. We infer the Schatten norm of single layer potentials:

$$\operatorname{tr}\{(K^*K)^p\} \leq \operatorname{tr}\left\{\left(\frac{1}{4}S^*S\right)^p\right\} + \operatorname{tr}\left\{\left(K - \frac{1}{2}S\right)^*\left(K - \frac{1}{2}S\right)\right\}^p.$$

It's also known that ([KPS, Theorem 8] and see also [EKS], [Re], [Ra1] and [Ra2]) :

Theorem 4.4. The following is true: for a ball in \mathbf{R}^3 the kernel of K is symmetric and $K = \frac{1}{2}S$, and balls are the only domains with this property.

Thus if one proves the single layer version of the above conjectures, simultaneously we obtain the proof for the double layer potentials.

5 Conclusion

Some fundamental properties of the eigenvalue and eigenfunctions of double layer potentials are discussed. Characteristic properties of the ball are given by the Hilbert-Schmidt norm and Schatten norms of double layer potentials. The fundamental estimates of decay rates of eigenvalues are also given by the regularity of the boundary.

With respect to eigenfunctions, the growth rates of nodal sets are characterized of the eigenvalues. Even less is known in $n = 3$. We want to mention about this in the future.

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