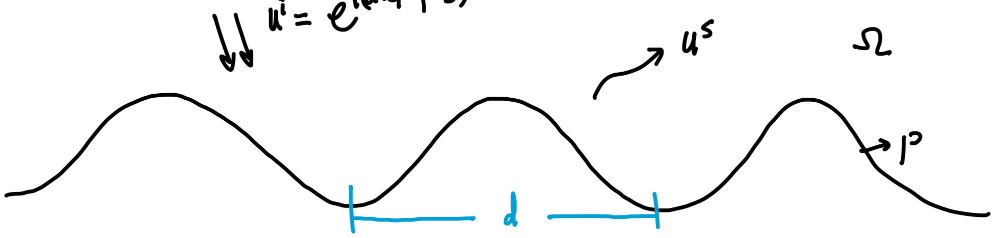
§1.6 Periodic problems and integral equations

Tuesday, February 26, 2019 4:35 PM

§1.6.1 periodic Green's function and layer potential



Consider the scattering by a periodic surface $\Gamma := \{(x_1, x_2) \mid x_2 = f(x_1)\}$.

The total field $u = u^i + u^s$, where u^s is the scattered field. u satisfies

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases} \qquad (1.6.1)$$

Due to the quasi-periodicity of the incident wave u^i, we also require that u is quasi-periodic in the sense that $\boxed{u(x_1, x_2) = e^{i\alpha x_1} v(x_1, x_2) \quad \forall x_1 \in \mathbb{R}}$ where $v(x_1, x_2)$ is a periodic function with period d. This is equivalent to the condition $\boxed{u(x_1 + d, x_2) = e^{i\alpha d} u(x_1, x_2) \quad \forall x_1 \in \mathbb{R}}$.

To impose the radiation condition at infinity, we consider the solution in the domain $\Omega_H := \{(x_1, x_2) \mid x_2 > H\}$. Due to the quasi-periodicity of u^s, we have $u^s(x_1, x_2) = e^{i\alpha x_1} v^s(x_1, x_2)$, where $v^s(x_1, x_2) = \sum_{n=-\infty}^{\infty} v_n^s(x_2)\, e^{i\frac{2\pi n}{d} x_1}$.

By substituting into the Helmholtz equation $\Delta u^s + k^2 u^s = 0$, we obtain

$$v_n''(x_2) + \beta_n^2 v_n(x_2) = 0, \quad \text{where } \alpha_n = \alpha + \frac{2\pi n}{d}, \quad \beta_n = \begin{cases} \sqrt{k^2 - \alpha_n^2}, & |\alpha_n| \le k, \\ i\sqrt{\alpha_n^2 - k^2}, & |\alpha_n| > k. \end{cases}$$

$\Rightarrow$ $v_n(x_2) = c_n^+ e^{i\beta_n x_2} + c_n^- e^{-i\beta_n x_2}$, and the scattered field

$$u^s(x_1, x_2) = \sum_{n=-\infty}^{\infty} c_n^+ \underbrace{e^{i(\alpha_n x_1 + \beta_n x_2)}}_{\text{outgoing wave mode}} + c_n^- \underbrace{e^{i(\alpha_n x_1 - \beta_n x_2)}}_{\text{incoming wave mode}} \quad \text{for } x_2 > H.$$

We enforce the outgoing condition by letting $\boxed{u^s(x_1, x_2) = \sum_{n=-\infty}^{\infty} c_n^+ e^{i(\alpha_n x_1 + \beta_n x_2)}}$

Remark: If $\beta_n > 0$ is real, the mode $e^{i(\alpha_n x_1 + \beta_n x_2)}$ is a propagating mode;

if $|\alpha_n| > k$ s.t. β_n is a complex number, $e^{i(\alpha_n x_1 + \beta_n x_2)}$ is an evanescent mode which decays exponentially in x_2 direction.

Quasi-periodic Green's function

Define $\Phi^{qp}(x,y)=\sum_{n=-\infty}^{\infty}\Phi_k(x,y^{(n)})\,e^{i\alpha nd}$, where $y^{(n)}=y+(nd,0)$, $\Phi_k(x,y)=\frac{i}{4}H_0^{(1)}(k|x-y|)$.

We have $\Phi^{qp}(x,y)=\frac{i}{4}\sum_{n=-\infty}^{\infty}H_0^{(1)}(k|x-y^{(n)}|)\,e^{i\alpha nd}$, and $|x-y^{(n)}|=\sqrt{(x_1-y_1-nd)^2+(x_2-y_2)^2}$.

Thus for convenience we rewrite $\Phi^{qp}(x,y)$ as

$$G(x_1-y_1,x_2-y_2)=\sum_{n=-\infty}^{\infty}G_k(x_1-y_1-nd,\,x_2-y_2)\,e^{i\alpha nd},\text{ where } G_k(x_1-y_1,x_2-y_2)=\frac{i}{4}H_0^{(1)}(k|x-y|).$$

From the above expression, without loss of generality we may assume that $y_1=y_2=0$.

The following holds for Green's function $G(x_1,x_2)$:

(i) $G(x_1,x_2)$ is quasi-periodic in x_1 s.t. $G(x_1+d,x_2)=e^{i\alpha d}G(x_1,x_2)$.

this follows by observing that

$$G(x_1+d,x_2)=\sum_{n=-\infty}^{\infty}G_k(x_1-(n-1)d,x_2)\,e^{i\alpha(n-1)d}\cdot e^{i\alpha d}=e^{i\alpha d}G(x_1,x_2).$$

(ii) $\Delta G(x_1,x_2)+k^2G(x_1,x_2)=-\sum_{n=-\infty}^{\infty}\delta(x-y^{(n)})\,e^{i\alpha nd}$, where $y^{(n)}=(nd,0)$

(iii) $G(x_1,x_2)=e^{i\alpha x_1}G_p(x_1,x_2)$, where $G_p(x_1,x_2)$ is periodic in x_1. In addition,

$$(\Delta+2i\alpha\partial_{x_1}+k^2-\alpha^2)G_p(x_1,x_2)=-\sum_{n=-\infty}^{\infty}\delta(x-y^{(n)}),\text{ where } y^{(n)}=(nd,0).$$

The quasi-periodicity follows from (i).

By substituting $G(x_1,x_2)=e^{i\alpha x_1}G_p(x_1,x_2)$ into (ii), we have

$$e^{i\alpha x_1}\cdot(\Delta+2i\alpha\partial_{x_1}+k^2-\alpha^2)G_p(x_1,x_2)=-\sum_{n=-\infty}^{\infty}\delta(x-y^{(n)})\,e^{i\alpha nd}=-\sum_{n=-\infty}^{\infty}\delta(x-y^{(n)})\,e^{i\alpha x_1}.$$

Thus the equation for $G_p(x_1,x_2)$ follows.

(iv) The quasi-periodic Green's function $G(x_1,x_2)$ adopts the following spectral decomposition:

$$G(x_1,x_2)=\frac{i}{2d}\sum_{n=-\infty}^{\infty}\frac{1}{\beta_n}e^{i(\alpha_n x_1+\beta_n|x_2|)},\quad \alpha_n=\alpha+\frac{2\pi n}{d},\ \beta_n=\begin{cases}\sqrt{k^2-\alpha_n^2} & |\alpha_n|<k,\\ i\sqrt{\alpha_n^2-k^2}, & |\alpha_n|>k.\end{cases}$$

In view of (iii), expand $G_p(x_1,x_2)=\sum_{n=-\infty}^{\infty}g_n(x_2)\,e^{i\frac{2\pi n}{d}x_1}$. (1.6.1)

Substitute into the equation in (iii) we have

$$\sum_{n=-\infty}^{\infty}\left[g_n''(x_2)+(k^2-\alpha_n^2)g_n(x_2)\right]e^{i\frac{2\pi n}{d}x_1}=-\sum_{n=-\infty}^{\infty}\delta(x_1-nd)\,\delta(x_2)$$

using the poisson formula $\sum_{n=-\infty}^{\infty}\delta(x_1-nd)=\frac{1}{d}\sum_{n=-\infty}^{\infty}e^{i\frac{2\pi n}{d}x_1}$, it follows that

$$g_n''(x_2) + (k^2 - \alpha_n^2)\, g_n(x_2) = -\frac{1}{d}\,\delta(x_2)$$

Solving the above equations gives $g_n(x_2) = \dfrac{i}{2d\beta_n} e^{i\beta_n|x_2|}$. (1.6.2)

The spectral decomposition of $G(x_1,x_2)$ follows by combining (1.6.1) (1.6.2) & (iii).

Exercise. Show that the outgoing Green's function for the equation

$$g_n''(x_2) + (k^2 - \alpha_n^2)\, g_n(x_2) = -\frac{1}{d}\,\delta(x_2)$$

is given by (1.6.2).

Layer potentials and boundary integral equations

Let $\Omega_0 := \{ x \in \Omega \mid 0 < x_1 < d \}$ be the domain of the reference period for the periodic problem (1.6.1). Let $\Gamma_0 := \{ x \in \Gamma \mid 0 < x_1 < d \}$.

Define the single-layer and double-layer potentials as

$$u(x) = \int_{\Gamma_0} \Phi^{qp}(x,y)\, \varphi(y)\, ds_y, \qquad x \in \Omega_0,$$

$$v(x) = \int_{\Gamma_0} \frac{\partial \Phi^{qp}(x,y)}{\partial n_y}\, \varphi(y)\, ds_y, \qquad x \in \Omega_0.$$

n_y: unit normal direction pointing to Ω.

Noting that $\Phi^{qp}(x,y) = \sum_{n=-\infty}^{\infty} \Phi_k(x, y^{(n)})\, e^{i\alpha n d}$, by a change of variable, we obtain

$$u(x) = \int_{\Gamma} \Phi_k(x,y)\, \tilde\varphi(y)\, ds_y,$$

where $\tilde\varphi(y)$ is the quasi-periodic extension of $\varphi(y)$ to the whole boundary Γ,

or $\tilde\varphi(y) = e^{i\alpha n d} \varphi(y_0)$, $y \in \Gamma_0 + (nd, 0)$, $y_0 = y - (nd,0) \in \Gamma_0$.

Therefore, $u(x)$ can be extended continuously to the boundary Γ_0.

Similarly, $v(x)$ can be written as $v(x) = \int_{\Gamma} \dfrac{\partial \Phi_k(x,y)}{\partial n_y} \tilde\varphi(y)\, ds_y$, and we obtain the usual jump relation:

$$v_\pm(x_0) = \lim_{x \to x_0 \in \Gamma_0} v(x) = \int_{\Gamma_0} \frac{\partial \Phi^{qp}(x_0,y)}{\partial n_y}\, \varphi(y)\, ds_y \pm \frac{1}{2}\varphi(x_0).$$

For the periodic problem (1.6.1), now we can formulate the integral equation

$$S\varphi = -u^i \quad \text{or} \quad \frac{1}{2}\varphi + K\varphi = -u^i \quad \text{on } \Gamma_0$$

by expressing the scattered field u^s as the single or double layer potential.

In the above,
$$[S\varphi](x) := \int_{\Gamma_0} \Phi^{qp}(x,y)\,\varphi(y)\,ds_y, \quad x \in \Gamma_0.$$
$$[K\varphi](x) := \int_{\Gamma_0} \frac{\partial \Phi^{qp}(x,y)}{\partial n_y}\,\varphi(y)\,ds_y, \quad x \in \Gamma_0.$$

§1.6.2 Computation of periodic Green's function

Accelerated computation by Kummer's transformation

Take the spectral decomposition of the Green's function
$$G(x_1,x_2) = \frac{i}{2d}\sum_{n=-\infty}^{\infty} \frac{1}{\beta_n} e^{i\alpha_n x_1 + i\beta_n |x_2|}.$$

The Kummer's transformation seeks to decompose $G(x_1,x_2)$ as

$G(x_1,x_2) = \underline{\text{the sum of slowly convergent series}} + \text{the sum of fast convergent series}$

Find equivalent analytic expressions or accelerated computation

To do that, we do the expansion of mode for $n \gg 1$. Let us set $d = 2\pi$ for simplicity.

$$\beta_n = i\sqrt{\alpha_n^2 - k^2} = i\sqrt{(\alpha+n)^2 - k^2} = i|n|\sqrt{(1+\frac{\alpha}{n})^2 - \frac{k^2}{n^2}} = i|n|\left(1 + \frac{\alpha}{n} - \frac{k^2}{2n^2} + O(\frac{1}{n^3})\right).$$

$$\frac{1}{\beta_n} = \frac{1}{i|n|}\left(1 - \frac{\alpha}{n} + \frac{k^2 + 2\alpha^2}{2n^2} + O(\frac{1}{n^3})\right).$$

$$e^{i\beta_n|x_2|} = e^{-|n|\left(1+\frac{\alpha}{n} - \frac{k^2}{2n^2} + O(\frac{1}{n^3})\right)|x_2|} = e^{-(|n| + \text{sign}(n)\alpha)|x_2|} \cdot e^{\frac{k^2}{2n^2}|x_2|} \cdot \left(1 + O(\frac{1}{n^2})\right).$$
$$= e^{-(|n| + \text{sign}(n)\alpha)|x_2|} \cdot \left(1 + \frac{k^2}{2|n|}|x_2| + O(\frac{1}{n^2})\right).$$

$$\Rightarrow \frac{i}{\beta_n} e^{i\alpha_n x_1 + i\beta_n|x_2|} = e^{i\alpha_n x_1 - |n x_2| - \text{sign}(n)\alpha|x_2|} \cdot \frac{1}{|n|}\left(1 - \frac{\alpha}{n} + \frac{k^2}{2|n|}|x_2| + O(\frac{1}{n^2})\right).$$

Define
$$G_1(x_1,x_2) := \frac{i}{2d}\sum_{n\neq 0} e^{i\alpha_n x_1 - |n x_2| - \text{sign}(n)\alpha|x_2|} \cdot \left(\frac{1}{|n|} - \frac{\alpha}{n|n|} + \frac{k^2}{2n^2}|x_2|\right)$$

$$G_2(x_1,x_2) = G(x_1,x_2) - G_1(x_1,x_2).$$

Then $G_2(x_1,x_2) = \sum_{n=-\infty}^{\infty} a_n(x_1,x_2)$, where $|a_n| = O(\frac{1}{n^3})$ for $n \gg 1$.

$$G_1(x_1,x_2) = \frac{1}{2d}\sum_{n=1}^{\infty} e^{i\alpha_n x_1 - n|x_2| - \alpha|x_2|}\left(\frac{1}{n} + \frac{k^2|x_2| - 2\alpha}{2n^2}\right) + \frac{1}{2d}\sum_{n=-1}^{-\infty} e^{i\alpha_n x_1 + n|x_2| + \alpha|x_2|}\left(-\frac{1}{n} + \frac{k^2 \cdots}{2n \cdots}\right.$$

$$= \frac{1}{2d} e^{idx_1 - d|x_2|} \sum_{n=1}^{\infty} \left[e^{ix_1 - |x_2|}\right]^n \left(\frac{1}{n} + \frac{1}{2n^2}\left(k^2|x_2| - 2d\right)\right)$$
$$+ \frac{1}{2d} e^{idx_1 + d|x_2|} \sum_{n=1}^{\infty} \left[e^{-ix_1 - |x_2|}\right]^n \left(\frac{1}{n} + \frac{1}{2n^2}\left(k^2|x_2| + 2d\right)\right).$$

Introduce the polylogarithm function $Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$. In particular, $Li_1(z) = -\log(1-z)$ and $Li_2(z)$ can be computed efficiently and with high-order accuracy for $|z| \le 1, z \ne 1$.

Then $G_1(x_1, x_2)$ can be written as

$$G_1(x_1, x_2) = \frac{1}{2d} e^{idx_1 - d|x_2|} \left[Li_1\left(e^{ix_1 - |x_2|}\right) + Li_2\left(e^{ix_1 - |x_2|}\right)\left(k^2|x_2| + 2d\right)\right]$$

$$+ \frac{1}{2d} e^{idx_1 - d|x_2|} \left[Li_1\left(e^{-ix_1 - |x_2|}\right) + Li_2\left(e^{-ix_1 - |x_2|}\right)\left(k^2|x_2| + 2d\right)\right].$$

Ewald representation ($d = 2\pi$)

Semigroup. Consider the time-dependent problem

$$\begin{cases} u'(t) = L u(t), & t \ge 0 \\ u(0) = u_0. \end{cases}$$

where $u: [0, \infty) \to$ Hilbert space H

L: linear operator from H to H.

Then the solution $u(t)$ can be expressed as $u(t) = S(t) u_0 = e^{Lt} u_0$, where $S(t): H \to H$ is continuous.

$\{S(t)\}_{t \ge 0}$ is called a semigroup if (i) $S(0)u = u$, (ii) $S(t_1 + t_2) u = S(t_1) S(t_2) u = S(t_2) S(t_1) u$.

The operator L is called the generator of the semigroup $\{S(t)\}_{t \ge 0}$.

The following holds for resolvent $(\lambda - L)^{-1}$ and the associated semigroup $\{S(t)\}_{t \ge 0}$.

If $\lambda \in \rho(L)$, then $(\lambda - L)^{-1} u = \int_0^{\infty} e^{-\lambda t} \cdot S(t)\, u \, dt = \int_0^{\infty} e^{(L-\lambda)t}\, u \, dt$ (1.6.3)

Let us now consider the computation of the Green's function

Let us now consider the computation of the Green's function $G(x_1,x_2)$:

$$G(x_1,x_2) = \frac{i}{2d}\sum_{n=-\infty}^{\infty}\frac{1}{\beta_n}e^{i\alpha_n x_1 + i\beta_n |x_2|} =: \frac{1}{d}\sum_{n=-\infty}^{\infty} g_n(x_2)\, e^{i\alpha_n x_1},$$

where $g_n(x_2)$ solves $-g_n''(x_2) - (k^2-\alpha_n^2)\, g_n(x_2) = \delta(x_2)$.

Define
$$L := \frac{d^2}{dx_2^2}, \quad \lambda_n := \alpha_n^2 - k^2.$$

We have $(\lambda_n - L)\, g_n(x_2) = \delta(x_2)$, and the resolvent kernel is given by $g(x_2-y_2)$.

Namely,
$$(\lambda_n - L)^{-1}\varphi = \int_{-\infty}^{\infty} g_n(x_2-y_2)\,\varphi(y_2)\,dy_2. \qquad (1.6.4)$$

On the other hand, from the relation (1.6.3), it follows that

$$(\lambda_n - L)^{-1}\varphi = \int_0^{\infty} e^{(L-\lambda_n)t}\varphi\, dt = \int_0^{\infty}\int_{-\infty}^{\infty} g(t;x_2-y_2)\,\varphi(y_2)\,dy_2\, e^{-\lambda_n t}\,dt, \qquad (1.6.5)$$

where $g(t;x_2-y_2) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x_2-y_2)^2}{4t}}$ is the fundamental solution for the time-dependent operator $\frac{\partial}{\partial t} - L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x_2^2}$.

(1.6.4) and (1.6.5) imply that
$$\boxed{g_n(x_2) = \int_0^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\lambda_n t - \frac{x_2^2}{4t}}\,dt}$$

Therefore, the quasi-periodic Green's function

$$G(x_1,x_2) = \frac{1}{d}\sum_{n=-\infty}^{\infty} e^{i\alpha_n x_1}\cdot\int_0^{\infty}\frac{1}{\sqrt{4\pi t}}e^{-\lambda_n t - \frac{x_2^2}{4t}}\,dt$$

$$= \frac{1}{d}\sum_{n=-\infty}^{\infty} e^{i\alpha_n x_1}\cdot\int_0^{E^2}\frac{1}{\sqrt{4\pi t}}e^{-\lambda_n t - \frac{x_2^2}{4t}}\,dt + \frac{1}{d}\sum_{n=-\infty}^{\infty} e^{i\alpha_n x_1}\int_{E^2}^{+\infty}\frac{1}{\sqrt{4\pi t}}e^{-\lambda_n t - \frac{x_2^2}{4t}}\,dt$$

$$=: G_1(x_1,x_2) + G_2(x_1,x_2),$$

where E is a constant.

First,
$$\int_{E^2}^{+\infty}\frac{1}{\sqrt{4\pi t}}e^{-\lambda_n t - \frac{x_2^2}{4t}}\,dt = \frac{1}{\sqrt{\pi}}\int_{E^2}^{+\infty} e^{-\left(\sqrt{\lambda_n t} + \frac{x_2}{2\sqrt{t}}\right)^2}\frac{dt}{2\sqrt{t}}\cdot e^{\sqrt{\lambda_n}x_2}$$

$$\overset{\boxed{\tau=\sqrt{t}}}{=} \frac{1}{\sqrt{\pi}}\int_E^{+\infty} e^{-\left(\sqrt{\lambda_n}\tau + \frac{x_2}{2\tau}\right)^2}d\tau\cdot e^{\sqrt{\lambda_n}x_2}$$

$\boxed{s = \sqrt{\lambda_n}\tau + \frac{x_2}{2\tau}}$ $e^{\sqrt{\lambda_n}x_2}\Big(\int^{+\infty} e^{-s^2} \ldots - e^{-2\sqrt{\lambda_n}x_2}\int e^{-s^2} \ldots\Big)$

$$\boxed{\text{Set} \quad s_2 = \sqrt{\lambda_n}\,\tau - \frac{x_2}{2\tau}} \quad = \frac{\cdots}{2\sqrt{\lambda_n}\cdot\sqrt{\pi}}\left(\int_{\sqrt{\lambda_n}E + \frac{x_2}{2E}} e^{\cdots}\, ds_1 + e^{\cdots} \int_{\sqrt{\lambda_n}E - \frac{x_2}{2E}} e^{\cdots}\, ds_2\right)$$

$$= \frac{e^{\sqrt{\lambda_n}x_2}}{4\sqrt{\lambda_n}}\,\mathrm{erfc}\left(\sqrt{\lambda_n}E + \frac{x_2}{2E}\right) + \frac{e^{-\sqrt{\lambda_n}x_2}}{4\sqrt{\lambda_n}}\,\mathrm{erfc}\left(\sqrt{\lambda_n}E - \frac{x_2}{2E}\right),$$

where $\mathrm{erfc}(z) = 1 - \mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_z^{+\infty} e^{-s^2}\, ds.$

$$\Rightarrow G_2(x_1, x_2) = \frac{1}{d}\sum_{n=-\infty}^{\infty} \frac{e^{i\alpha_n x_1}}{4\sqrt{\lambda_n}}\Big[\underbrace{e^{\sqrt{\lambda_n}x_2}\,\mathrm{erfc}\left(\sqrt{\lambda_n}E + \frac{x_2}{2E}\right) + e^{-\sqrt{\lambda_n}x_2}\,\mathrm{erfc}\left(\sqrt{\lambda_n}E - \frac{x_2}{2E}\right)}_{\text{The coefficient decays exponentially w.r.t } n.}\Big]$$

To compute $G_1(x_1, x_2)$ efficiently, let us introduce the theta function

$$V(z, q) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2 q + i2\pi n z}$$

the following Jacobi identity holds: $V\left(\frac{z}{q}, -\frac{1}{q}\right) = \sqrt{-iq}\, e^{i\pi\frac{z^2}{q}}\, V(z, q)$. (1.6.6)

$$G_1(x_1, x_2) = \frac{1}{d}\sum_{n=-\infty}^{\infty} e^{i\alpha_n x_1}\int_0^{E^2} \frac{1}{\sqrt{4\pi t}}\, e^{-\lambda_n t - \frac{x_2^2}{4t}}\, dt$$

$$= \frac{e^{i\alpha x_1}}{d}\int_0^{E^2} \frac{1}{\sqrt{4\pi t}}\, e^{k^2 t - \frac{x_2^2}{4t}} \underbrace{\boxed{\sum_{n=-\infty}^{\infty} e^{inx_1}\cdot e^{-(n+\alpha)^2 t}}}_{\text{theta function}}\, dt$$

From (1.6.6), it can be shown that

$$\sum_{n=-\infty}^{\infty} e^{inx_1}\, e^{-(n+\alpha)^2 t} = \sqrt{\frac{\pi}{t}}\, e^{-i\alpha x_1 - \frac{x_1^2}{4t}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi^2 n^2}{t} + \frac{n\pi x_1}{t} + i2\pi\alpha n} \quad (1.6.7)$$

$$= \sqrt{\frac{\pi}{t}}\, e^{-i\alpha x_1} \sum_{n=-\infty}^{\infty} e^{-\frac{(x_1 - 2\pi n)^2}{4t}}\cdot e^{i2\pi\alpha n}$$

It follows that $G_1(x_1, x_2) = \sum_{n=-\infty}^{\infty} e^{i2\pi\alpha n}\underbrace{\int_0^{E} e^{k^2 t - \frac{(x_1 + 2\pi n)^2 + x_2^2}{4t}}\cdot\frac{1}{4\pi t}\, dt}_{\text{decays exponentially w.r.t. } n.}$

Exercises. 1. Show that $\sum_{n=1}^{\infty} \frac{z^n}{n} = -\log(1-z)$ for $|z| \le 1$ and $z \ne 1$.

2. Show that (1.6.7) holds by using the Jacobi identity (1.6.6).