§2.3 General properties of the spectrum for Neumann-Poincaré operator

<u>Theorem 2.3.1</u> If λ_n is an eigenvalue of K' in $L_0^2(\Gamma)$, then $-\lambda_n$ is also an eigenvalue of K'.

<u>proof</u> By Thm 2.2.1, if λ_n is an eigenvalue of K' in $L_0^2(\Gamma)$, & φ_n is the eigenfunction.

Then $u(x)=\int_\Gamma \Phi(x,y)\varphi_n(y)\,ds_y$ solves
$$\begin{cases}\Delta u=0 \text{ in } \Omega_\pm \\ [u]=0 \text{ on } \Gamma \\ \left[\varepsilon\frac{\partial u}{\partial n}\right]=0 \text{ on } \Gamma\end{cases} \quad \text{where } \varepsilon=\begin{cases}\gamma_n & \text{in } \Omega_-, \\ 1 & \text{in } \Omega_+.\end{cases} \quad \text{and } \lambda_n=\frac{1}{2}\frac{1+\gamma_n}{1-\gamma_n}.$$

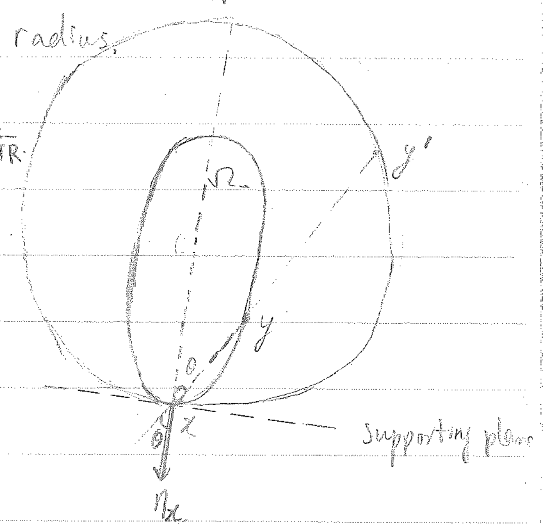
$\vec{E}=-\nabla u$ is the electric field. $\vec{D}=-\varepsilon\nabla u$ is the electric displacement.

From $\nabla\cdot\vec{D}=0$, $\exists$ a function V s.t. $\vec{D}=\left[-\frac{\partial V}{\partial x_2}, \frac{\partial V}{\partial x_1}\right]$ in $\Omega_\pm$.

$$\Rightarrow \begin{cases}\Delta V=0 \text{ in } \Omega_\pm, \\ [V]=0 \text{ on } \Gamma, \\ \left[\frac{1}{\varepsilon}\frac{\partial V}{\partial n}\right]=0 \text{ on } \Gamma\end{cases}$$
From Thm 2.2.1, $\exists$ eigenvalue $\tilde{\lambda}_j$ of K' s.t. $\tilde{\lambda}_n=\frac{1}{2}\frac{1+\frac{1}{\gamma_n}}{1-\frac{1}{\gamma_n}}=-\lambda_n$.

[Mayergoyz, 2013]

<u>Theorem 2.3.2</u> If Ω_- is convex, there holds $|\lambda|\le\frac{1}{2}\left(1-\frac{|\Gamma|}{2\pi R}\right)$ for the eigenvalue of K' in $L_0^2(\Gamma)$, where R is the curvature radius.



<u>Proof</u>
$$\frac{(x-y)\cdot n_x}{|x-y|^2}=\frac{\cos\theta}{|x-y|}=\frac{1}{|x-y|}\frac{|x-y'|}{2R}\ge\frac{1}{2R} \Rightarrow -\frac{\partial\Phi(x,y)}{\partial n_x}\ge\frac{1}{4\pi R}$$

Let λ be an eigenvalue of K' s.t.
$$(\lambda I-K')\varphi=0, \text{ and } \langle\varphi,1\rangle=0, \qquad (1)$$
$$\langle\varphi,1\rangle=0 \Rightarrow \int_{\Gamma_+}\varphi\,ds=-\int_{\Gamma_-}\varphi\,ds, \quad (2)$$

where $\Gamma_+=\{x\in\Gamma \mid \varphi(x)>0\}$ and $\Gamma_-=\{x\in\Gamma \mid \varphi(x)<0\}$

$$\overset{(1)(2)}{\Rightarrow} \lambda\int_\Gamma|\varphi(x)|\,ds_x=-2\lambda\int_{\Gamma_-}\varphi(x)\,ds=-2\int_{\Gamma_-}K'\varphi\,ds_x \qquad (3)$$

$$-2\int_{\Gamma_-}K'\varphi\,ds_x=-2\int_{\Gamma_-}\int_\Gamma\frac{\partial\Phi(x,y)}{\partial n_x}\varphi(y)\,ds_y\,ds_x=-2\int_\Gamma\int_{\Gamma_-}\frac{\partial\Phi(x,y)}{\partial n_x}ds_x\,\varphi(y)\,ds_y$$
$$=2\int_{\Gamma_+}\int_{\Gamma_-}\underbrace{-\frac{\partial\Phi(x,y)}{\partial n_x}}_{>0}ds_x\,\varphi(y)\,ds_y+2\underbrace{\int_{\Gamma_-}\int_{\Gamma_-}-\frac{\partial\Phi(x,y)}{\partial n_x}ds_x\,\varphi(y)\,ds_y}_{<0} \qquad (4)$$

using $\int_\Gamma \frac{\partial \Phi(x,y)}{\partial n_x} ds_x = -\frac{1}{2}$

we obtain

$$2\int_{\Gamma_+}\int_{\Gamma_-} -\frac{\partial \Phi(x,y)}{\partial n_x} ds_x \, ds_y = \int_{\Gamma_+} \varphi(y) ds_y - 2\int_{\Gamma_+}\int_{\Gamma_+} -\frac{\partial \Phi(x,y)}{\partial n_x} ds_x \, \varphi(y) ds_y$$

$$\leq \int_{\Gamma_+} \varphi(y) ds_y - 2 \cdot \frac{|\Gamma_+|}{4\pi R} \int_{\Gamma_+} \varphi(y) ds_y$$

$$= \left(\frac{1}{2} - \frac{|\Gamma_+|}{4\pi R}\right) \int_\Gamma |\varphi(y)| ds_y \qquad (5)$$

$$2\int_{\Gamma_-}\int_{\Gamma_-} -\frac{\partial \Phi(x,y)}{\partial n_x} ds_x \, \varphi(y) ds_y \leq 2 \frac{|\Gamma_-|}{4\pi R} \int_{\Gamma_-} \varphi(y) ds_y = -\frac{|\Gamma_-|}{4\pi R} \int_\Gamma |\varphi(y)| ds_y \qquad (6)$$

$(3)-(6) \Rightarrow \lambda \leq \frac{1}{2} - \frac{|\Gamma|}{4\pi R} = \frac{1}{2}\left(1 - \frac{|\Gamma|}{2\pi R}\right).$

Theorem 2.3.3 Let Γ be of C^α $(\alpha \geq 2)$, then for any $\beta > \frac{3}{2} - \alpha$, the following holds for the eigenvalue of K' in $L_0^2(\Gamma)$: $\lambda_n = o(n^\beta)$.

In particular, $\lambda_n = O(n^{-\infty})$ if Γ is smooth.

The proof is given in [Miyanishi & Suzuki, 2017].