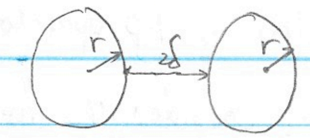
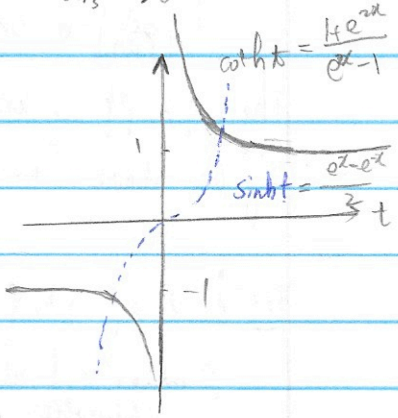
Two nano particles

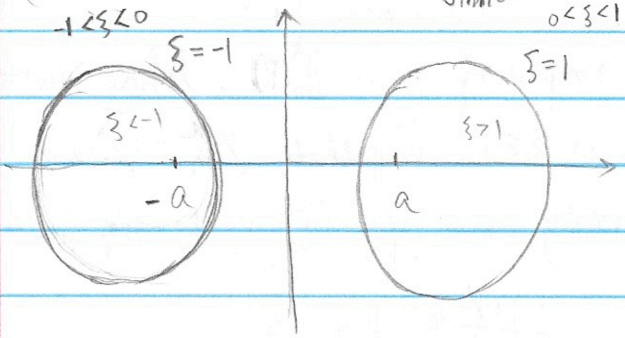


$\boxed{\text{Bipolar coordinate}}$ $x_1 = a\dfrac{\sinh\xi}{\cosh\xi-\cos\theta}$, $x_2 = a\dfrac{\sin\theta}{\cosh\xi-\cos\theta}$, $\xi\in\mathbb{R}$, $\theta\in(-\pi,\pi]$



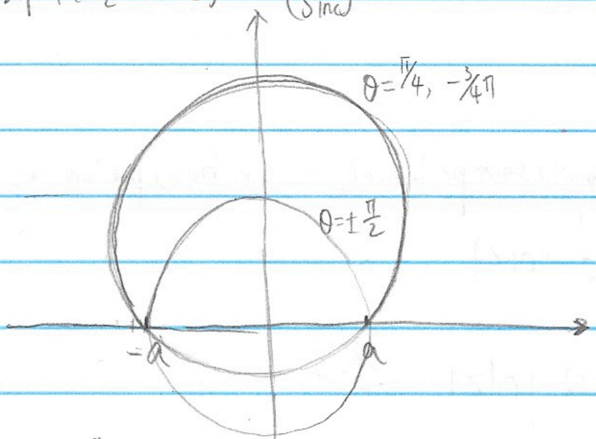
The curve $\{\xi = c\}$ is given by

$$(x_1 - a\coth c)^2 + x_2^2 = \left(\frac{a}{\sinh c}\right)^2$$



Let $c>0$, The curve $\{\theta=c\}\cup\{\theta=c-\pi\}$ is given by

$$x_1^2+(x_2-\cot c)^2 = \left(\frac{a}{\sin c}\right)^2$$

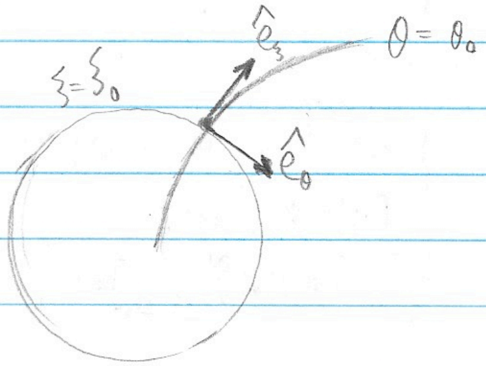


define the scaling factor $h(\xi,\theta)=\dfrac{\cosh\xi-\cos\theta}{a}$

$x_1=\dfrac{\sinh\xi}{h}$, $x_2=\dfrac{\sin\theta}{h}$. $\quad \dfrac{\partial x_1}{\partial\xi}=\dfrac{1}{a}\,\dfrac{1-\cosh\xi\cdot\cos\theta}{h^2}$, $\quad \dfrac{\partial x_2}{\partial\xi}=\dfrac{1}{a}\,\dfrac{-\sinh\xi\cdot\sin\theta}{h^2}$

$\dfrac{\partial x_1}{\partial\theta}=\dfrac{1}{a}\,\dfrac{-\sinh\xi\cdot\sin\theta}{h^2}$, $\quad \dfrac{\partial x_2}{\partial\theta}=\dfrac{1}{a}\,\dfrac{\cosh\xi\cdot\cos\theta-1}{h^2}$

or $\dfrac{\partial(x_1,x_2)}{\partial(\xi,\theta)}=\dfrac{a}{h^2}\begin{bmatrix}1-\cosh\xi\cdot\cos\theta & -\sinh\xi\cdot\sin\theta\\ -\sinh\xi\cdot\sin\theta & \cosh\xi\cdot\cos\theta-1\end{bmatrix} \Rightarrow \dfrac{\partial(\xi,\theta)}{\partial(x_1,x_2)}=\dfrac{1}{a}\begin{bmatrix}1-\cosh\xi\cos\theta & -\sinh\xi\cdot\sin\theta\\ -\sinh\xi\cdot\sin\theta & \cosh\xi\cos\theta-1\end{bmatrix}.$



$=: h\,[\hat{e}_\xi, \hat{e}_\theta]$,

where $\|\hat{e}_\xi\|=\|\hat{e}_\theta\|=1$

$$\nabla_x u=\frac{\partial(\xi,\theta)}{\partial(x_1,x_2)}\nabla_{\xi,\theta}u=\begin{bmatrix}\frac{\partial\xi}{\partial x_1} & \frac{\partial\theta}{\partial x_1}\\ \frac{\partial\xi}{\partial x_1} & \frac{\partial\theta}{\partial x_2}\end{bmatrix}\begin{bmatrix}\frac{\partial u}{\partial\xi}\\ \frac{\partial\theta}{\partial\theta}\end{bmatrix}=h\left(\frac{\partial u}{\partial\xi}\hat{e}_\xi+\frac{\partial u}{\partial\theta}\hat{e}_\theta\right)$$

$$\left.\frac{\partial u}{\partial n}\right|_{\xi=c}=-\operatorname{sign}(c)\,\nabla u\cdot\hat{e}_\xi=-\operatorname{sign}(c)\,h(c,\theta)\left.\frac{\partial u}{\partial\xi}\right|_{\xi=c}$$

$$\Delta_x u=h^2\left(\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\theta^2}\right)$$

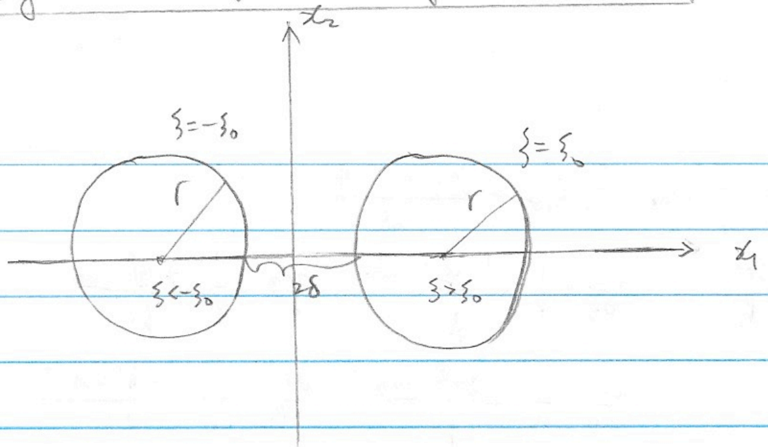
$\dfrac{\partial u}{\partial x_1}=\dfrac{\partial u}{\partial\xi}\dfrac{\partial\xi}{\partial x_1}+\dfrac{\partial u}{\partial\theta}\dfrac{\partial\theta}{\partial x_1}$, $\quad \dfrac{\partial^2 u}{\partial x_1^2}=\left(\dfrac{\partial^2 u}{\partial\xi^2}\dfrac{\partial\xi}{\partial x_1}+\dfrac{\partial^2 u}{\partial\xi\partial\theta}\dfrac{\partial\theta}{\partial x_1}\right)\dfrac{\partial\xi}{\partial x_1}+\left(\dfrac{\partial^2 u}{\partial\theta\partial\xi}\dfrac{\partial\xi}{\partial x_1}+\dfrac{\partial^2 u}{\partial\theta^2}\dfrac{\partial\theta}{\partial x_1}\right)\dfrac{\partial\theta}{\partial x_1}+\dfrac{\partial u}{\partial\xi}\dfrac{\partial^2\xi}{\partial x_1^2}+\dfrac{\partial u}{\partial\theta}\dfrac{\partial^2\theta}{\partial x_1^2}$

$\dfrac{\partial^2 u}{\partial x_2^2}=\left(\dfrac{\partial^2 u}{\partial\xi^2}\dfrac{\partial\xi}{\partial x_2}+\dfrac{\partial^2 u}{\partial\xi\partial\theta}\dfrac{\partial\theta}{\partial x_2}\right)\dfrac{\partial\xi}{\partial x_2}+\left(\dfrac{\partial^2 u}{\partial\theta\partial\xi}\dfrac{\partial\xi}{\partial x_2}+\dfrac{\partial^2 u}{\partial\theta^2}\dfrac{\partial\theta}{\partial x_2}\right)\dfrac{\partial\theta}{\partial x_2}+\dfrac{\partial u}{\partial\xi}\dfrac{\partial^2\xi}{\partial x_2^2}+\dfrac{\partial u}{\partial\theta}\dfrac{\partial^2\theta}{\partial x_2^2}$

① Sum of "$\sim$" $=0$. ② $\Delta\xi=\Delta\theta=0$

$$\Rightarrow \Delta u=h^2\left(\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\theta^2}\right)$$

$\boxed{\text{Eigenvalues \& Eigenfunctions of the } N_p \text{ operator}}$



$$\left.\begin{array}{l} \dfrac{a}{\sinh\xi_0} = r \Rightarrow \xi_0 = \sinh^{-1}\dfrac{a}{r} \\[2ex] a\coth\xi_0 = r+\delta \Rightarrow a\,\dfrac{r}{a}\cdot\cosh\xi_0 = r+\delta \Rightarrow \cosh\xi_0 = \dfrac{r+\delta}{r} \end{array}\right\} \quad a=\sqrt{(r+\delta)^2-r^2}$$

$$\begin{cases} \Delta u = 0 \quad \text{in } B_-\cup B_1\cup\left(\mathbb{R}^2\setminus\overline{B_-\cup B_2}\right) \\ [u]=0 \quad \text{on } \partial B_-\cup\partial B_+ \\ \left[\gamma\dfrac{\partial u}{\partial n}\right]=0 \quad \text{on } \partial B_-\cup\partial B_+ \end{cases} \qquad \begin{cases} \Delta u = 0 \quad \text{in } \{(\xi,\theta) \mid \xi\neq\pm\xi_0,\ \theta\in(-\pi,\pi]\} \\ [u]=0, \quad \xi=\pm\xi_0 \\ \left[\gamma\cdot \operatorname{sign}(\pm\xi_0)\dfrac{\partial u}{\partial\xi}\Big|_{\xi=\pm\xi_0}\right]=0, \quad \xi=\pm\xi_0 \end{cases}$$

$$u(\xi,\theta) = \begin{cases} a_0^{(1)}+b_0^{(1)}\xi+c_0^{(1)}\theta + \sum_{n\neq0}\left[C_{n,-}^{(1)}e^{-|n|(\xi+\xi_0)} + C_{n,+}^{(1)}e^{|n|(\xi-\xi_0)}\right]e^{in\theta}, & -\xi_0<\xi<\xi_0 \\ a_0^{(2)}+b_0^{(2)}\xi+c_0^{(2)}\theta + \sum_{n\neq0} C_{n,+}^{(2)}e^{|n|(\xi+\xi_0)}e^{in\theta}, & \xi<-\xi_0, \\ a_0^{(3)}+b_0^{(3)}\xi+c_0^{(3)}\theta + \sum_{n\neq0} C_{n,-}^{(3)}e^{-|n|(\xi-\xi_0)}e^{in\theta}, & \xi>\xi_0. \end{cases}$$

For each mode, the normal derivative

$$\begin{cases} \dfrac{\partial u_n^+}{\partial\xi}\Big|_{\xi=-\xi_0} = -|n|\,C_{n,-}^{(1)} + |n|e^{-2n\xi_0}C_{n,+}, & \dfrac{\partial u_n^-}{\partial\xi}\Big|_{\xi=-\xi_0} = n\,C_{n,+}^{(2)}, \\[2ex] \dfrac{\partial u_n^+}{\partial\xi}\Big|_{\xi=\xi_0} = -n\,e^{-2n\xi_0}C_{n,-}^{(1)} + |n|\,C_{n,+}^{(1)}, & \dfrac{\partial u_n^-}{\partial\xi}\Big|_{\xi=\xi_0} = -n\,C_{n,-}^{(3)} \end{cases}$$

The continuity relation gives

$$\begin{cases} C_{n,-}^{(1)} + e^{-2|n|\xi_0}C_{n,+}^{(1)} = C_{n,+}^{(2)} \\ e^{-2|n|\xi_0}C_{n,-}^{(1)} + C_{n,+}^{(1)} = C_{n,-}^{(3)} \end{cases} \quad \& \quad \begin{cases} -C_{n,-}^{(1)} + e^{-2|n|\xi_0}C_{n,+}^{(1)} = \gamma\, C_{n,+}^{(2)} \\ -e^{-2|n|\xi_0}C_{n,-}^{(1)} + C_{n,+}^{(1)} = -\gamma\, C_{n,-}^{(3)} \end{cases}$$

$$\Rightarrow \begin{cases} 2C^{(1)}_{n,-} = (1-\nu)\, C^{(2)}_{n,+}, & 2e^{-2|n|\xi_0} C^{(1)}_{n,-} = (1+\nu)\, C^{(3)}_{n,-} \\ 2C^{(1)}_{n,+} = (1-\nu)\, C^{(3)}_{n,-}, & 2e^{-2|n|\xi_0} C^{(1)}_{n,+} = (1+\nu)\, C^{(2)}_{n,+} \end{cases} \Rightarrow \begin{cases} e^{-2|n|\xi_0}(1-\nu)\, C^{(2)}_{n,+} = (1+\nu)\, C^{(3)}_{n,-} \\ e^{-2|n|\xi_0}(1-\nu)\, C^{(3)}_{n,-} = (1+\nu)\, C^{(2)}_{n,+} \end{cases}$$

$$\Rightarrow e^{-4|n|\xi_0}(1-\nu)^2 = (1+\nu)^2 \quad \text{or} \quad \frac{1+\nu}{1-\nu} = \pm e^{-2|n|\xi_0} \Rightarrow \boxed{\lambda_n^{\pm} = \pm \frac{1}{2} e^{-2|n|\xi_0}}$$

$$e^{-2|n|\xi_0}\frac{C^{(1)}_{n,+}}{C^{(1)}_{n,-}} = \frac{1+\nu}{1-\nu} = 2\lambda_n^{\pm} = \pm e^{-2|n|\xi_0} \Rightarrow C^{(1)}_{n,+} = \pm C^{(1)}_{n,-}, \text{ set } C^{(1)}_{n,-} = 1,\ C^{(1)}_{n,+} = \pm 1$$

$$C^{(2)}_{n,+} = \frac{2}{1-\nu} C^{(1)}_{n,-} = 1 \pm e^{-2|n|\xi_0}, \quad C^{(3)}_{n,-} = \frac{2}{1-\nu} C^{(1)}_{n,+} = \pm\left(1 \pm e^{-2|n|\xi_0}\right) = e^{-2|n|\xi_0} \pm 1.$$

$$\Rightarrow u_n^{\pm} = \begin{cases} \left(e^{-|n|(\xi+\xi_0)} \pm e^{|n|(\xi-\xi_0)}\right) e^{in\theta} & -\xi_0 < \xi < \xi_0, \\ \left(e^{n\xi_0} \pm e^{-n\xi_0}\right) e^{n\xi} e^{in\theta}, & \xi < -\xi_0, \\ \left(e^{-n\xi_0} \pm e^{n\xi_0}\right) e^{-n\xi} e^{in\theta}, & \xi > \xi_0. \end{cases}$$

$$\Rightarrow \begin{cases} \varphi^{\pm}_{n,1} = -\frac{\partial u^-}{\partial n} - \frac{\partial u^+}{\partial n} = h\left(\frac{\partial u^-}{\partial \xi} - \frac{\partial u^+}{\partial \xi}\right) = 2|n|\, h(-\xi_0,\theta)\, e^{in\theta}, & \xi = -\xi_0, \\ \varphi^{\pm}_{n,2} = \frac{\partial u^-}{\partial n} - \frac{\partial u^+}{\partial n} = h\left(\frac{\partial u^+}{\partial \xi} - \frac{\partial u^-}{\partial \xi}\right) = \mp 2|n|\, h(\xi_0,\theta)\, e^{in\theta}, & \xi = \xi_0. \end{cases}$$

$$\boxed{\varphi_n^{\pm} = \begin{bmatrix} \varphi^{\pm}_{n,1} \\ \varphi^{\pm}_{n,2} \end{bmatrix} = 2|n| \begin{bmatrix} h(-\xi_0,\theta)\, e^{in\theta} \\ \mp h(\xi_0,\theta)\, e^{in\theta} \end{bmatrix}}$$

eigenfunctions of K'.