§2.2 Spectra of NP operator for Nano-particles at special geometries

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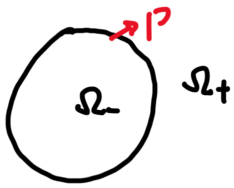
Theorem 2.2.1 Consider the homogeneous problem

$$\begin{cases} \Delta u = 0 \text{ in } \Omega_+ \cup \Omega_- \\ [u] = 0 \text{ on } \Gamma \\ [\varepsilon \frac{\partial u}{\partial n}] = 0 \text{ on } \Gamma \\ u(x) \to 0 \text{ as } |x| \to \infty \end{cases} \quad \text{where } \varepsilon(x) = \begin{cases} 1, & x \in \Omega^+, \\ \gamma, & x \in \Omega^-. \end{cases}$$



Then $\{\lambda, \varphi\}$ is an eigenpair of K' in $H_0^*(\Gamma)$ if and only if $\{\gamma, u\}$ is a solution of the above homogeneous problem, where $\lambda = \frac{1}{2}\frac{1+\gamma}{1-\gamma}$ and $\varphi = \frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n}$.

proof $\Rightarrow$ If $\{\lambda, \varphi\}$ is an eigenpair of K', define $u(x) = \int_\Gamma \Phi(x,y)\varphi(y)\, ds_y$ for $x \notin \Gamma$.

First $\varphi = \frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n}$ follows from the jump relation in Theorem 1.2.3. Now

$$(\lambda - K')\varphi = \frac{1}{2}\frac{1+\gamma}{1-\gamma}\varphi - K'\varphi = 0$$

$$\Rightarrow \frac{1}{2}(1+\gamma)\varphi - (1-\gamma)K'\varphi = 0 \Rightarrow \gamma\left(\tfrac{1}{2}\varphi + K'\varphi\right) = K'\varphi - \tfrac{1}{2}\varphi.$$

$$\Rightarrow \gamma\frac{\partial u_-}{\partial n} = \frac{\partial u_+}{\partial n}.$$ Thus $\{\gamma, u\}$ solves the homogeneous problem.

$\Leftarrow$ If $\{\gamma, u\}$ solves the homogeneous problem, then by the Green's formula, we have

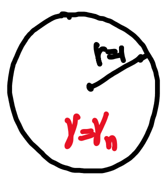
$$u(x) = \int_\Gamma \Phi(x,y)\varphi(y)\, ds_y, \text{ where } \varphi = \frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n}.$$

The continuity condition $\gamma\frac{\partial u^-}{\partial n} = \frac{\partial u^+}{\partial n}$ leads to the equation

$$\frac{1}{2}\frac{1+\gamma}{1-\gamma}\varphi - K'\varphi = 0$$

The assertion follows by setting $\lambda = \frac{1}{2}\frac{1+\gamma}{1-\gamma}$.

Circular nano-particle



The solution of the homogeneous boundary value problem can be expressed as

$$u(x) = \begin{cases} \sum_{n\neq 0} C_n^- r^{|n|} e^{in\theta} + C_0^-, & 0<r<1, \end{cases}$$

$$\Big| \sum_{n\neq 0} C_n^+ r^{-|n|} e^{in\theta}, \quad r>1.$$

Apply the Continuity condition on the boundary of disk gives $C_0^-=0$, and

$$\begin{cases} C_n^- = C_n^+, \\ \gamma_n |n| C_n^- = -|n| C_n^+ \end{cases} \Rightarrow \gamma_n = -1, \quad u_n(x) = \begin{cases} r^{|n|} e^{in\theta}, \\ r^{-|n|} e^{in\theta}. \end{cases} \quad n \neq 0$$

From Theorem 2.2.1 $\Rightarrow$ $\lambda_n = 0$ and $\varphi_n = e^{in\theta}$, $n = \pm 1, \pm 2, \dots$

Elliptical nano-particle

Apply the elliptical coordinate $x_1 = a\cosh\xi\cos\theta$, $x_2 = a\sinh\xi\sin\theta$. Note that

$$\frac{x_1^2}{(a\cosh\xi)^2} + \frac{x_2^2}{(a\sinh\xi)^2} = 1.$$

$\xi = \xi_0$ corresponds to an ellipse with semi-axes length $a\cosh\xi_0$ and $a\sinh\xi_0$, respectively. It can also be shown that $\Delta_{\xi,\theta} u = 0$. Thus by applying seperation of variables in the (ξ,θ) coordinate, the homogeneous boundary value problem can be solved.

In particular, for an ellipse with semi-axes length a_1 and a_2, it can be computed that the eigenvalues of the Neumann-poincare operator are

$$\lambda_n = \pm \frac{1}{2}\left(\frac{a_1 - a_2}{a_1 + a_2}\right)^n, \quad n = 1, 2, 3, \dots \qquad (2.2.1)$$

Exercise: show that the eigenvalues the N-P operator for an ellipse are given by (2.2.1).