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On the numerical solution of a hypersingular integral equation in scattering theory

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Abstract

We describe a fully discrete method for the numerical solution of the hypersingular integral equation arising from the combined double- and single-layer approach for the solution of the exterior Neumann problem for the two-dimensional Helmholtz equation in smooth domains. We develop an error analysis in a Hölder space setting with pointwise estimates and prove an exponential convergence rate for analytic boundaries and boundary data.

Keywords: Helmholtz equation; Neumann problem; Boundary integral equations; Hölder spaces; Trigonometric interpolation; Collocation methods

1. Introduction

The mathematical treatment of the scattering of time-harmonic acoustic or electromagnetic waves by an infinitely long cylindrical obstacle with a simply connected bounded cross-section $D \subset \mathbb{R}^2$ leads to exterior boundary value problems for the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D} \tag{1.1}$$

with wave number $k > 0$. In the subsequent analysis we denote by Γ the boundary of D and by ν the outward unit normal to Γ. For the time being, we assume that the boundary Γ is C^2. The field u is decomposed, $u = u^{\mathrm{i}} + u^{\mathrm{s}}$, into the given incident field u^{i}, which is assumed to be an entire solution to the Helmholtz equation, and the unknown scattered field u^{s}, which is required to satisfy the Sommerfeld radiation condition

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u^{\mathrm{s}}}{\partial r} - \mathrm{i}ku^{\mathrm{s}} \right) = 0, \quad r = |x|, \tag{1.2}$$

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uniformly in all directions. Depending on the physical nature of the scattering obstacle, the total field u has to satisfy a boundary condition on Γ. The Neumann condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma \tag{1.3}$$

in acoustics corresponds to scattering from a sound-hard obstacle whereas in electromagnetics it models scattering from a perfect conductor with the electromagnetic field H-polarized.

After renaming the unknown function, the scattering problem (1.1)–(1.3) is a special case of the following exterior Neumann problem: Given a function $g \in C^{0,\alpha}(\Gamma)$, $0 < \alpha < 1$, find a solution $u \in C^2(\mathbb{R}^2 \setminus \bar{D}) \cap C^{1,\alpha}(\mathbb{R}^2 \setminus D)$ to the Helmholtz equation which satisfies the Sommerfeld radiation condition and the boundary condition

$$\frac{\partial u}{\partial \nu} = g \quad \text{on } \Gamma. \tag{1.4}$$

For details of the uniqueness and the existence analysis for this boundary value problem we refer to [3, 4]. (We also use the notations of [3,4] in the subsequent analysis.) The uniqueness is ensured through the radiation condition via Rellich's lemma. The existence of a solution can be based on boundary integral equations. We denote the fundamental solution to the Helmholtz equation in $\mathbb{R}^2$ by

$$\Phi(x, y) := \tfrac{1}{4}\,\mathrm{i}\, H_0^{(1)}(k|x - y|),$$

where $H_0^{(1)}$ is the Hankel function of order zero and of the first kind. In order to arrive at a uniquely solvable integral equation, we seek the solution to the exterior Neumann problem in the form of a combined acoustic double- and single-layer potential

$$u(x) = \int_\Gamma \left\{ \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \mathrm{i}\eta\, \Phi(x, y) \right\} \varphi(y)\, \mathrm{d}s(y), \quad x \in \mathbb{R}^2 \setminus \bar{D}, \tag{1.5}$$

with unknown density $\varphi \in C^{1,\alpha}(\Gamma)$ and some real coupling parameter η. Then from the jump relations for single- and double-layer potentials it follows that (1.5) solves the exterior Neumann problem provided the density is a solution of the integral equation

$$T\varphi - \mathrm{i}\eta K'\varphi + \mathrm{i}\eta\varphi = 2g, \tag{1.6}$$

where K' and T denote the integral operators defined by

$$(K'\varphi)(x) := 2\int_\Gamma \frac{\partial \Phi(x, y)}{\partial \nu(x)}\, \varphi(y)\, \mathrm{d}s(y), \quad x \in \Gamma,$$

and

$$(T\varphi)(x) := 2\,\frac{\partial}{\partial \nu(x)} \int_\Gamma \frac{\partial \Phi(x, y)}{\partial \nu(y)}\, \varphi(y)\, \mathrm{d}s(y), \quad x \in \Gamma.$$

In our analysis we will also need the integral operator S defined by

$$(S\varphi)(x) := 2\int_{\Gamma} \Phi(x,y)\,\varphi(y)\,\mathrm{d}s(y), \quad x \in \Gamma.$$

The approach (1.5) for the solution of exterior boundary value problems for the Helmholtz equation was introduced independently by Leis [9], Brakhage and Werner [1], and Panich [15] in order to remedy the nonuniqueness deficiency of the classical integral equations when k is a so-called irregular wave number or internal resonance. Whereas for the exterior Dirichlet problem the combined double- and single-layer approach does not pose any difficulty in the discussion of the resulting integral equation of the second kind in the framework of the Riesz theory for compact operators, for the exterior Neumann problem technical difficulties arise due to the singular behavior of the normal derivative of the double-layer potential, i.e., the fact that the integral equation (1.6) contains the hypersingular operator T. This requires the use of regularization techniques in order to allow the application of the Riesz theory, as described for example in [3,4].

From a numerical point of view, since the regularizations (or its discretized versions) involve additional computational costs, it is preferable to use the unregularized equation (1.6) for numerical approximations. Despite the fact that the boundary integral equation (1.6) or modifications of it have been widely used in the literature for numerical approximations for a long time already (see [2, 8] among others), there seems to be a lack of a rigorous error and convergence analysis due to the hypersingular behavior of T. In this paper we describe a very efficient fully discrete method for the numerical solution of the hypersingular integral equation (1.6) for analytic boundary curves Γ by a quadrature method based on trigonometric interpolation including a convergence analysis which mimics a regularization procedure. After we gave a complete description of the corresponding numerical method based on the combined double- and single-layer approach (1.5) for the exterior Dirichlet problem in the monograph [4], we felt the need and obligation to give an analogous presentation for the exterior Neumann problem.

Our error and convergence analysis is closely related to the more general results on collocation methods for singular integral equations and pseudodifferential equations via trigonometric interpolation by McLean et al. [12, 13]. However, the convergence analysis for our fully discrete method is not included in the results of [13], since we consider a periodic pseudodifferential operator where the compact perturbation of the principal part is only marginally smoother than the principal part itself. A trigonometric Galerkin method for the hypersingular integral equation in the limiting case $k = 0$ of Laplace's equation has been considered by Rathsfeld et al. [18]. Our analysis contains a condensed and simplified version of the work of Mönch [14] who gave the first pointwise error analysis for the approximation of the hypersingular operator T for the Helmholtz equation via trigonometric interpolation and collocation. An error analysis for our numerical method in a Sobolev space setting can be worked out by means of the results in [7].

The plan of the paper is as follows. In Section 2 we give the parametrized form of the integral equation and describe an appropriate splitting of the various singularities. Then in Section 3 we present our fully discrete quadrature method based on trigonometric interpolation. The error and convergence analysis is carried out in Section 4. Section 5 of the paper concludes with a numerical example illustrating the fast convergence rate of our method for analytic data.

2. Parametrization of the integral equations

We proceed by describing the parametrization of the integral equation (1.6). From now on, we assume that the boundary curve Γ is analytic and given through

$$\Gamma = \{x(t) = (x_1(t), x_2(t)) \colon 0 \leqslant t \leqslant 2\pi\},$$

where $x \colon \mathbb{R} \to \mathbb{R}^2$ is analytic and 2π-periodic with $|x'(t)| > 0$ for all t, such that the orientation of Γ is counterclockwise. Using $H_1^{(1)} = -H_0^{(1)\prime}$, where $H_1^{(1)}$ denotes the Hankel function of order one and of the first kind, we see that the kernel H in

$$(K'\varphi)(x(t)) = \frac{1}{|x'(t)|} \int_0^{2\pi} H(t,\tau)\, \varphi(x(\tau))\, \mathrm{d}\tau \tag{2.1}$$

is given by

$$H(t,\tau) := \frac{\mathrm{i}k}{2}\, n(t)\cdot[x(\tau) - x(t)]\, \frac{H_1^{(1)}(k|x(t) - x(\tau)|)}{|x(t) - x(\tau)|}\, |x'(\tau)|$$

where we have set $n(t) := |x'(t)|\, \nu(x(t)) = (x_2'(t), -x_1'(t))$. We decompose the fundamental solution $H_0^{(1)} = J_0 + \mathrm{i}N_0$, and use the power series

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n} \tag{2.2}$$

for the Bessel function of order zero and

$$N_0(z) = \frac{2}{\pi}\left(\ln\frac{z}{2} + C\right) J_0(z) + \frac{2}{\pi} \sum_{n=1}^{\infty} \left\{\sum_{m=1}^{n} \frac{1}{m}\right\} \frac{(-1)^{n+1}}{(n!)^2} \left(\frac{z}{2}\right)^{2n} \tag{2.3}$$

for the Neumann function of order zero with Euler's constant $C = 0.57721\ldots$. From these series we can see that the kernel H can be written in the form

$$H(t,\tau) = H_1(t,\tau) \ln\left(4\sin^2\frac{t-\tau}{2}\right) + H_2(t,\tau),$$

where

$$H_1(t,\tau) := -\frac{k}{2\pi}\, n(t)\cdot[x(\tau) - x(t)]\, \frac{J_1(k|x(t) - x(\tau)|)}{|x(t) - x(\tau)|}\, |x'(\tau)|,$$

$$H_2(t,\tau) := H(t,\tau) - H_1(t,\tau) \ln\left(4\sin^2\frac{t-\tau}{2}\right)$$

turn out to be analytic with the diagonal terms

$$H_1(t,t) = 0, \qquad H_2(t,t) = \frac{1}{2\pi}\, \frac{n(t)\cdot x''(t)}{|x'(t)|}.$$

(Of course, $J_1 = -J_0'$ denotes the Bessel function of order one.)

For the parametrization of the hypersingular operator T we make use of the identity

$$T = \frac{\mathrm{d}}{\mathrm{d}s} S \frac{\mathrm{d}\varphi}{\mathrm{d}s} + k^2 \nu \cdot S(\nu\varphi) \tag{2.4}$$

for the normal derivative of the double-layer potential for densities $\varphi \in C^{1,\alpha}(\Gamma)$ which is due to Maue [11]. For a derivation of (2.4) we refer to [3, p. 57] for the three-dimensional case and to [6, p. 102] for the two-dimensional case with $k = 0$. The identity (2.4) indicates that we need a parametrization of S. From the expansions (2.2) and (2.3) we see that the kernel

$$M(t,\tau) := \frac{\mathrm{i}}{2} H_0^{(1)}(k|x(t) - x(\tau)|)$$

of

$$(S\varphi)(x(t)) = \int_0^{2\pi} M(t,\tau)|x'(\tau)|\,\varphi(x(\tau))\,\mathrm{d}\tau \tag{2.5}$$

can be expressed in the form

$$M(t,\tau) = M_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + M_2(t,\tau),$$

where

$$M_1(t,\tau) := -\frac{1}{2\pi} J_0(k|x(t) - x(\tau)|),$$

$$M_2(t,\tau) := M(t,\tau) - M_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)$$

again are analytic with diagonal terms

$$M_1(t,t) = -\frac{1}{2\pi}, \qquad M_2(t,t) = \frac{\mathrm{i}}{2} - \frac{C}{\pi} - \frac{1}{\pi}\ln\frac{k|x'(t)|}{2}.$$

We define

$$N(t,\tau) := \frac{\partial^2}{\partial t\,\partial\tau}\left\{\frac{\mathrm{i}}{2} H_0^{(1)}(k|x(t) - x(\tau)|) + \frac{1}{2\pi}\ln\left(4\sin^2\frac{t-\tau}{2}\right)\right\}$$

and can deduce from the expansions (2.2) and (2.3) that

$$N(t,\tau) = N_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + N_2(t,\tau),$$

where

$$N_1(t,\tau) := -\frac{1}{2\pi}\frac{\partial^2}{\partial t\,\partial\tau} J_0(k|x(t)-x(\tau)|),$$

$$N_2(t,\tau) := N(t,\tau) - N_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)$$

are analytic functions with diagonal terms

$$N_1(t,t) = -\frac{k^2|x'(t)|^2}{4\pi}$$

and

$$N_2(t,t) = \left(\pi\mathrm{i} - 1 - 2C - 2\ln\frac{k|x'(t)|}{2}\right)\frac{k^2|x'(t)|^2}{4\pi}$$
$$+\frac{1}{12\pi} + \frac{[x'(t)\cdot x''(t)]^2}{2\pi|x'(t)|^4} - \frac{|x''(t)|^2}{4\pi|x'(t)|^2} - \frac{x'(t)\cdot x'''(t)}{6\pi|x'(t)|^2}.$$

Therefore, we can carry out a partial integration in

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}S\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)(x(t)) = \frac{\mathrm{i}}{2|x'(t)|}\int_0^{2\pi}\frac{\partial}{\partial t}H_0^{(1)}(k|x(t)-x(\tau)|)\frac{\mathrm{d}\varphi(x(\tau))}{\mathrm{d}\tau}\,\mathrm{d}\tau$$

to arrive at

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}S\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)(x(t)) = \frac{1}{|x'(t)|}\int_0^{2\pi}\left\{\frac{1}{2\pi}\cot\frac{\tau-t}{2}\frac{\mathrm{d}\varphi(x(\tau))}{\mathrm{d}\tau} - N(t,\tau)\,\varphi(x(\tau))\right\}\mathrm{d}\tau. \tag{2.6}$$

Using the Bessel differential equation for $H_0^{(1)}$, we can compute the following explicit expressions:

$$N(t,\tau) = \frac{\mathrm{i}}{2}\tilde{N}(t,\tau)\left\{k^2 H_0^{(1)}(k|x(t)-x(\tau)|) - \frac{2kH_1^{(1)}(k|x(t)-x(\tau)|)}{|x(t)-x(\tau)|}\right\}$$
$$+\frac{\mathrm{i}k\,x'(t)\cdot x'(\tau)}{2|x(t)-x(\tau)|}H_1^{(1)}(k|x(t)-x(\tau)|) + \frac{1}{4\pi}\frac{1}{\sin^2\frac{1}{2}(t-\tau)}$$

and

$$N_1(t,\tau) = \frac{-1}{2\pi}\tilde{N}(t,\tau)\left\{k^2 J_0(k|x(t)-x(\tau)|) - \frac{2kJ_1(k|x(t)-x(\tau)|)}{|x(t)-x(\tau)|}\right\}$$
$$-\frac{k\,x'(t)\cdot x'(\tau)}{2\pi|x(t)-x(\tau)|}J_1(k|x(t)-x(\tau)|),$$

where we have set

$$\tilde{N}(t,\tau) := \frac{x'(t)\cdot(x(t)-x(\tau))\,x'(\tau)\cdot(x(t)-x(\tau))}{|x(t)-x(\tau)|^2}.$$

If we now piece (2.1) and (2.4)–(2.6) together, we see that the parametrized integral equation (1.6) is of the form

$$\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{\tau-t}{2}\,\psi'(\tau)\mathrm{d}\tau+\int_0^{2\pi}K(t,\tau)\psi(\tau)\,\mathrm{d}\tau+a(t)\psi(t)=f(t),\quad 0\leqslant t\leqslant 2\pi,\tag{2.7}$$

for the unknown function $\psi(t) := \varphi(x(t))$ and the right-hand side given by $f(t) := 2|x'(t)|g(x(t))$. We have set $a(t) := \mathrm{i}\eta|x'(t)|$ and the kernel

$$K(t,\tau) := k^2 M(t,\tau)x'(t)\cdot x'(\tau) - N(t,\tau) - \mathrm{i}\eta H(t,\tau)$$

can be written as

$$K(t,\tau) = K_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + K_2(t,\tau)$$

with 2π-periodic analytic functions K_1 and K_2. After introducing bounded operators T_0, A_1, A_2, $A_3: C^{1,\alpha}[0,2\pi] \to C^{0,\alpha}[0,2\pi]$ by

$$(T_0\psi)(t) := \frac{1}{2\pi}\int_0^{2\pi}\cot\frac{\tau-t}{2}\,\psi'(\tau)\,\mathrm{d}\tau,$$

$$(A_1\psi)(t) := \int_0^{2\pi}K_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)\psi(\tau)\,\mathrm{d}\tau,$$

$$(A_2\psi)(t) := \int_0^{2\pi}K_2(t,\tau)\psi(\tau)\mathrm{d}\tau,$$

$$(A_3\psi)(t) := a(t)\psi(t),$$

and setting $A := A_1 + A_2 + A_3$ we can rewrite the integral equation (2.7) in the short form

$$T_0\psi + A\psi = f.\tag{2.8}$$

We note that all our function spaces on the interval $[0,2\pi]$ have to be understood as spaces of 2π-periodic functions. The operator T_0 corresponds to the normal derivative of the double-layer potential for $k=0$ and Γ the unit circle. It is also related to the bounded operator $H_0: C^{0,\alpha}[0,2\pi] \to C^{0,\alpha}[0,2\pi]$ with Hilbert kernel

$$(H_0\psi)(t) := \frac{1}{2\pi\mathrm{i}}\int_0^{2\pi}\left\{\cot\frac{\tau-t}{2}+\mathrm{i}\right\}\psi(\tau)\,\mathrm{d}\tau$$

which satisfies $H_0^2 = I$ (cf. [6, p. 91]). From this it is obvious that the modified operator $\tilde{T}_0: C^{1,\alpha}[0,2\pi] \to C^{0,\alpha}[0,2\pi]$ defined by

$$\tilde{T}_0\psi := T_0\psi + \int_0^{2\pi}\psi(\tau)\mathrm{d}\tau,$$

has a bounded inverse $\tilde{T}_0^{-1}: C^{0,\alpha}[0,2\pi] \to C^{1,\alpha}[0,2\pi]$. This result can also be derived from the fact that for the trigonometric monomials $u_m(t) := \mathrm{e}^{\mathrm{i}mt}$ we have

$$T_0u_m = -|m|u_m,\quad m = 0, \pm 1, \pm 2, \ldots .\tag{2.9}$$

Hence, since the operator $A: C^{1,\alpha}[0,2\pi] \to C^{0,\alpha}[0,2\pi]$ can be seen to be compact (see Lemma 4.1), the inverse $\tilde{T}_0^{-1}$ may serve as an equivalent regularizer of (2.8). In particular, from the Riesz theory for compact operators we have that $T_0 + A: C^{1,\alpha}[0,2\pi] \to C^{0,\alpha}[0,2\pi]$ has a bounded inverse if and only if $T_0 + A$ is injective.

3. The numerical method

Our quadrature method is based on trigonometric interpolation. We choose $n \in \mathbb{N}$ and an equidistant mesh by setting

$$t_j^{(n)} := \frac{j\pi}{n}, \quad j = 0, \ldots, 2n-1.$$

The interpolation problem with respect to the $2n$-dimensional space T_n of trigonometric polynomials of the form

$$v(t) = \sum_{m=0}^{n} a_m \cos mt + \sum_{m=1}^{n-1} b_m \sin mt$$

and the nodal points $t_j^{(n)}$, $j = 0, \ldots, 2n-1$, is uniquely solvable. We denote by $P_n: C[0,2\pi] \to T_n$ the corresponding interpolation operator. For our error analysis we will use the estimate

$$\| P_n f - f \|_{p,\alpha} \leqslant C \frac{\ln n}{n^{q-p+\beta-\alpha}} \| f \|_{q,\beta}, \tag{3.1}$$

which is valid for all $f \in C^{q,\beta}[0,2\pi]$ for $0 \leqslant p \leqslant q$ and $0 < \alpha \leqslant \beta < 1$ and some constant C depending only on p, q, α and β (cf. [16, p. 40; 17, p. 78]). By $\| \cdot \|_{p,\alpha}$ we denote the usual Hölder norm. For the trigonometric interpolation of 2π-periodic analytic functions f we have a stronger error estimate of the form (cf. [6, p. 160])

$$\| P_n f - f \|_{1,\infty} \leqslant c\mathrm{e}^{-n\sigma} \tag{3.2}$$

for some positive constants c and σ depending on f. By $\| \cdot \|_{p,\infty}$ we denote the norm on the space of p-times continuously differentiable functions (given by the sum of the maximum norms of the function and its pth derivative).

We will use the following interpolatory quadrature rules:

$$\frac{1}{2\pi} \int_0^{2\pi} \cot \frac{\tau - t}{2} f'(\tau)\, \mathrm{d}\tau \approx \frac{1}{2\pi} \int_0^{2\pi} \cot \frac{\tau - t}{2} (P_n f)'(\tau)\, \mathrm{d}\tau, \tag{3.3}$$

$$\int_0^{2\pi} \ln \left(4 \sin^2 \frac{t - \tau}{2} \right) f(\tau)\, \mathrm{d}\tau \approx \int_0^{2\pi} \ln \left(4 \sin^2 \frac{t - \tau}{2} \right) (P_n f)(\tau)\, \mathrm{d}\tau, \tag{3.4}$$

that is,

$$\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{\tau-t}{2}f'(\tau)\,d\tau \approx \sum_{j=0}^{2n-1} T_j^{(n)}(t)f(t_j^{(n)}),$$

$$\int_0^{2\pi}\ln\left(4\sin^2\frac{t-\tau}{2}\right)f(\tau)\,d\tau \approx \sum_{j=0}^{2n-1} R_j^{(n)}(t)f(t_j^{(n)}),$$

where by using the explicit form of the trigonometric interpolation polynomial and elementary integrals (see (2.9)) the quadrature weights can be seen to be given by

$$T_j^{(n)}(t) = -\frac{1}{n}\sum_{m=1}^{n-1} m\cos m(t-t_j^{(n)}) - \tfrac{1}{2}\cos n(t-t_j^{(n)}),$$

$$R_j^{(n)}(t) = -\frac{2\pi}{n}\sum_{m=1}^{n-1}\frac{1}{m}\cos m(t-t_j^{(n)}) - \frac{\pi}{n^2}\cos n(t-t_j^{(n)}).$$

In addition, we use the trapezoidal rule

$$\int_0^{2\pi} f(\tau)\,d\tau \approx \int_0^{2\pi}(P_nf)(\tau)\,d\tau = \frac{\pi}{n}\sum_{j=0}^{2n-1} f(t_j^{(n)}). \tag{3.5}$$

The approximation (3.3) can be considered as a modification of the quadrature rule for the singular integral operator with Hilbert kernel due to Wittich [19]. The logarithmic quadrature formula (3.4) was first used by Martensen [10] and Kussmaul [8].

We apply the quadrature rules (3.3)–(3.5) to the integral equation (2.7) and obtain the approximating equation

$$\sum_{j=0}^{2n-1}\tilde{\psi}_n(t_j^{(n)})\left\{T_j^{(n)}(t) + R_j^{(n)}(t)K_1(t,t_j^{(n)}) + \frac{\pi}{n}K_2(t,t_j^{(n)})\right\} + a(t)\tilde{\psi}_n(t) = f(t)$$

which we solve for $\tilde{\psi}_n \in T_n$. Using the fact that

$$T_0P_n\psi = T_0\psi = P_nT_0\psi, \quad \psi\in T_n, \tag{3.6}$$

which follows from (2.9), we can write this in operator notation as

$$T_0\tilde{\psi}_n + A_{1,n}\tilde{\psi}_n + A_{2,n}\tilde{\psi}_n + A_3\tilde{\psi}_n = f \tag{3.7}$$

with the numerical quadrature operators

$$(A_{1,n}\psi)(t) := \int_0^{2\pi}\ln\left(4\sin^2\frac{t-\tau}{2}\right)(P_nK_1(t,\cdot)\psi)(\tau)\,d\tau,$$

$$(A_{2,n}\psi)(t) := \int_0^{2\pi}(P_nK_2(t,\cdot)\psi)(\tau)\,d\tau,$$

that is,

$$(A_{1,n}\psi)(t) = \sum_{j=0}^{2n-1} R_j^{(n)}(t) K_1(t, t_j^{(n)}) \psi(t_j^{(n)}),$$

$$(A_{2,n}\psi)(t) = \frac{\pi}{n} \sum_{j=0}^{2n-1} K_2(t, t_j^{(n)}) \psi(t_j^{(n)}).$$

In order to arrive at an approximating equation which can be reduced to solving a finite dimensional linear system we collocate (3.7) with the interpolation operator P_n. Hence, in view of (3.6), our approximation scheme finally consists in solving

$$T_0\psi_n + P_n A_{1,n}\psi_n + P_n A_{2,n}\psi_n + P_n A_3\psi_n = P_n f \tag{3.8}$$

for $\psi_n \in C^{1,\alpha}[0, 2\pi]$. We note that due to (2.9) any solution to (3.8) automatically belongs to the trigonometric polynomial space T_n. Clearly, (3.8) is equivalent to the linear system

$$\sum_{j=0}^{2n-1} \psi_n(t_j^{(n)}) \left\{ T_{|k-j|}^{(n)} + R_{|k-j|}^{(n)} K_1(t_k^{(n)}, t_j^{(n)}) + \frac{\pi}{n} K_2(t_k^{(n)}, t_j^{(n)}) \right\}$$

$$+ a(t_k^{(n)}) \psi_n(t_k^{(n)}) = f(t_k^{(n)}), \quad k = 0, 1, \ldots, 2n-1, \tag{3.9}$$

which we have to solve for the nodal values $\psi_n(t_k^{(n)})$ of $\psi_n \in T_n$, and where

$$T_j^{(n)} := T_j^{(n)}(0) = \begin{cases} \dfrac{1}{2n \sin^2(t_j^{(n)}/2)}, & j \text{ odd}, \\ 0, & j \text{ even}, \ j \neq 0, \\ -\dfrac{n}{2}, & j = 0, \end{cases}$$

$$R_j^{(n)} := R_j^{(n)}(0) = -\frac{2\pi}{n} \sum_{m=1}^{n-1} \frac{1}{m} \cos\frac{mj\pi}{n} + \frac{(-1)^j \pi}{n^2}.$$

4. Error and convergence analysis

The following lemma, of course, is standard. However, we include its proof in our analysis since we need the explicit form of the dependence of the estimates on the kernel function φ.

Lemma 4.1. *Let $\varphi: \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be a continuous function which is 2π-periodic with respect to both variables and continuously differentiable with respect to the first variable. Then for*

$$u(t) := \int_0^{2\pi} \ln\left(4 \sin^2 \frac{t-\tau}{2}\right) \varphi(t, \tau)\, d\tau, \quad 0 \leqslant t \leqslant 2\pi,$$

we have that

$$\| u \|_{0,\alpha} \leqslant \gamma \left(\| \varphi \|_\infty + \left\| \frac{\partial \varphi}{\partial t} \right\|_\infty \right) \tag{4.1}$$

for all $0 < \alpha < 1$ *and some constant* γ *depending only on* α.

Proof. For the weakly singular logarithmic kernel we clearly have that

$$\| u \|_\infty \leqslant c_1 \| \varphi \|_\infty \tag{4.2}$$

with some constant c_1. In order to estimate the Hölder semi-norm, we write

$$u(t_1) - u(t_2) = v_1 + v_2,$$

where

$$v_1 := \int_0^{2\pi} \ln \left(4 \sin^2 \frac{t_1 - \tau}{2} \right) \{ \varphi(t_1, \tau) - \varphi(t_2, \tau) \} \, \mathrm{d}\tau,$$

$$v_2 := \int_0^{2\pi} \left\{ \ln \left(4 \sin^2 \frac{t_1 - \tau}{2} \right) - \ln \left(4 \sin^2 \frac{t_2 - \tau}{2} \right) \right\} \varphi(t_2, \tau) \, \mathrm{d}\tau.$$

By the mean value theorem we can estimate

$$| v_1 | \leqslant c_1 | t_1 - t_2 | \left\| \frac{\partial \varphi}{\partial t} \right\|_\infty \tag{4.3}$$

for all $t_1, t_2 \in [0, 2\pi]$. For the second integral we set $s := \frac{1}{2}(t_2 - t_1)$ and may assume that $0 < s \leqslant \frac{1}{2}\pi$. Using periodicity we obtain

$$| v_2 | \leqslant I \| \varphi \|_\infty ,$$

where

$$I := \int_0^{2\pi} \left| \ln \frac{\sin^2 \frac{1}{2}(\tau + s)}{\sin^2 \frac{1}{2}(\tau - s)} \right| \mathrm{d}\tau = 4 \int_0^{\pi} \ln \left| \frac{\sin \frac{1}{2}(\tau + s)}{\sin \frac{1}{2}(\tau - s)} \right| \mathrm{d}\tau.$$

Since for $0 \leqslant \tau \leqslant 2s$ we have that

$$\left| \frac{\sin \frac{1}{2}(\tau + s)}{\sin \frac{1}{2}(\tau - s)} \right| \leqslant \frac{\pi}{2} \left| \frac{\tau + s}{\tau - s} \right| ,$$

we can estimate

$$\int_0^{2s} \ln \left| \frac{\sin \frac{1}{2}(\tau + s)}{\sin \frac{1}{2}(\tau - s)} \right| \mathrm{d}\tau \leqslant \int_0^{2s} \ln \frac{\pi}{2} \left| \frac{\tau + s}{\tau - s} \right| \mathrm{d}\tau = s \int_0^{2} \ln \frac{\pi}{2} \left| \frac{z + 1}{z - 1} \right| \mathrm{d}z.$$

From the inequality

$$\ln \frac{1+z}{1-z} \leqslant \tfrac{8}{3} z, \quad 0 \leqslant z \leqslant \tfrac{1}{2},$$

we deduce that

$$\ln \left| \frac{\sin \frac{1}{2}(\tau + s)}{\sin \frac{1}{2}(\tau - s)} \right| = \ln \frac{1 + \tan(\frac{1}{2}s) \cot(\frac{1}{2}\tau)}{1 - \tan(\frac{1}{2}s) \cot(\frac{1}{2}\tau)} \leqslant \tfrac{8}{3} \tan(\tfrac{1}{2}s) \cot(\tfrac{1}{2}\tau), \quad 2s \leqslant \tau \leqslant \pi.$$

Hence

$$\int_{2s}^{\pi} \ln \left| \frac{\sin \frac{1}{2}(\tau + s)}{\sin \frac{1}{2}(\tau - s)} \right| d\tau \leqslant \tfrac{8}{3} \tan(\tfrac{1}{2}s) \int_{2s}^{\pi} \cot(\tfrac{1}{2}\tau)\, d\tau = -\tfrac{16}{3} \tan(\tfrac{1}{2}s) \ln \sin s$$

and consequently

$$|v_2| \leqslant c_2 |t_1 - t_2| (1 + |\ln |t_1 - t_2||) \, \| \varphi \|_\infty \tag{4.4}$$

for all $t_1, t_2 \in [0, 2\pi]$ and some constant c_2. Now the statement follows by putting (4.2)–(4.4) together. □

If we assume that φ is continuously differentiable with respect to both variables, then we can substitute $s = \tau - t$ in order to establish that u is continuously differentiable with

$$\frac{du}{dt}(t) = \int_0^{2\pi} \ln \left(4 \sin^2 \frac{t - \tau}{2} \right) \left\{ \frac{\partial \varphi(t, \tau)}{\partial t} + \frac{\partial \varphi(t, \tau)}{\partial \tau} \right\} d\tau, \quad 0 \leqslant t \leqslant 2\pi.$$

Therefore, by induction, from Lemma 4.1 we can deduce the following corollary.

Corollary 4.2. *Under the assumptions of Lemma* 4.1, *let φ be p-times continuously differentiable with respect to both variables and $(p + 1)$-times continuously differentiable with respect to the first variable where $p \in \mathbb{N}$. Then we have that*

$$\| u \|_{p,\alpha} \leqslant \gamma \left(\| \varphi \|_{p,\infty} + \left\| \frac{\partial \varphi}{\partial t} \right\|_{p,\infty} \right) \tag{4.5}$$

for all $0 < \alpha < 1$ and some constant γ depending only on p and α.

We recall $A := A_1 + A_2 + A_3$, abbreviate $A_n := A_{1,n} + A_{2,n} + A_3$ and establish the following convergence result.

Theorem 4.3. *Assume that the kernels K_1 and K_2 both are analytic and 2π-periodic. Then the operator sequence $P_n A_n : C^{1,\alpha}[0, 2\pi] \to C^{0,\alpha}[0, 2\pi]$ is norm convergent with limit operator $A : C^{1,\alpha}[0, 2\pi] \to C^{0,\alpha}[0, 2\pi]$ for all $0 < \alpha < 1$.*

Proof. Throughout the proof, by c we denote a generic constant (depending on K_1, K_2, α and β) which may differ in each formula. An application of Lemma 4.1, with $\varphi(t, \tau) =$

$(P_n K_1(t,\cdot)\psi)(\tau) - K_1(t,\tau)\psi(\tau)$, yields

$$\|(A_{1,n} - A_1)\psi\|_{0,\alpha} \leqslant c \max_{0\leqslant t,\tau\leqslant 2\pi} |(P_n K_1(t,\cdot)\psi)(\tau) - K_1(t,\tau)\psi(\tau)|$$
$$+ c \max_{0\leqslant t,\tau\leqslant 2\pi} \left| \left(P_n \frac{\partial K_1(t,\cdot)}{\partial t} \psi \right)(\tau) - \frac{\partial K_1(t,\tau)}{\partial t} \psi(\tau) \right|.$$

Inserting the error estimate (3.1) for the trigonometric interpolation, we obtain

$$\|(A_{1,n} - A_1)\psi\|_{0,\alpha} \leqslant c \frac{\ln n}{n^{1+\alpha-\beta}} \|\psi\|_{1,\alpha} \tag{4.6}$$

for all $0 < \beta \leqslant \alpha < 1$. Similarly, combining Corollary 4.2 and the estimate (3.1) we find

$$\|(A_{1,n} - A_1)\psi\|_{1,\alpha} \leqslant c \frac{\ln n}{n^{\alpha-\beta}} \|\psi\|_{1,\alpha}$$

for all $0 < \beta \leqslant \alpha < 1$. This, in particular, implies

$$\|A_{1,n}\psi\|_{1,\alpha} \leqslant c\|\psi\|_{1,\alpha}, \quad n \in \mathbb{N}.$$

Using this uniform boundedness, we apply again the estimate (3.1) to obtain

$$\|(P_n A_{1,n} - A_{1,n})\psi\|_{0,\alpha} \leqslant c \frac{\ln n}{n} \|A_{1,n}\psi\|_{1,\alpha} \leqslant c \frac{\ln n}{n} \|\psi\|_{1,\alpha}. \tag{4.7}$$

Now from (4.6) and (4.7) and the triangle inequality we have

$$\|(P_n A_{1,n} - A_1)\psi\|_{0,\alpha} \leqslant c \frac{\ln n}{n} \|\psi\|_{1,\alpha}.$$

By the same technique, this estimate can be seen to be valid also for the operator A_2 with analytic kernel K_2. Finally (3.1) implies

$$\|(P_n A_3 - A_3)\psi\|_{0,\alpha} \leqslant c \frac{\ln n}{n} \|A_3\psi\|_{1,\alpha} \leqslant c \frac{\ln n}{n} \|\psi\|_{1,\alpha}.$$

Therefore, we have that

$$\|(P_n A_n - A)\psi\|_{0,\alpha} \leqslant c \frac{\ln n}{n} \|\psi\|_{1,\alpha} \tag{4.8}$$

for all $\psi \in C^{1,\alpha}[0,2\pi]$ and the proof is finished. □

We are now in a position to formulate our main convergence result.

Theorem 4.4. *For sufficiently large n the approximating equation* (3.8) *has a unique solution* ψ_n *and for the unique solution* ψ *to the original equation* (2.8) *we have the error estimate*

$$\|\psi_n - \psi\|_{1,\alpha} \leqslant C(\|P_n f - f\|_{0,\alpha} + \|P_n A_n \psi - A\psi\|_{0,\alpha}) \tag{4.9}$$

for some constant $C = C(\alpha)$ *and* $0 < \alpha < 1$.

Proof. Since $T_0 + A: C^{1,\alpha}[0,2\pi] \to C^{0,\alpha}[0,2\pi]$ has a bounded inverse and since by Theorem 4.3 we have norm convergence of the approximating sequence $P_n A_n: C^{1,\alpha}[0,2\pi] \to C^{0,\alpha}[0,2\pi]$ to $A: C^{1,\alpha}[0,2\pi] \to C^{0,\alpha}[0,2\pi]$, by standard convergence analysis through the Neumann series, for sufficiently large n the operators $T_0 + P_n A_n: C^{1,\alpha}[0,2\pi] \to C^{0,\alpha}[0,2\pi]$ are invertible and the inverse operators are uniformly bounded. The error estimate follows by writing

$$\psi_n - \psi = (T_0 + P_n A_n)^{-1}\{(P_n f - f) + (A - P_n A_n)\psi\}$$

and using the uniform boundedness of the inverse operators. □

The error estimate (4.9), in view of (3.1) and (4.8), illustrates that the term $P_n f - f$ is decisive for whether we have convergence. Note that this term reflects the approximation of the principal part T_0 by $P_n T_0$. Thus our approximation of the perturbation A is chosen in a manner which does not affect the convergence order for the principal part. From (3.1) and (4.8) we obtain the error estimate

$$\|\psi_n - \psi\|_{1,\alpha} \leqslant c\,\frac{\ln n}{n^{\beta-\alpha}}\,\|f\|_{0,\beta}$$

for $0 < \alpha \leqslant \beta < 1$, i.e., convergence if $f \in C^{0,\beta}[0,2\pi]$ for $\beta > \alpha$. Using the analogue of (4.8) for higher-order Hölder norms, which can be obtained with the aid of Corollary 4.2, we obtain the estimates

$$\|\psi_n - \psi\|_{1,\alpha} \leqslant c\,\frac{\ln n}{n^{q+\beta-\alpha}}\,\|f\|_{q,\beta}$$

for $q \in \mathbb{N}$ and $0 < \alpha \leqslant \beta < 1$. In addition to these low-order estimates, we wish to point out that if the exact solution is analytic (and this is the case if the boundary and the boundary data are analytic), then from (3.2), (4.1) and (4.9) it can be derived that

$$\|\psi_n - \psi\|_{1,\alpha} \leqslant c\mathrm{e}^{-n\sigma},$$

i.e., the error decreases exponentially.

5. A numerical example

For a numerical example, we consider the scattering of a plane wave u^{i} by a sound-hard cylinder with a non-convex kite-shaped cross-section with boundary Γ illustrated in Fig. 1 and described by the parametric representation

$$x(t) = (\cos t + 0.65\cos 2t - 0.65,\ 1.5\sin t), \quad 0 \leqslant t \leqslant 2\pi.$$

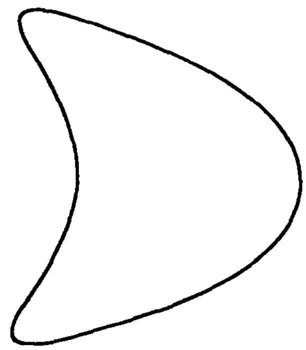

Fig. 1. Kite-shaped domain for numerical example.

Table 1
Numerical results

	n	$\mathrm{Re}\,u_\infty(d)$	$\mathrm{Im}\,u_\infty(d)$	$\mathrm{Re}\,u_\infty(-d)$	$\mathrm{Im}\,u_\infty(-d)$
$k=1$	8	0.13973626	0.18093027	− 1.11011577	− 0.50681485
	16	0.15158507	0.19159181	− 1.10228822	− 0.50925214
	32	0.15153740	0.19153454	− 1.10234229	− 0.50918721
	64	0.15153740	0.19153454	− 1.10234230	− 0.50918720
$k=3$	8	0.11163356	0.91056703	− 1.67222931	− 0.86694951
	16	− 0.03641571	0.71129456	− 1.63684382	− 0.82343826
	32	− 0.03646654	0.71122115	− 1.63689151	− 0.82335680
	64	− 0.03646654	0.71122115	− 1.63689151	− 0.82335679
$k=5$	8	0.53567309	0.12509154	− 1.92379981	− 1.40068649
	16	− 0.27473745	− 0.29834046	− 1.95374230	− 1.27549747
	32	− 0.28067233	0.29817977	− 1.94749252	− 1.27590706
	64	− 0.28067233	0.29817977	− 1.94749251	− 1.27590706

The incident wave is given by $u^{\mathrm{i}}(x) = \mathrm{e}^{ikd\cdot x}$ where d denotes a unit vector giving the direction of propagation.

For the scattered wave u^{s} we have to solve an exterior Neumann problem with boundary values $g = -\partial u^{\mathrm{i}}/\partial\nu$ on Γ. The far-field pattern u_∞ is defined by the asymptotic behavior of the scattered wave

$$u^{\mathrm{s}}(x) = \frac{\mathrm{e}^{ik|x|}}{\sqrt{|x|}}\left\{u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty,$$

uniformly for all directions $\hat{x} := x/|x|$ (see [3, 4]). From the asymptotics for the Hankel function for large argument, we see that the far-field pattern of the combined double- and single-layer potential (1.5) is given by

$$u_\infty(\hat{x}) = \frac{e^{-i\pi/4}}{\sqrt{8\pi k}} \int_\Gamma \{k\hat{x}\cdot n(y) + \eta\} e^{-ik\hat{x}\cdot y} \varphi(y)\, ds(y). \tag{5.1}$$

After solving the integral equation (1.6) numerically by the method of this paper, the integral (5.1) is evaluated by the trapezoidal rule. Table 1 gives some approximate values for the far-field pattern $u_\infty(d)$ and $u_\infty(-d)$ in the forward direction d and the backward direction $-d$. The direction d of the incident wave is $d = (1,0)$ and, as recommended in [5], the coupling parameter was chosen by $\eta = k$. Note that the fast convergence is clearly exhibited.

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