§2.4 plasmon for nano-particle with corners

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$$[K'\varphi](x) = \int_\Gamma \frac{\partial \Phi(x,y)}{\partial n_x} \varphi(y)\, ds_y, \quad x \in \Gamma$$

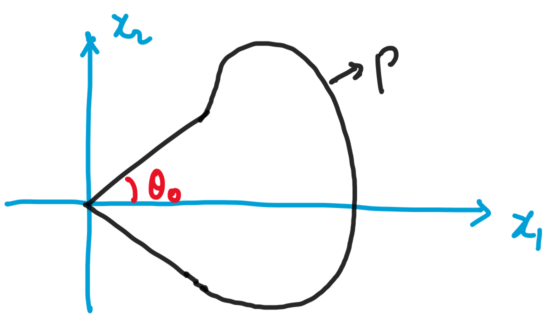
The Neumann-poincaré operator K' is not compact for domain with corners, and its spectrum now also contains essential spectrum.

Let $\sigma_d(A)$ be the discrete spectrum that contains all eigenvalues of A with finite algebraic multiplicity and which are isolated points of the spectrum.

The essential spectrum of A is the complement of $\sigma_d(A)$ defined by

$$\sigma_{ess}(A) = \sigma(A) \setminus \sigma_d(A).$$

Consider a wedge domain $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = \{(r,\theta) \mid 0 < r < r_0, -\theta_0 < \theta < \theta_0\}$.
We assume that the only corner appears in Ω_1. The angle $0 < \theta_0 < \frac{\pi}{2}$.



Theorem 2.4.1 [Perfekt & Putinar, 2017]

The operator $K' : H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ attains the essential spectrum $\sigma_{ess} = \left[-\left(\frac{1}{2} - \frac{\theta_0}{\pi}\right), \frac{1}{2} - \frac{\theta_0}{\pi}\right]$.

We will give an alternative proof of the above theorem based on Weyl's criterion.

Definition Weyl sequence: A sequence $\{u_n\}_{n=1}^\infty$ in the Hilbert space H is called a Weyl sequence for the operator $A: H \to H$ and $\lambda \in \mathbb{C}$ if $\|u_n\| = 1$, $u_n \xrightarrow{w} 0$, and $\|(A-\lambda)u_n\| \to 0$ as $n \to \infty$

Theorem 2.4.2 (Weyl's criterion) Let $A: H \to H$ be a self-adjoint operator, then $\lambda \in \sigma_{ess}(A)$ if and only if there exist a Weyl sequence for A and λ.

The proof the theorem can be found, for instance, in [Hislop & Sigal]. Here we give a proof on the sufficient condition.

Lemma 2.4.3 If there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset H$ such that $\|u_n\| = 1$ and $\|(A-\lambda)u_n\| \to 0$ as $n \to \infty$, then $\lambda \in \sigma(A)$.

proof. If $\lambda \notin \sigma(A)$, then $\lambda \in \rho(A)$ and $A - \lambda I$ has a bounded inverse. Thus $\forall u \in H$, $\|u\| \leq \|(A-\lambda I)^{-1}\| \, \|(A-\lambda)u\|$, which implies that no sequence in the statement of the lemma would exist.

Lemma 2.4.4 Let $A: H \to H$ be self-adjoint. If $\lambda \in \sigma_d(A)$, then $(A - \lambda I)\big|_{Ker(A-\lambda I)^{\perp}}$ has a bounded inverse.

Proof Decompose H as $H = Ker(A-\lambda I) \oplus Ker(A-\lambda I)^{\perp}$.

Let $A_1 := A\big|_{Ker(A-\lambda I)^{\perp}}$. then A_1 is self-adjoint on $Ker(A-\lambda I)^{\perp}$ and $Ker(A_1 - \lambda I) = \{0\}$. We deduce that $\lambda \in \rho(A_1)$, since λ is an isolated point of $\sigma(A)$. Hence A_1 has a bounded inverse.

proof of Theorem 2.4.2 $\Leftarrow$

If $\{u_n\}_{n=1}^{\infty}$ is a Weyl sequence, from Lemma 2.4.3 we deduce that $\lambda \in \sigma(A)$. We only need to show that $\lambda \notin \sigma_d(A)$. If not, λ is an isolated eigenvalue of A and $\dim(\mathrm{Ker}(A-\lambda I)) < +\infty$. Let $\mathrm{Ker}(A-\lambda I) = \mathrm{Span}\{\phi_1, \phi_2, \cdots, \phi_m\}$, where $\phi_1, \cdots, \phi_m$ is an orthonormal basis of $\mathrm{Ker}(A-\lambda I)$. Let P be the projection on $\mathrm{Ker}(A-\lambda I)$.

$$Pu_n = \sum_{j=1}^{m} \langle u_n, \phi_j \rangle \phi_j \quad \text{and} \quad \| Pu_n \|^2 = \sum_{j=1}^{m} |\langle u_n, \phi_j \rangle|^2 \to 0 \quad \text{as } n \to \infty$$

since $u_n \xrightarrow{w} 0$. Thus $\|(I-P)u_n\| \to 1$ as $n \to \infty$.

Define $v_n = \dfrac{(I-P)u_n}{\|(I-P)u_n\|}$. Then $v_n \in \mathrm{Ker}(A-\lambda I)^{\perp}$ and $\|v_n\| = 1$.

$$(A-\lambda)v_n = \frac{(A-\lambda)u_n}{\|(I-P)u_n\|} \to 0 \quad \text{as } n \to \infty.$$

$\Rightarrow$ The inverse of $(A-\lambda I)\big|_{\mathrm{Ker}(A-\lambda I)^{\perp}}$ is unbounded. This contradicts with Lemma 2.4.4. Therefore, $\lambda \in \sigma_{ess}(A)$.