

# *Spectral Theory of a Neumann–Poincaré-Type Operator and Analysis of Cloaking Due to Anomalous Localized Resonance*

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## **Abstract**

The aim of this paper is to give a mathematical justification of cloaking due to anomalous localized resonance (CALR). We consider the dielectric problem with a source term in a structure with a layer of plasmonic material. Using layer potentials and symmetrization techniques, we give a necessary and sufficient condition on the fixed source term for electromagnetic power dissipation to blow up as the loss parameter of the plasmonic material goes to zero. This condition is written in terms of the Newtonian potential of the source term. In the case of concentric disks, we make the condition even more explicit. Using the condition, we are able to show that for any source supported outside a critical radius, CALR does not take place, and for sources located inside the critical radius satisfying certain conditions, CALR does take place as the loss parameter goes to zero.

## **1. Introduction**

In recent years much interest has been aroused by the possibility of cloaking objects from interrogation by electromagnetic waves. Many schemes are under active current investigation [1, 3, 7, 8, 11–13, 18–22, 24, 26, 35]. One such scheme, which is the focus of our study, relies on resonant interaction to mask the electromagnetic signature of the object to be cloaked [5, 6, 25, 27–29, 31, 32, 34].

We consider the dielectric problem with a source term  $\alpha f$ , proportional to  $f$ , which models the quasi-static (zero-frequency) transverse magnetic regime. The cloaking of the source is achieved in a region external to a plasmonic structure. The plasmonic structure consists of a shell having relative permittivity  $-1 + i\delta$ , with  $\delta$  modelling losses.

The cloaking issue is directly linked to the existence of anomalous localized resonance (ALR), which is tied to the fact that an elliptic system of equations can exhibit localization effects near the boundary of ellipticity. The plasmonic structure exhibits ALR if, as the loss parameter  $\delta$  goes to zero, the magnitude of the

quasi-static in-plane electric field diverges throughout a specific region (with sharp boundary not defined by any discontinuities in the relative permittivity), called the anomalous resonance region, but converges to a smooth field outside that region. The convergence to a smooth field outside the region was shown in [33], where the first numerical evidence for ALR was also presented. A proof of ALR for a dipolar source outside a plasmonic annulus was given in [30].

ALEXEI EFROS (2005 private communication to GWM) made the key observation that for a fixed dipolar source within a critical distance of the plasmonic structure, the total electrical power absorbed would become infinite as  $\delta \rightarrow 0$ , which is unphysical. The anomalously resonant fields interact with the source creating a sort of “electromagnetic molasses” against which the source has to do a huge amount of work to maintain its amplitude; in fact, an infinite amount of work in the limit  $\delta \rightarrow 0$ . Therefore, it makes sense to normalize the source term (by adjusting  $\alpha$ , letting it depend on  $\delta$ ) so the source supplies power at a constant rate independent of  $\delta$ . Then outside the region where ALR occurs the field tends to zero as  $\delta \rightarrow 0$ : the source becomes cloaked. Cloaking also extends to finite collections of polarizable dipoles (dipole sources whose strength is proportional the field acting on them) within a critical radius around a plasmonic annulus [27,34], and to a sufficiently small dielectric disk (with radius which goes to zero as  $\delta \rightarrow 0$ ) lying within this critical radius [5]. However, numerical evidence suggests that a small dielectric disk with  $\delta$  independent radius is only partially cloaked in the limit  $\delta \rightarrow 0$  [6]. We also mention that opposing sources on opposite sides of a planar superlens can be cloaked [4], but this is due to cancellation of fields, rather than anomalous resonance.

To mathematically state the problem, let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$  and let  $D$  be a domain whose closure is contained in  $\Omega$ . Throughout this paper, we assume that  $\Omega$  and  $D$  are of class  $C^{1,\mu}$  for some  $0 < \mu < 1$ . For a given loss parameter  $\delta > 0$ , the permittivity distribution in  $\mathbb{R}^2$  is given by

$$\varepsilon_\delta = \begin{cases} 1 & \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\ -1 + i\delta & \text{in } \Omega \setminus \overline{D}, \\ 1 & \text{in } D. \end{cases} \tag{1}$$

We may consider the configuration as a core with permittivity 1 coated by the shell  $\Omega \setminus \overline{D}$  with permittivity  $-1 + i\delta$ . For a given function  $f$  compactly supported in  $\mathbb{R}^2$  satisfying

$$\int_{\mathbb{R}^2} f dx = 0 \tag{2}$$

(which physically is required by conservation of charge), we consider the following dielectric problem:

$$\nabla \cdot \varepsilon_\delta \nabla V_\delta = \alpha f \quad \text{in } \mathbb{R}^2, \tag{3}$$

with the decay condition  $V_\delta(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ .

A fundamental problem is to identify those sources  $f$  such that when  $\alpha = 1$ , then, first,

$$E_\delta := \int_{\Omega \setminus \bar{D}} \delta |\nabla V_\delta|^2 dx \rightarrow \infty \quad \text{as } \delta \rightarrow 0, \quad (4)$$

and, second,  $V_\delta$  remains bounded outside some radius  $a$ :

$$|V_\delta(x)| < C, \quad \text{when } |x| > a \quad (5)$$

for some constants  $C$  and  $a$  independent of  $\delta$  (which necessitates that the ball  $B_a$  contains the entire region of anomalous localized resonance). The quantity  $E_\delta$  is proportional to the electromagnetic power dissipated into heat by the time harmonic electrical field averaged over time. Hence (4) implies an infinite amount of energy dissipated per unit time in the limit  $\delta \rightarrow 0$ , which is unphysical. If instead we choose  $\alpha = 1/\sqrt{E_\delta}$ , then the source  $\alpha f$  will produce the same power independent of  $\delta$ , and the new associated solution  $V_\delta$  (which is the previous solution  $V_\delta$  multiplied by  $\alpha$ ) will approach zero outside the radius  $a$ : cloaking due to anomalous localized resonance (CALR) occurs. The conditions (4) and (5) are sufficient to ensure CALR; a necessary and sufficient condition is that (with  $\alpha = 1$ )  $V_\delta/\sqrt{E_\delta}$  goes to zero outside some radius as  $\delta \rightarrow 0$ . We also consider a weaker blow-up of the energy dissipation, namely,

$$\limsup_{\delta \rightarrow 0} E_\delta = \infty. \quad (6)$$

We say that weak CALR takes place if (6) holds (in addition to (5)). Then the (renormalized) source  $f/\sqrt{E_\delta}$  will be essentially invisible at an infinite sequence of small values of  $\delta$  tending to zero (but would be quite visible for values of  $\delta$  interspersed between this sequence if CALR does not additionally hold).

The aim of this paper is to develop a general method based on potential theory for studying cloaking due to anomalous resonance. Using layer potential techniques, we reduce the problem to a singularly perturbed system of integral equations. The system is non-self-adjoint. A symmetrization technique is introduced in order to express the solution in terms of the eigenfunctions of a self-adjoint compact operator. The symmetrization technique is based on a generalization of a Calderón identity to the system of integral equations under consideration and a general theorem on symmetrization of non-selfadjoint operators obtained in a recent paper by KHAVINSON et al. [17].

Using the technique developed in this paper, we are able to provide a necessary and sufficient condition on the source term under which the blowup (4) of the power dissipation takes place. The condition is given in terms of the Newtonian potential of the source, which is the solution for the potential in the absence of the plasmonic structure.

In the case of an annulus ( $D$  is the disk of radius  $r_i$  and  $\Omega$  is the concentric disk of radius  $r_e$ ), it is known [27] that there exists a critical radius (the cloaking radius),

$$r_* = \sqrt{r_e^3 r_i^{-1}}, \quad (7)$$

such that any finite collection of dipole sources located at fixed positions within the annulus  $B_{r_*} \setminus \overline{B}_e$  is cloaked. We show (see Theorem 5.3 below) that if  $f$  is an integrable function supported in  $E \subset B_{r_*} \setminus \overline{B}_e$  satisfying (2) and the Newtonian potential of  $f$  does not extend as a harmonic function in  $B_{r_*}$ , then weak CALR takes place. Moreover, we show that if the Fourier coefficients of the Newtonian potential of  $f$  satisfy a mild gap condition, then CALR takes place. Using this result, we are able to show that a quadrupole source inside the annulus  $B_{r_*} \setminus \overline{B}_e$  would be cloaked, in agreement with the numerical results of [34]. Conversely, we show that if the source function  $f$  is supported outside  $B_{r_*}$ , then (4) does not happen and no cloaking occurs. We stress that we assume  $f$  does not depend on  $\delta$ ; the results of [6] strongly suggest that there exist sequences of sources  $f_\delta$  supported in  $E \subset B_{r_*} \setminus \overline{B}_e$  with non-trivial Newtonian potentials outside  $E$ , such that the power dissipation does not blow up, and such that  $V_\delta$  does not go to zero outside  $B_{r_*}$  as  $\delta \rightarrow 0$ .

This paper is organized as follows. In Section 2 we transform the problem into a system of integral equations using layer potentials. In Section 3, we develop a spectral theory for the relevant integral operators and derive a necessary and sufficient condition for CALR to take place. Section 4 treats the special case of an annulus.

### 2. Layer Potential Formulation

Let  $G$  be the fundamental solution to the Laplacian in two dimensions which is given by

$$G(x) = \frac{1}{2\pi} \ln |x|.$$

Let  $\Gamma_i := \partial D$  and  $\Gamma_e := \partial \Omega$ . For  $\Gamma = \Gamma_i$  or  $\Gamma_e$ , we denote, respectively, the single and double layer potentials of a function  $\varphi \in L^2(\Gamma)$  as  $\mathcal{S}_\Gamma[\varphi]$  and  $\mathcal{D}_\Gamma[\varphi]$ , where

$$\begin{aligned} \mathcal{S}_\Gamma[\varphi](x) &:= \int_\Gamma G(x - y)\varphi(y) \, d\sigma(y), \quad x \in \mathbb{R}^2, \\ \mathcal{D}_\Gamma[\varphi](x) &:= \int_\Gamma \frac{\partial}{\partial \nu(y)} G(x - y)\varphi(y) \, d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \Gamma. \end{aligned}$$

Here,  $\nu(y)$  is the outward unit normal to  $\Gamma$  at  $y$ .

We also define a boundary integral operator  $\mathcal{K}_\Gamma$  on  $L^2(\Gamma)$  by

$$\mathcal{K}_\Gamma[\varphi](x) := \frac{1}{2\pi} \int_\Gamma \frac{\langle y - x, \nu(y) \rangle}{|x - y|^2} \varphi(y) \, d\sigma(y),$$

and let  $\mathcal{K}_\Gamma^*$  be the  $L^2$ -adjoint of  $\mathcal{K}_\Gamma$ . Hence, the operator  $\mathcal{K}_\Gamma^*$  is given by

$$\mathcal{K}_\Gamma^*[\varphi](x) = \frac{1}{2\pi} \int_\Gamma \frac{\langle x - y, \nu(x) \rangle}{|x - y|^2} \varphi(y) \, d\sigma(y), \quad \varphi \in L^2(\Gamma).$$

Here and throughout this paper,  $\langle \cdot, \cdot \rangle$  denotes the scalar product in  $\mathbb{R}^2$ . The operators  $\mathcal{K}_\Gamma$  and  $\mathcal{K}_\Gamma^*$  are sometimes called Neumann–Poincaré operators. These operators are compact in  $L^2(\Gamma)$  if  $\Gamma$  is  $\mathcal{C}^{1,\alpha}$  for some  $\alpha > 0$ .

The following notation will be used throughout this paper. For a function  $u$  defined on  $\mathbb{R}^2 \setminus \Gamma$ , we denote

$$u|_{\pm}(x) := \lim_{t \rightarrow 0^+} u(x \pm t\nu(x)), \quad x \in \Gamma,$$

and

$$\frac{\partial u}{\partial \nu} \Big|_{\pm}(x) := \lim_{t \rightarrow 0^+} \langle \nabla u(x \pm t\nu(x)), \nu(x) \rangle, \quad x \in \Gamma,$$

if the limits exist.

The following jump formulas relate the traces of the double layer potential and the normal derivative of the single layer potential to the operators  $\mathcal{K}_\Gamma$  and  $\mathcal{K}_\Gamma^*$ . We have

$$(\mathcal{D}_\Gamma[\varphi])|_{\pm}(x) = (\mp \frac{1}{2}I + \mathcal{K}_\Gamma)[\varphi](x), \quad x \in \Gamma, \quad (8)$$

$$\frac{\partial}{\partial \nu} \mathcal{S}_\Gamma[\varphi] \Big|_{\pm}(x) = (\pm \frac{1}{2}I + \mathcal{K}_\Gamma^*)[\varphi](x), \quad x \in \Gamma. \quad (9)$$

See, for example, [2,9].

Let  $F$  be the Newtonian potential of  $f$ , that is,

$$F(x) = \int_{\mathbb{R}^2} G(x-y)f(y)dy, \quad x \in \mathbb{R}^2. \quad (10)$$

Then  $F$  satisfies  $\Delta F = f$  in  $\mathbb{R}^2$ , and the solution  $V_\delta$  to (3) may be represented as

$$V_\delta(x) = F(x) + \mathcal{S}_{\Gamma_i}[\varphi_i](x) + \mathcal{S}_{\Gamma_e}[\varphi_e](x) \quad (11)$$

for some functions  $\varphi_i \in L_0^2(\Gamma_i)$  and  $\varphi_e \in L_0^2(\Gamma_e)$  ( $L_0^2$  is the collection of all square integrable functions with the integral zero). The transmission conditions along the interfaces  $\Gamma_e$  and  $\Gamma_i$  satisfied by  $V_\delta$  read

$$\begin{aligned} (-1 + i\delta) \frac{\partial V_\delta}{\partial \nu} \Big|_+ &= \frac{\partial V_\delta}{\partial \nu} \Big|_- \quad \text{on } \Gamma_i \\ \frac{\partial V_\delta}{\partial \nu} \Big|_+ &= (-1 + i\delta) \frac{\partial V_\delta}{\partial \nu} \Big|_- \quad \text{on } \Gamma_e. \end{aligned}$$

Hence the pair of potentials  $(\varphi_i, \varphi_e)$  is the solution to the following system of integral equations:

$$\begin{cases} (-1 + i\delta) \frac{\partial \mathcal{S}_{\Gamma_i}[\varphi_i]}{\partial \nu_i} \Big|_+ - \frac{\partial \mathcal{S}_{\Gamma_i}[\varphi_i]}{\partial \nu_i} \Big|_- + (-2 + i\delta) \frac{\partial \mathcal{S}_{\Gamma_e}[\varphi_e]}{\partial \nu_i} = (2 - i\delta) \frac{\partial F}{\partial \nu_i} & \text{on } \Gamma_i, \\ (2 - i\delta) \frac{\partial \mathcal{S}_{\Gamma_i}[\varphi_i]}{\partial \nu_e} + \frac{\partial \mathcal{S}_{\Gamma_e}[\varphi_e]}{\partial \nu_e} \Big|_+ - (-1 + i\delta) \frac{\partial \mathcal{S}_{\Gamma_e}[\varphi_e]}{\partial \nu_e} \Big|_- = (-2 + i\delta) \frac{\partial F}{\partial \nu_e} & \text{on } \Gamma_e. \end{cases}$$

Note that we have used the notation  $\nu_i$  and  $\nu_e$  to indicate the outward normal on  $\Gamma_i$  and  $\Gamma_e$ , respectively. Using the jump formula (9) for the normal derivative of the single layer potentials, the above equations can be rewritten as

$$\begin{bmatrix} -z_\delta I + \mathcal{K}_{\Gamma_i}^* & \frac{\partial}{\partial \nu_i} \mathcal{S}_{\Gamma_e} \\ \frac{\partial}{\partial \nu_e} \mathcal{S}_{\Gamma_i} & z_\delta I + \mathcal{K}_{\Gamma_e}^* \end{bmatrix} \begin{bmatrix} \varphi_i \\ \varphi_e \end{bmatrix} = - \begin{bmatrix} \frac{\partial F}{\partial \nu_i} \\ \frac{\partial F}{\partial \nu_e} \end{bmatrix} \tag{12}$$

on  $L_0^2(\Gamma_i) \times L_0^2(\Gamma_e)$ , where we set

$$z_\delta = \frac{i\delta}{2(2 - i\delta)}. \tag{13}$$

Note that the operator in (12) can be viewed as a compact perturbation of the operator

$$R_\delta := \begin{bmatrix} -z_\delta I + \mathcal{K}_{\Gamma_i}^* & 0 \\ 0 & z_\delta I + \mathcal{K}_{\Gamma_e}^* \end{bmatrix}. \tag{14}$$

We now recall KELLOGG’s result in [16] on the spectrums of  $\mathcal{K}_{\Gamma_i}^*$  and  $\mathcal{K}_{\Gamma_e}^*$ . The eigenvalues of  $\mathcal{K}_{\Gamma_i}^*$  and  $\mathcal{K}_{\Gamma_e}^*$  lie in the interval  $[-\frac{1}{2}, \frac{1}{2}]$ . Observe that  $z_\delta \rightarrow 0$  as  $\delta \rightarrow 0$  and that there are sequences of eigenvalues of  $\mathcal{K}_{\Gamma_i}^*$  and  $\mathcal{K}_{\Gamma_e}^*$  approaching 0 since  $\mathcal{K}_{\Gamma_i}^*$  and  $\mathcal{K}_{\Gamma_e}^*$  are compact. So 0 is the essential singularity of the operator valued meromorphic function

$$\lambda \in \mathbb{C} \mapsto (\lambda I + \mathcal{K}_{\Gamma_e}^*)^{-1}.$$

This causes a serious difficulty in dealing with (12). We emphasize that  $\mathcal{K}_{\Gamma_e}^*$  is not self-adjoint in general. In fact,  $\mathcal{K}_{\Gamma_e}^*$  is self-adjoint only when  $\Gamma_e$  is a circle or a sphere (see [23]).

Let  $\mathcal{H} = L^2(\Gamma_i) \times L^2(\Gamma_e)$ . We write (12) in a slightly different form. We first apply the operator

$$\begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} : \mathcal{H} \rightarrow \mathcal{H}$$

to (12), then the equation becomes

$$\begin{bmatrix} z_\delta I - \mathcal{K}_{\Gamma_i}^* & -\frac{\partial}{\partial \nu_i} \mathcal{S}_{\Gamma_e} \\ \frac{\partial}{\partial \nu_e} \mathcal{S}_{\Gamma_i} & z_\delta I + \mathcal{K}_{\Gamma_e}^* \end{bmatrix} \begin{bmatrix} \varphi_i \\ \varphi_e \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial \nu_i} \\ -\frac{\partial F}{\partial \nu_e} \end{bmatrix}. \tag{15}$$

Let the Neumann–Poincaré-type operator  $\mathbb{K}^* : \mathcal{H} \rightarrow \mathcal{H}$  be defined by

$$\mathbb{K}^* := \begin{bmatrix} -\mathcal{K}_{\Gamma_i}^* & -\frac{\partial}{\partial \nu_i} \mathcal{S}_{\Gamma_e} \\ \frac{\partial}{\partial \nu_e} \mathcal{S}_{\Gamma_i} & \mathcal{K}_{\Gamma_e}^* \end{bmatrix}, \tag{16}$$

and let

$$\Phi := \begin{bmatrix} \varphi_i \\ \varphi_e \end{bmatrix}, \quad g := \begin{bmatrix} \frac{\partial F}{\partial v_i} \\ \frac{\partial F}{\partial v_e} \end{bmatrix}. \tag{17}$$

Then, (15) can be rewritten in the form

$$(z_\delta \mathbb{I} + \mathbb{K}^*)\Phi = g, \tag{18}$$

where  $\mathbb{I}$  is given by

$$\mathbb{I} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{19}$$

### 3. Properties of $\mathbb{K}^*$

In the following we provide some properties of  $\mathbb{K}^*$ . In particular, we compute the adjoint operator  $\mathbb{K}$  of  $\mathbb{K}^*$ , study the spectrum of  $\mathbb{K}^*$ , and show that  $\mathbb{K}^*$  is symmetrizable on the space  $\mathcal{H} = L^2(\Gamma_i) \times L^2(\Gamma_e)$ .

#### 3.1. Adjoint Operator of $\mathbb{K}^*$

We first compute the adjoint of  $\mathbb{K}^*$ . Denote by  $\langle \cdot, \cdot \rangle_{L^2(\Gamma)}$  the Hermitian product on  $L^2(\Gamma)$  for  $\Gamma = \Gamma_i$  or  $\Gamma_e$ . It is easy to see that

$$\left\langle \frac{\partial}{\partial v_i} \mathcal{S}_{\Gamma_e}[\varphi_e], \Psi_i \right\rangle_{L^2(\Gamma_i)} = \langle \varphi_e, \mathcal{D}_{\Gamma_i}[\Psi_i] \rangle_{L^2(\Gamma_e)}, \tag{20}$$

and

$$\left\langle \frac{\partial}{\partial v_e} \mathcal{S}_{\Gamma_i}[\varphi_i], \Psi_e \right\rangle_{L^2(\Gamma_e)} = \langle \varphi_i, \mathcal{D}_{\Gamma_e}[\Psi_e] \rangle_{L^2(\Gamma_i)}. \tag{21}$$

Thus the  $L^2$ -adjoint of  $\mathbb{K}^*$ ,  $\mathbb{K}$ , is given by

$$\mathbb{K} = \begin{bmatrix} -\mathcal{K}_{\Gamma_i} & \mathcal{D}_{\Gamma_e} \\ -\mathcal{D}_{\Gamma_i} & \mathcal{K}_{\Gamma_e} \end{bmatrix}. \tag{22}$$

We emphasize that the operators  $\mathcal{D}_{\Gamma_e}$  and  $\mathcal{D}_{\Gamma_i}$  in the off-diagonal entries are those from  $L^2(\Gamma_e)$  into  $L^2(\Gamma_i)$ , and from  $L^2(\Gamma_i)$  into  $L^2(\Gamma_e)$ , respectively.

3.2. Spectrum of  $\mathbb{K}^*$ 

We now look into the spectrum of  $\mathbb{K}^*$ . We have the following proposition which is a generalization of KELLOGG's result in [16] on the spectrum of the operator  $\mathcal{K}_\Gamma^*$  on  $L^2(\Gamma)$ .

**Lemma 3.1** *The spectrum of  $\mathbb{K}^*$  lies in the interval  $[-1/2, 1/2]$ .*

**Proof.** Let  $\lambda$  be a point in the spectrum of  $\mathbb{K}^*$ . Then there exists  $\Phi = (\varphi_i, \varphi_e)$  with  $\varphi_i \in L^2(\Gamma_i)$  and  $\varphi_e \in L^2(\Gamma_e)$  such that

$$\begin{cases} \mathcal{K}_{\Gamma_i}^*[\varphi_i] + \frac{\partial}{\partial \nu_i} \mathcal{S}_{\Gamma_e}[\varphi_e] = -\lambda \varphi_i & \text{on } \Gamma_i, \\ \frac{\partial}{\partial \nu_e} \mathcal{S}_{\Gamma_i}[\varphi_i] + \mathcal{K}_{\Gamma_e}^*[\varphi_e] = \lambda \varphi_e & \text{on } \Gamma_e. \end{cases}$$

By integrating the above equations on  $\Gamma_i$  and  $\Gamma_e$ , respectively, and using (20) and (21), we obtain

$$\begin{cases} \left(\lambda + \frac{1}{2}\right) \int_{\Gamma_i} \varphi_i d\sigma = 0, \\ \left(\lambda - \frac{1}{2}\right) \int_{\Gamma_e} \varphi_e d\sigma = - \int_{\Gamma_i} \varphi_i d\sigma. \end{cases}$$

Here, we used the facts that  $\mathcal{K}_{\Gamma_i}[1] = 1/2$ ,  $\mathcal{K}_{\Gamma_e}[1] = 1/2$ ,  $\mathcal{D}_{\Gamma_e}[1] = 1$  on  $\Gamma_i$ , and  $\mathcal{D}_{\Gamma_i}[1] = 0$  on  $\Gamma_e$ . Thus, either  $\lambda = \pm 1/2$  or  $\lambda \neq \pm 1/2$  with  $\varphi_i \in L_0^2(\Gamma_i)$  and  $\varphi_e \in L_0^2(\Gamma_e)$  holds. We assume that  $\lambda \neq \pm 1/2$  and consider

$$u(x) := \mathcal{S}_{\Gamma_i}[\varphi_i](x) + \mathcal{S}_{\Gamma_e}[\varphi_e](x), \quad x \in \mathbb{R}^2.$$

Since  $\varphi_i \in L_0^2(\Gamma_i)$  and  $\varphi_e \in L_0^2(\Gamma_e)$ , we have  $u(x) = O(|x|^{-1})$  as  $|x| \rightarrow \infty$ , and hence the following integrals are finite:

$$A = \int_D |\nabla u|^2 dx, \quad B = \int_{\Omega \setminus \bar{D}} |\nabla u|^2 dx, \quad C = \int_{\mathbb{R}^2 \setminus \bar{\Omega}} |\nabla u|^2 dx.$$

Since  $\lambda$  is an eigenvalue of  $\mathbb{K}^*$ , we obtain from Green's formulas and the jump relation (9) that

$$\begin{aligned} A &= -\left(\lambda + \frac{1}{2}\right) \int_{\Gamma_i} \bar{u} \varphi_i d\sigma, \\ B &= \left(\lambda - \frac{1}{2}\right) \int_{\Gamma_i} \bar{u} \varphi_i d\sigma + \left(\lambda - \frac{1}{2}\right) \int_{\Gamma_e} \bar{u} \varphi_e d\sigma, \end{aligned}$$

and

$$C = -\left(\lambda + \frac{1}{2}\right) \int_{\Gamma_e} \bar{u} \varphi_e d\sigma.$$



Thus, we get

$$\frac{\lambda - \frac{1}{2}}{\lambda + \frac{1}{2}}(A + C) = -B,$$

which implies

$$\lambda = \frac{1}{2} - \frac{B}{A + B + C}.$$

Since  $A, B, C \geq 0$  and  $A + B + C > 0$ , we conclude that  $-1/2 < \lambda < 1/2$ . This completes the proof.  $\square$

### 3.3. Calderón's Identity

We prove that there exists a positive self-adjoint operator  $-\mathbb{S}$  such that  $\mathbb{S}\mathbb{K}^* = \mathbb{K}\mathbb{S}$  on  $\mathcal{H} = L^2(\Gamma_i) \times L^2(\Gamma_e)$ . This is a Calderón-type identity. It will be used to prove that  $\mathbb{K}^*$  is symmetrizable.

In fact,  $\mathbb{S}$  is given by

$$\mathbb{S} = \begin{bmatrix} \mathcal{S}_{\Gamma_i} & \mathcal{S}_{\Gamma_e} \\ \mathcal{S}_{\Gamma_i} & \mathcal{S}_{\Gamma_e} \end{bmatrix}. \tag{23}$$

Again we emphasize that the operator  $\mathcal{S}_{\Gamma_e}$  off the diagonal is the one from  $L^2(\Gamma_e)$  into  $L^2(\Gamma_i)$ , and likewise for  $\mathcal{S}_{\Gamma_i}$  off the diagonal.

**Lemma 3.2** *The operator  $-\mathbb{S}$  is self-adjoint and  $-\mathbb{S} \geq 0$  on  $\mathcal{H}$ .*

**Proof.** It is clear that  $\begin{bmatrix} \mathcal{S}_{\Gamma_i} & 0 \\ 0 & \mathcal{S}_{\Gamma_e} \end{bmatrix}$  is self-adjoint. On the other hand, from the relations

$$\langle \mathcal{S}_{\Gamma_i}[\varphi_i], \varphi_e \rangle_{L^2(\Gamma_e)} = \langle \varphi_i, \mathcal{S}_{\Gamma_e}[\varphi_e] \rangle_{L^2(\Gamma_i)}$$

and

$$\langle \mathcal{S}_{\Gamma_e}[\varphi_e], \varphi_i \rangle_{L^2(\Gamma_i)} = \langle \varphi_e, \mathcal{S}_{\Gamma_i}[\varphi_i] \rangle_{L^2(\Gamma_e)},$$

it follows that  $\begin{bmatrix} 0 & \mathcal{S}_{\Gamma_e} \\ \mathcal{S}_{\Gamma_i} & 0 \end{bmatrix}$  is self-adjoint and hence  $\mathbb{S}$  is self-adjoint.

Let  $\Phi = (\varphi_i, \varphi_e) \in \mathcal{H}$  and define

$$u(x) = \mathcal{S}_{\Gamma_i}[\varphi_i](x) + \mathcal{S}_{\Gamma_e}[\varphi_e](x). \tag{24}$$

Then we have

$$\begin{aligned} \int_D |\nabla u|^2 dx &= \int_{\partial D} \bar{u} \left( -\frac{1}{2} \varphi_i + \mathcal{K}_{\Gamma_i}^*[\varphi_i] + \frac{\partial}{\partial \nu_i} \mathcal{S}_{\Gamma_e}[\varphi_e] \right) d\sigma, \\ \int_{\Omega \setminus \bar{D}} |\nabla u|^2 dx &= - \int_{\partial D} \bar{u} \left( \frac{1}{2} \varphi_i + \mathcal{K}_{\Gamma_i}^*[\varphi_i] + \frac{\partial}{\partial \nu_i} \mathcal{S}_{\Gamma_e}[\varphi_e] \right) d\sigma \\ &\quad + \int_{\partial \Omega} \bar{u} \left( -\frac{1}{2} \varphi_e + \mathcal{K}_{\Gamma_e}^*[\varphi_e] + \frac{\partial}{\partial \nu_e} \mathcal{S}_{\Gamma_i}[\varphi_i] \right) d\sigma, \end{aligned}$$

and

$$\int_{\mathbb{R}^2 \setminus \bar{\Omega}} |\nabla u|^2 dx = - \int_{\partial\Omega} \bar{u} \left( \frac{1}{2} \varphi_e + \mathcal{K}_{\Gamma_e}^*[\varphi_e] + \frac{\partial}{\partial \nu_e} \mathcal{S}_{\Gamma_i}[\varphi_i] \right) d\sigma.$$

Summing up the above three identities we find

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla u|^2 dx &= - \int_{\partial D} \bar{u} \varphi_i d\sigma - \int_{\partial\Omega} \bar{u} \varphi_e d\sigma \\ &= \langle \Phi, -\mathbb{S}[\Phi] \rangle_{\mathcal{H}}. \end{aligned}$$

Thus  $-\mathbb{S} \geq 0$ . This completes the proof.  $\square$

To prove that  $\mathbb{K}^*$  is symmetrizable, we shall make use of the following lemma, which can be proved by Green’s formulas.

**Lemma 3.3** *Let  $E \subset \mathbb{R}^2$  be a bounded domain.*

(i) *If  $u$  is a solution of  $\Delta u = 0$  in  $E$ , then*

$$\mathcal{S}_{\partial E} \left[ \frac{\partial u}{\partial \nu} \Big|_{-} \right] (x) = \mathcal{D}_{\partial E} \left[ u \Big|_{-} \right] (x), \quad x \in \mathbb{R}^2 \setminus \bar{E}. \tag{25}$$

(ii) *If  $u$  is a solution of*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \setminus \bar{E}, \\ u(x) \rightarrow 0, & |x| \rightarrow \infty, \end{cases} \tag{26}$$

*then*

$$\mathcal{S}_{\partial E} \left[ \frac{\partial u}{\partial \nu} \Big|_{+} \right] (x) = \mathcal{D}_{\partial E} \left[ u \Big|_{+} \right] (x), \quad x \in E.$$

Note that the well-known Calderón’s identity (also known as Plemelj’s symmetrization principle)

$$\mathcal{S}_{\partial E} \mathcal{K}_{\partial E}^* = \mathcal{K}_{\partial E} \mathcal{S}_{\partial E} \tag{27}$$

is an immediate consequence of Lemma 3.3. In fact, if we put  $u = \mathcal{S}_{\partial E}[\varphi]$  in (25), we have

$$-\frac{1}{2} \mathcal{S}_{\partial E}[\varphi](x) + \mathcal{S}_{\partial E} \mathcal{K}_{\partial E}^*[\varphi](x) = \mathcal{D}_{\partial E} \mathcal{S}_{\partial E}[\varphi](x), \quad x \in \mathbb{R}^2 \setminus \bar{E}.$$

By taking the limit as  $x \rightarrow \partial E$  from outside  $E$ , we obtain (27) using the jump relation (8) of the double layer potential.

The following lemma is a generalization of Calderón’s identity.

**Lemma 3.4** *Let  $\mathbb{S}$  and  $\mathbb{K}$  be given by (23) and (16), respectively. Then*

$$\mathbb{S} \mathbb{K}^* = \mathbb{K} \mathbb{S}, \tag{28}$$

*that is,  $\mathbb{S} \mathbb{K}^*$  is self-adjoint.*

**Proof.** Notice that

$$\mathbb{S}\mathbb{K}^* = \begin{bmatrix} -\mathcal{S}_{\Gamma_i}\mathcal{K}_{\Gamma_i}^* + \mathcal{S}_{\Gamma_e}\frac{\partial}{\partial v_e}\mathcal{S}_{\Gamma_i} & -\mathcal{S}_{\Gamma_i}\frac{\partial}{\partial v_i}\mathcal{S}_{\Gamma_e} + \mathcal{S}_{\Gamma_e}\mathcal{K}_{\Gamma_e}^* \\ -\mathcal{S}_{\Gamma_i}\mathcal{K}_{\Gamma_i}^* + \mathcal{S}_{\Gamma_e}\frac{\partial}{\partial v_e}\mathcal{S}_{\Gamma_i} & -\mathcal{S}_{\Gamma_i}\frac{\partial}{\partial v_i}\mathcal{S}_{\Gamma_e} + \mathcal{S}_{\Gamma_e}\mathcal{K}_{\Gamma_e}^* \end{bmatrix}$$

and

$$\mathbb{K}\mathbb{S} = \begin{bmatrix} -\mathcal{K}_{\Gamma_i}\mathcal{S}_{\Gamma_i} + \mathcal{D}_{\Gamma_e}\mathcal{S}_{\Gamma_i} & -\mathcal{K}_{\Gamma_i}\mathcal{S}_{\Gamma_e} + \mathcal{D}_{\Gamma_e}\mathcal{S}_{\Gamma_e} \\ -\mathcal{D}_{\Gamma_i}\mathcal{S}_{\Gamma_i} + \mathcal{K}_{\Gamma_e}\mathcal{S}_{\Gamma_i} & -\mathcal{D}_{\Gamma_i}\mathcal{S}_{\Gamma_e} + \mathcal{K}_{\Gamma_e}\mathcal{S}_{\Gamma_e} \end{bmatrix}.$$

We now check the following.

- $(\mathbb{S}\mathbb{K}^*)_{11} = (\mathbb{K}\mathbb{S})_{11}$ : by (27) it follows that  $\mathcal{S}_{\Gamma_i}\mathcal{K}_{\Gamma_i}^* = \mathcal{K}_{\Gamma_i}\mathcal{S}_{\Gamma_i}$  on  $\Gamma_i$ . If we set  $u(x) = \mathcal{S}_{\Gamma_i}[\varphi_i](x)$  and  $E = \Omega$  in Lemma 3.3 (ii), we have

$$\mathcal{S}_{\Gamma_e}\frac{\partial}{\partial v_e}\mathcal{S}_{\Gamma_i}[\varphi_i] = \mathcal{D}_{\Gamma_e}\mathcal{S}_{\Gamma_i}[\varphi_i] \quad \text{on } \Gamma_i.$$

This implies  $(\mathbb{S}\mathbb{K}^*)_{11} = (\mathbb{K}\mathbb{S})_{11}$ .

- $(\mathbb{S}\mathbb{K}^*)_{12} = (\mathbb{K}\mathbb{S})_{12}$ : from Lemma 3.3 (ii), by setting  $u(x) = \mathcal{S}_{\Gamma_e}[\varphi_e](x)$  and  $E = D$ , we find

$$\mathcal{S}_{\Gamma_i}\frac{\partial}{\partial v_i}\mathcal{S}_{\Gamma_e}[\varphi_e](x) = \mathcal{D}_{\Gamma_i}\mathcal{S}_{\Gamma_e}[\varphi_e](x), \quad x \in \mathbb{R}^2 \setminus \bar{D}.$$

By taking the limit as  $x \rightarrow \Gamma_i|_+$ , we find

$$\mathcal{S}_{\Gamma_i}\frac{\partial}{\partial v_i}\mathcal{S}_{\Gamma_e}[\varphi_e] = -\frac{1}{2}\mathcal{S}_{\Gamma_e}[\varphi_e] + \mathcal{K}_{\Gamma_i}\mathcal{S}_{\Gamma_e}[\varphi_e] \quad \text{on } \Gamma_i. \quad (29)$$

Now, we use Lemma 3.3 (ii) by taking  $u = \mathcal{S}_{\Gamma_e}[\varphi_e]$  and  $E = \Omega$  and find

$$\mathcal{S}_{\Gamma_e}\left[\frac{\partial\mathcal{S}_{\Gamma_e}[\varphi_e]}{\partial v_e}\Big|_+\right](x) = \mathcal{D}_{\Gamma_e}\mathcal{S}_{\Gamma_e}[\varphi_e](x) \quad \text{for } x \in \Omega,$$

and thus we have

$$\frac{1}{2}\mathcal{S}_{\Gamma_e}[\varphi_e] + \mathcal{S}_{\Gamma_e}\mathcal{K}_{\Gamma_e}^*[\varphi_e] = \mathcal{D}_{\Gamma_e}\mathcal{S}_{\Gamma_e}[\varphi_e] \quad \text{on } \Gamma_i. \quad (30)$$

Summing up (29) and (30), we find that  $(\mathbb{S}\mathbb{K}^*)_{12} = (\mathbb{K}\mathbb{S})_{12}$ .

- $(\mathbb{S}\mathbb{K}^*)_{21} = (\mathbb{K}\mathbb{S})_{21}$ : we use Lemma 3.3 (i) by setting  $u = \mathcal{S}_{\Gamma_i}[\varphi_i]$  and  $E = D$  and find

$$\mathcal{S}_{\Gamma_i}\left[\frac{\partial\mathcal{S}_{\Gamma_i}[\varphi_i]}{\partial v_i}\Big|_-\right](x) = \mathcal{D}_{\Gamma_i}\mathcal{S}_{\Gamma_i}[\varphi_i](x) \quad \text{for } x \in \mathbb{R}^2 \setminus \bar{D},$$

and thus we have

$$-\frac{1}{2}\mathcal{S}_{\Gamma_i}[\varphi_i] + \mathcal{S}_{\Gamma_i}\mathcal{K}_{\Gamma_i}^*[\varphi_i] = \mathcal{D}_{\Gamma_i}\mathcal{S}_{\Gamma_i}[\varphi_i] \quad \text{on } \Gamma_e. \quad (31)$$

By setting  $u = \mathcal{S}_{\Gamma_i}[\varphi_i]$  and  $E = \Omega$  in Lemma 3.3 (ii), we find

$$\mathcal{S}_{\Gamma_e} \frac{\partial}{\partial v_e} \mathcal{S}_{\Gamma_i}[\varphi_i](x) = \mathcal{D}_{\Gamma_e} \mathcal{S}_{\Gamma_i}[\varphi_i](x), \quad x \in \Omega,$$

and by taking the limit as  $x \rightarrow \Gamma_e|_-,$  we find

$$\mathcal{S}_{\Gamma_e} \frac{\partial}{\partial v_e} \mathcal{S}_{\Gamma_i}[\varphi_i] = \frac{1}{2} \mathcal{S}_{\Gamma_i}[\varphi_i] + \mathcal{K}_{\Gamma_e} \mathcal{S}_{\Gamma_i}[\varphi_i], \quad \text{on } \Gamma_e. \tag{32}$$

Summing up (31) and (32), we find that  $(\mathbb{S}\mathbb{K}^*)_{21} = (\mathbb{K}\mathbb{S})_{21}.$

- $(\mathbb{S}\mathbb{K}^*)_{22} = (\mathbb{K}\mathbb{S})_{22}:$  by (27) it follows that  $\mathcal{S}_{\Gamma_e} \mathcal{K}_{\Gamma_e}^* = \mathcal{K}_{\Gamma_e} \mathcal{S}_{\Gamma_e}$  on  $\Gamma_e.$  Thus, we have only to prove that

$$\mathcal{S}_{\Gamma_i} \frac{\partial}{\partial v_i} \mathcal{S}_{\Gamma_e}[\varphi_e] = \mathcal{D}_{\Gamma_i} \mathcal{S}_{\Gamma_e}[\varphi_e] \quad \text{on } \Gamma_e,$$

which follows from Lemma 3.3 (i) by setting  $u(x) = \mathcal{S}_{\Gamma_e}[\varphi_e](x)$  and  $E = D.$

This completes the proof.  $\square$

### 3.4. $\mathbb{K}^*$ Is Symmetrizable

Let  $\mathcal{C}_p(\mathcal{H}), 1 \leq p < \infty,$  be the Schatten–von Neumann class of compact operators acting on  $\mathcal{H}$  (see [10]). We recall that a compact operator  $A$  on  $\mathcal{H}$  is in the Schatten–von Neumann class  $\mathcal{C}_p(\mathcal{H}),$  with  $1 \leq p < \infty,$  if the sequence of its singular values is in  $l_p = \{(\mu_n)_{n \in \mathbb{Z}} : \sum_{n \in \mathbb{Z}} |\mu_n|^p < \infty\}.$  An equivalent characterization is  $\sum_n \|A\Phi_n\|^p < \infty$  for any orthonormal basis  $(\Phi_n)$  of  $\mathcal{H}.$  The elements of  $\mathcal{C}_2(\mathcal{H})$  are the Hilbert–Schmidt operators. It is proved in [17] that  $\mathcal{K}_{\Gamma_i}^* \in \mathcal{C}_2(L^2(\Gamma_i))$  and  $\mathcal{K}_{\Gamma_e}^* \in \mathcal{C}_2(L^2(\Gamma_e))$  are Hilbert–Schmidt operators. On the other hand,  $\frac{\partial}{\partial v_i} \mathcal{S}_{\Gamma_e}$  and  $\frac{\partial}{\partial v_e} \mathcal{S}_{\Gamma_i}$  are Hilbert–Schmidt operators on  $L^2(\Gamma_i)$  and  $L^2(\Gamma_e),$  respectively, because they have smooth integral kernels. Thus they belong to  $\mathcal{C}_2.$  So we easily have the following lemma.

**Lemma 3.5**  $\mathbb{K}^* \in \mathcal{C}_2(\mathcal{H}).$

By Lemma 3.2,  $-\mathbb{S}$  is self-adjoint and  $-\mathbb{S} \geq 0$  on  $\mathcal{H}.$  Thus there exists a unique square root of  $-\mathbb{S},$  which we denote by  $\sqrt{-\mathbb{S}};$  furthermore,  $\sqrt{-\mathbb{S}}$  is self-adjoint and  $\sqrt{-\mathbb{S}} \geq 0$  (see for instance Theorem 13.31 in [36]). We now look into the kernel of  $\mathbb{S}.$  If  $\Phi = (\varphi_i, \varphi_e) \in \text{Ker}(\mathbb{S}),$  then the function  $u$  defined by

$$u(x) := \mathcal{S}_{\Gamma_i}[\varphi_i](x) + \mathcal{S}_{\Gamma_e}[\varphi_e](x), \quad x \in \mathbb{R}^2$$

satisfies  $u = 0$  on  $\Gamma_i$  and  $\Gamma_e.$  Therefore,  $u(x) = 0$  for all  $x \in \Omega.$  It then follows from (9) that  $\varphi_i = 0$  and

$$\mathcal{K}_{\Gamma_e}^*[\varphi_e] = \frac{1}{2} \varphi_e \quad \text{on } \Gamma_e. \tag{33}$$

If  $\varphi_e \in L^2_0(\Gamma_e),$  then  $u(x) \rightarrow 0$  as  $|x| \rightarrow \infty,$  and hence  $u(x) = 0$  for  $x \in \mathbb{R}^2 \setminus \Omega$  as well. Thus  $\varphi_e = 0.$  The eigenfunctions of (33) make a one-dimensional subspace of  $L^2(\Gamma_e),$  which means that  $\text{Ker}(\mathbb{S})$  is of at most one dimension.

We now recall a result of KHAVINSON et al. [17, proof of Theorem 1]: let  $M \in \mathcal{C}_p(\mathcal{H})$ . If there exists a strictly positive bounded self-adjoint operator  $R$  such that  $R^2M$  is self adjoint, then there is a bounded self-adjoint operator  $A \in \mathcal{C}_p(\mathcal{H})$  such that

$$AR = RM. \tag{34}$$

We use this result and (28) to show that there is a bounded self-adjoint operator  $\mathbb{A}$  on  $\text{Ran}(\mathbb{S})$  such that

$$\mathbb{A}\sqrt{-\mathbb{S}} = \sqrt{-\mathbb{S}}\mathbb{K}^*. \tag{35}$$

By defining  $\mathbb{A}$  to be 0 on  $\text{Ker}(\mathbb{S})$ , we extend  $\mathbb{A}$  to  $\mathcal{H}$ . We note that (35) still holds and the extended operator is self-adjoint in  $\mathcal{H}$ . In fact, if  $\Phi \in \text{Ker}(\mathbb{S})$ , then  $\mathbb{K}^*[\Phi] = \frac{1}{2}\Phi$  because of (33), and hence  $\sqrt{-\mathbb{S}}\mathbb{K}^*[\Phi] = 0$ . Moreover, if  $\Phi, \Psi \in \mathcal{H}$ , then we can decompose them as  $\Phi = \Phi_1 + \Phi_2$  and  $\Psi = \Psi_1 + \Psi_2$ , where  $\Phi_1, \Psi_1 \in \text{Ran}(\mathbb{S})$  and  $\Phi_2, \Psi_2 \in \text{Ker}(\mathbb{S})$ . Let  $\tilde{\Phi}_1 = \sqrt{-\mathbb{S}}\Phi_1$  and  $\tilde{\Psi}_1 = \sqrt{-\mathbb{S}}\Psi_1$ . We then get

$$\begin{aligned} \langle \mathbb{A}\Phi, \Psi \rangle &= \langle \mathbb{A}\Phi_1, \Psi \rangle = \langle \mathbb{A}\sqrt{-\mathbb{S}}\tilde{\Phi}_1, \Psi \rangle = \langle \sqrt{-\mathbb{S}}\mathbb{K}^*\tilde{\Phi}_1, \Psi \rangle \\ &= \langle \sqrt{-\mathbb{S}}\mathbb{K}^*\tilde{\Phi}_1, \Psi_1 \rangle = \langle \mathbb{A}\Phi_1, \Psi_1 \rangle = \langle \Phi_1, \mathbb{A}\Psi_1 \rangle = \langle \Phi, \mathbb{A}\Psi \rangle, \end{aligned}$$

and hence  $\mathbb{A}$  is self-adjoint on  $\mathcal{H}$ .

We obtain the following theorem.

**Theorem 3.6** *There exists a bounded self-adjoint operator  $\mathbb{A} \in \mathcal{C}_2(\mathcal{H})$  such that*

$$\mathbb{A}\sqrt{-\mathbb{S}} = \sqrt{-\mathbb{S}}\mathbb{K}^*. \tag{36}$$

#### 4. Limiting Properties of the Solution and the Electromagnetic Power Dissipation

Let  $V_\delta$  be the solution to (3) with  $\alpha = 1$ . In this section we derive a necessary and sufficient condition on the source  $f$ , which is supported outside  $\overline{\Omega}$ , such that the blow-up (4) of the power dissipation takes place.

The solution  $V_\delta$  can be represented as

$$V_\delta(x) = F(x) + \mathcal{S}_{\Gamma_i}[\varphi_i^\delta](x) + \mathcal{S}_{\Gamma_e}[\varphi_e^\delta](x), \tag{37}$$

where  $\Phi_\delta = (\varphi_i^\delta, \varphi_e^\delta) \in L_0^2(\Gamma_i) \times L_0^2(\Gamma_e)$  is the solution to (18). Since  $\int_{\Omega \setminus \overline{D}} |\nabla F|^2 dx < \infty$ , (4) occurs if and only if

$$\delta \int_{\Omega \setminus \overline{D}} |\nabla(\mathcal{S}_{\Gamma_i}[\varphi_i^\delta] + \mathcal{S}_{\Gamma_e}[\varphi_e^\delta])|^2 dx \rightarrow \infty \quad \text{as } \delta \rightarrow \infty. \tag{38}$$

One can use (9) to obtain

$$\int_{\Omega \setminus \overline{D}} |\nabla(\mathcal{S}_{\Gamma_i}[\varphi_i^\delta] + \mathcal{S}_{\Gamma_e}[\varphi_e^\delta])|^2 dx = -\frac{1}{2}\langle \Phi_\delta, \mathbb{S}\Phi_\delta \rangle + \langle \mathbb{K}^*\Phi_\delta, \mathbb{S}\Phi_\delta \rangle,$$

where  $\langle \cdot, \cdot \rangle$  is the Hermitian product on  $\mathcal{H}$ . We then get from (36)

$$\begin{aligned} & \int_{\Omega \setminus \bar{D}} |\nabla(S_{\Gamma_i}[\varphi_i^\delta] + S_{\Gamma_e}[\varphi_e^\delta])|^2 dx \\ &= \frac{1}{2} \langle \sqrt{-\mathbb{S}}\Phi_\delta, \sqrt{-\mathbb{S}}\Phi_\delta \rangle - \langle \mathbb{A}\sqrt{-\mathbb{S}}\Phi_\delta, \sqrt{-\mathbb{S}}\Phi_\delta \rangle. \end{aligned} \tag{39}$$

Since  $\mathbb{A}$  is self-adjoint, we have an orthogonal decomposition

$$\mathcal{H} = \text{Ker}\mathbb{A} \oplus (\text{Ker}\mathbb{A})^\perp, \tag{40}$$

and  $(\text{Ker}\mathbb{A})^\perp = \overline{\text{Range}\mathbb{A}}$ . Let  $P$  and  $Q = I - P$  be the orthogonal projections from  $\mathcal{H}$  onto  $\text{Ker}\mathbb{A}$  and  $(\text{Ker}\mathbb{A})^\perp$ , respectively. Let  $\lambda_1, \lambda_2, \dots$  with  $|\lambda_1| \geq |\lambda_2| \geq \dots$  be the nonzero eigenvalues of  $\mathbb{A}$  and let  $\Psi_n$  be the corresponding (normalized) eigenfunctions. Since  $\mathbb{A} \in \mathcal{C}_2(\mathcal{H})$ , we have

$$\sum_{n=1}^\infty \lambda_n^2 < \infty, \tag{41}$$

and

$$\mathbb{A}\Phi = \sum_{n=1}^\infty \lambda_n \langle \Phi, \Psi_n \rangle \Psi_n, \quad \Phi \in \mathcal{H}. \tag{42}$$

We apply  $\sqrt{-\mathbb{S}}$  to (18) to obtain

$$(z_\delta \sqrt{-\mathbb{S}} + \sqrt{-\mathbb{S}}\mathbb{K}^*)\Phi_\delta = \sqrt{-\mathbb{S}}g.$$

Then (36) yields

$$(z_\delta \mathbb{I} + \mathbb{A})\sqrt{-\mathbb{S}}\Phi_\delta = \sqrt{-\mathbb{S}}g, \tag{43}$$

and hence

$$\begin{aligned} P\sqrt{-\mathbb{S}}\Phi_\delta &= \frac{1}{z_\delta} P\sqrt{-\mathbb{S}}g, \\ z_\delta Q\sqrt{-\mathbb{S}}\Phi_\delta + \mathbb{A}Q\sqrt{-\mathbb{S}}\Phi_\delta &= Q\sqrt{-\mathbb{S}}g. \end{aligned}$$

Thus we get

$$Q\sqrt{-\mathbb{S}}\Phi_\delta = \sum_n \frac{\langle Q\sqrt{-\mathbb{S}}g, \Psi_n \rangle}{\lambda_n + z_\delta} \Psi_n.$$

We also get

$$\mathbb{A}\sqrt{-\mathbb{S}}\Phi_\delta = \sum_n \frac{\lambda_n \langle Q\sqrt{-\mathbb{S}}g, \Psi_n \rangle}{\lambda_n + z_\delta} \Psi_n.$$

Thus we have

$$\langle \sqrt{-\mathbb{S}}\Phi_\delta, \sqrt{-\mathbb{S}}\Phi_\delta \rangle = \frac{1}{|z_\delta|^2} \|P\sqrt{-\mathbb{S}}g\|^2 + \sum_n \frac{|\langle Q\sqrt{-\mathbb{S}}g, \Psi_n \rangle|^2}{|\lambda_n + z_\delta|^2}, \tag{44}$$

and

$$\langle \mathbb{A}\sqrt{-\mathbb{S}}\Phi_\delta, \sqrt{-\mathbb{S}}\Phi_\delta \rangle = \sum_n \frac{\lambda_n |\langle Q\sqrt{-\mathbb{S}}g, \Psi_n \rangle|^2}{|\lambda_n + z_\delta|^2}. \quad (45)$$

Since

$$|\lambda_n + z_\delta|^2 = \left( \lambda_n - \frac{\delta^2}{2(4 + \delta^2)} \right)^2 + \frac{\delta^2}{(4 + \delta^2)^2} \approx \lambda_n^2 + \delta^2$$

and  $\lambda_n \rightarrow 0$  as  $n \rightarrow \infty$ , we have

$$\int_{\Omega \setminus \bar{D}} |\nabla(S_{\Gamma_i}[\varphi_i^\delta] + S_{\Gamma_e}[\varphi_e^\delta])|^2 dx \approx \frac{1}{\delta^2} \|P\sqrt{-\mathbb{S}}g\|^2 + \sum_n \frac{|\langle Q\sqrt{-\mathbb{S}}g, \Psi_n \rangle|^2}{|\lambda_n|^2 + \delta^2}. \quad (46)$$

Here and throughout this paper  $A \approx B$  means that there are constants  $C_1$  and  $C_2$  such that

$$C_1 A \leq B \leq C_2 A.$$

We note that if  $\text{Ker}(\mathbb{K}^*) = \{0\}$ , then  $P\sqrt{-\mathbb{S}} = 0$ . To see this, let  $\Phi_0$  be a basis of  $\text{Ker}(\mathbb{S})$ . Then we have  $\mathbb{K}^*\Phi_0 = \frac{1}{2}\Phi_0$ . If  $\mathbb{A}\sqrt{-\mathbb{S}}\Phi = 0$ , then  $\sqrt{-\mathbb{S}}\mathbb{K}^*\Phi = 0$  by (36). Therefore  $\mathbb{K}^*\Phi \in \text{Ker}(\mathbb{S})$ . If  $\text{Ker}(\mathbb{K}^*) = \{0\}$ , then  $\Phi = c\Phi_0$  for some constant  $c$ . This means that  $P\sqrt{-\mathbb{S}} = 0$ .

We obtain the following theorem:

**Theorem 4.1** *If  $P\sqrt{-\mathbb{S}}g \neq 0$ , then (4) takes place. If  $\text{Ker}(\mathbb{K}^*) = \{0\}$ , then (4) takes place if and only if*

$$\delta \sum_n \frac{|\langle \sqrt{-\mathbb{S}}g, \Psi_n \rangle|^2}{\lambda_n^2 + \delta^2} \rightarrow \infty \quad \text{as } \delta \rightarrow 0. \quad (47)$$

The condition (47) gives a necessary and sufficient condition on the source term  $f$  for the blow up of the electromagnetic power dissipation in  $\Omega \setminus \bar{D}$  when  $\alpha = 1$ . This condition is in terms of the Newtonian potential of  $f$ . In the next section, we explicitly compute the eigenvalues and eigenfunctions of  $\mathbb{A}$  for the case of an annulus configuration. In particular, we show the existence of a cloaking region such that if  $f$  is supported outside that region, then there is no blow up, while if it is supported inside and satisfies certain conditions, there is a blow up and CALR occurs.

## 5. Anomalous Resonance in an Annulus

In this section we consider the anomalous resonance when the domains  $\Omega$  and  $D$  are concentric disks. We calculate the explicit form of the limiting solution. Throughout this section, we set  $\Omega = B_e = \{|x| < r_e\}$  and  $D = B_i = \{|x| < r_i\}$ , where  $r_e > r_i$ .

Let  $\Gamma = \{|x| = r_0\}$ . One can easily see that for each integer,  $n$ ,

$$\mathcal{S}_\Gamma[e^{in\theta}](x) = \begin{cases} -\frac{r_0}{2|n|} \left(\frac{r}{r_0}\right)^{|n|} e^{in\theta} & \text{if } |x| = r < r_0, \\ -\frac{r_0}{2|n|} \left(\frac{r_0}{r}\right)^{|n|} e^{in\theta} & \text{if } |x| = r > r_0, \end{cases} \tag{48}$$

and hence

$$\frac{\partial}{\partial r} \mathcal{S}_\Gamma[e^{in\theta}](x) = \begin{cases} -\frac{1}{2} \left(\frac{r}{r_0}\right)^{|n|-1} e^{in\theta} & \text{if } |x| = r < r_0, \\ \frac{1}{2} \left(\frac{r_0}{r}\right)^{|n|+1} e^{in\theta} & \text{if } |x| = r > r_0. \end{cases} \tag{49}$$

It then follows from (9) that

$$\mathcal{K}_\Gamma^*[e^{in\theta}] = 0 \quad \forall n \neq 0. \tag{50}$$

It is worth mentioning that this property was observed in [15] and immediately follows from the fact that

$$\mathcal{K}_\Gamma^*[\varphi] = \frac{1}{4\pi r_0} \int_\Gamma \varphi d\sigma.$$

We also get from (20) and (21)

$$\mathcal{D}_\Gamma[e^{in\theta}](x) = \begin{cases} \frac{1}{2} \left(\frac{r}{r_0}\right)^{|n|} e^{in\theta} & \text{if } |x| = r < r_0, \\ -\frac{1}{2} \left(\frac{r_0}{r}\right)^{|n|} e^{in\theta} & \text{if } |x| = r > r_0. \end{cases}$$

Because of (50) it follows that

$$\mathbb{K}^* = \begin{bmatrix} 0 & -\frac{\partial}{\partial v_i} \mathcal{S}_{\Gamma_e} \\ \frac{\partial}{\partial v_e} \mathcal{S}_{\Gamma_i} & 0 \end{bmatrix},$$

and hence we have from (49) that

$$\mathbb{K}^* \begin{bmatrix} e^{in\theta} \\ 0 \end{bmatrix} = \frac{1}{2} \rho^{|n|+1} \begin{bmatrix} 0 \\ e^{in\theta} \end{bmatrix} \tag{51}$$

and

$$\mathbb{K}^* \begin{bmatrix} 0 \\ e^{in\theta} \end{bmatrix} = \frac{1}{2} \rho^{|n|-1} \begin{bmatrix} e^{in\theta} \\ 0 \end{bmatrix} \tag{52}$$

for all  $n \neq 0$ , where

$$\rho = \frac{r_i}{r_e}.$$



Thus  $\mathbb{K}^*$  as an operator on  $\mathcal{H}$  has the trivial kernel, that is,

$$\text{Ker } \mathbb{K}^* = \{0\}. \quad (53)$$

According to (51) and (52), if  $\Phi$  is given by

$$\Phi = \sum_{n \neq 0} \begin{bmatrix} \varphi_i^n \\ \varphi_e^n \end{bmatrix} e^{in\theta},$$

then

$$\mathbb{K}^* \Phi = \sum_{n \neq 0} \begin{bmatrix} \frac{\rho^{|n|-1}}{2} \varphi_e^n \\ \frac{\rho^{|n|+1}}{2} \varphi_i^n \end{bmatrix} e^{in\theta}.$$

Thus, if  $g$  is given by

$$g = \sum_{n \neq 0} \begin{bmatrix} g_i^n \\ g_e^n \end{bmatrix} e^{in\theta},$$

the integral equations (18) are equivalent to

$$\begin{cases} z_\delta \varphi_i^n + \frac{\rho^{|n|-1}}{2} \varphi_e^n = g_i^n, \\ z_\delta \varphi_e^n + \frac{\rho^{|n|+1}}{2} \varphi_i^n = g_e^n, \end{cases} \quad (54)$$

for every  $|n| \geq 1$ . It is readily seen that the solution  $\Phi = (\varphi_i, \varphi_e)$  to (54) is given by

$$\begin{aligned} \varphi_i &= 2 \sum_{n \neq 0} \frac{2z_\delta g_i^n - \rho^{|n|-1} g_e^n}{4z_\delta^2 - \rho^{2|n|}} e^{in\theta}, \\ \varphi_e &= 2 \sum_{n \neq 0} \frac{2z_\delta g_e^n - \rho^{|n|+1} g_i^n}{4z_\delta^2 - \rho^{2|n|}} e^{in\theta}. \end{aligned}$$

If the source is located outside the structure, that is,  $f$  is supported in  $\mathbb{R}^2 \setminus \overline{B_e}$ , then the Newtonian potential of  $f$ ,  $F$ , is harmonic in  $B_{r_e}$  and

$$F(x) = c - \sum_{n \neq 0} \frac{g_e^n}{|n| r_e^{|n|-1}} r^{|n|} e^{in\theta}, \quad (55)$$

for  $|x| \leq r_e$ , where  $g$  is defined by (17). Thus we have

$$g_i^n = -g_e^n \rho^{|n|-1}. \quad (56)$$

Here,  $g_e^n$  is the Fourier coefficient of  $-\frac{\partial F}{\partial \nu_e}$  on  $\Gamma_e$ , or in other words,

$$-\frac{\partial F}{\partial \nu_e} = \sum_{n \neq 0} g_e^n e^{in\theta}. \quad (57)$$

We then get

$$\begin{cases} \varphi_i = -2 \sum_{n \neq 0} \frac{(2z_\delta + 1)\rho^{|n|-1} g_e^n}{4z_\delta^2 - \rho^{2|n|}} e^{in\theta}, \\ \varphi_e = 2 \sum_{n \neq 0} \frac{(2z_\delta + \rho^{2|n|})g_e^n}{4z_\delta^2 - \rho^{2|n|}} e^{in\theta}. \end{cases} \tag{58}$$

Therefore, from (48) we find that

$$S_{\Gamma_i}[\varphi_i](x) + S_{\Gamma_e}[\varphi_e](x) = \sum_{n \neq 0} \frac{2(r_i^{2|n|} - r_e^{2|n|})z_\delta}{|n|r_e^{|n|-1}(4z_\delta^2 - \rho^{2|n|})} \frac{g_e^n}{r^{|n|}} e^{in\theta}, \quad r_e < r = |x|, \tag{59}$$

and

$$S_{\Gamma_i}[\varphi_i](x) = - \sum_{n \neq 0} \frac{r_i^{2|n|}(2z_\delta + 1)}{|n|r_e^{|n|-1}(\rho^{2|n|} - 4z_\delta^2)} \frac{g_e^n}{r^{|n|}} e^{in\theta}, \quad r_i < r = |x| < r_e, \tag{60}$$

$$S_{\Gamma_e}[\varphi_e](x) = g \sum_{n \neq 0} \frac{(2z_\delta + \rho^{2|n|})}{|n|r_e^{|n|-1}(\rho^{2|n|} - 4z_\delta^2)} g_e^n r^{|n|} e^{in\theta}, \quad r_i < r = |x| < r_e. \tag{61}$$

We next obtain the following lemma which provides essential estimates for the investigation of this section.

**Lemma 5.1** *There exists  $\delta_0$  such that*

$$E_\delta := \int_{B_e \setminus \bar{B}_i} \delta |\nabla V_\delta|^2 \approx \sum_{n \neq 0} \frac{\delta |g_e^n|^2}{|n|(\frac{\delta^2}{4} + \rho^{2|n|})} \tag{62}$$

uniformly in  $\delta \leq \delta_0$ .

**Proof.** Using (55), (60), and (61), one can see that

$$V_\delta(x) = c + r_e \sum_{n \neq 0} \left[ \frac{r_i^{2|n|}}{r^{|n|}} (2z_\delta + 1) - (4z_\delta^2 + 2z_\delta)r^{|n|} \right] \frac{g_e^n e^{in\theta}}{|n|r_e^{|n|}(4z_\delta^2 - \rho^{2|n|})}.$$

Then straightforward computations yield that

$$E_\delta \approx r_e^2 \sum_{n \neq 0} \delta (1 - \rho^{2|n|}) \left| \frac{2z_\delta + 1}{4z_\delta^2 - \rho^{2|n|}} \right|^2 (4|z_\delta|^2 + \rho^{2|n|}) \frac{|g_e^n|^2}{|n|}.$$

If  $\delta$  is sufficiently small, then one can also easily show that

$$|4z_\delta^2 - \rho^{2|n|}| \approx \frac{\delta^2}{4} + \rho^{2|n|}.$$

Therefore we get (62) and the proof is complete.  $\square$

It is worth noticing that estimate (62) is exactly the same as the one from Theorem 4.1, since the eigenvalues of  $\mathbb{A}$  are  $\{\pm\rho^{|n|}/2\}$ . To see this fact, we restrict the identity  $\mathbb{A}\sqrt{-\mathbb{S}} = \sqrt{-\mathbb{S}}\mathbb{K}^*$  to the vectorial space spanned by  $\begin{bmatrix} 0 \\ e^{in\theta} \end{bmatrix}$  and  $\begin{bmatrix} e^{in\theta} \\ 0 \end{bmatrix}$ . Taking the trace and the determinant of the restricted identity and using (51) and (52) proves that the set of eigenvalues of  $\mathbb{A}$  is  $\{\pm\rho^{|n|}/2\}$ .

Now, we turn to Lemma 5.1. We investigate the behavior of the series in the right-hand side of (62). Let

$$N_\delta = \frac{\log(\delta/2)}{\log \rho}. \quad (63)$$

If  $|n| \leq N_\delta$ , then  $\delta/2 \leq \rho^{|n|}$ , and hence

$$\sum_{n \neq 0} \frac{\delta |g_e^n|^2}{|n|(\frac{\delta^2}{4} + \rho^{2|n|})} \geq \sum_{0 \neq |n| \leq N_\delta} \frac{\delta |g_e^n|^2}{|n|(\frac{\delta^2}{4} + \rho^{2|n|})} \geq \frac{1}{2} \sum_{0 \neq |n| \leq N_\delta} \frac{\delta |g_e^n|^2}{|n|\rho^{2|n|}}. \quad (64)$$

Suppose that

$$\limsup_{|n| \rightarrow \infty} \frac{|g_e^n|^2}{|n|\rho^{|n|}} = \infty. \quad (65)$$

Then there is a subsequence  $\{n_k\}$  with  $|n_1| < |n_2| < \dots$  such that

$$\lim_{k \rightarrow \infty} \frac{|g_e^{n_k}|^2}{|n_k|\rho^{|n_k|}} = \infty. \quad (66)$$

If we take  $\delta = 2\rho^{|n_k|}$ , then  $N_\delta = |n_k|$  and

$$\sum_{0 \neq |n| \leq N_\delta} \frac{\delta |g_e^n|^2}{|n|\rho^{2|n|}} = \rho^{|n_k|} \sum_{0 \neq |n| \leq |n_k|} \frac{|g_e^n|^2}{|n|\rho^{2|n|}} \geq \frac{|g_e^{n_k}|^2}{|n_k|\rho^{|n_k|}}. \quad (67)$$

Thus we obtain from (62) that

$$\lim_{k \rightarrow \infty} E_{\rho^{|n_k|}} = \infty. \quad (68)$$

We emphasize that (65) is not enough to guarantee (4), as pointed out by JIANFENG LU and JENS JORGENSEN (private communication). In fact, if we let

$$g_e^n = \begin{cases} n\rho^{n/2}, & \text{if } n = 2^j, \quad j = 1, 2, \dots, \\ 0, & \text{otherwise,} \end{cases} \quad (69)$$

and  $\delta_k = \rho^{n_k}$  with  $n_k = 2^k + 2^{k-1}$  for  $k = 1, 2, \dots$ , then

$$\limsup_{n \rightarrow \infty} \frac{|g_e^n|^2}{|n|\rho^{|n|}} = \infty.$$

But one can easily see that  $|2^j - n_k| \geq 2^{j-2}$  and

$$\frac{\rho^{n_k+2^j}}{\rho^{2n_k} + \rho^{2^{j+1}}} < \rho^{|n_k-2^j|}, \quad j, k = 1, 2, \dots$$

Thus we obtain

$$\sum_{n \neq 0} \frac{\delta_k |g_e^n|^2}{|n|(\delta_k^2 + \rho^{2|n|})} = \sum_{j=1}^{\infty} \frac{2^j \rho^{n_k+2^j}}{\rho^{2n_k} + \rho^{2^{j+1}}} \leq \sum_{j=1}^{\infty} 2^j \rho^{|n_k-2^j|} \leq \sum_{j=1}^{\infty} 2^j \rho^{2^{j-2}} < \infty,$$

which means that

$$E_{\delta_k} \leq C, \tag{70}$$

regardless of  $k$ . It is worth mentioning that the  $g_e^n$  defined by (69) are certainly Fourier coefficients of  $-\frac{\partial F}{\partial v_e}$  on  $\Gamma_e$  for an  $F$  which is harmonic in  $B_{r_*}$ , given by (55) when  $|x| \leq r_*$ . Also, there is a source function which generates these Fourier coefficients. To see this, choose  $r_1$  and  $r_2$  with  $r_e < r_1 < r_2 < r_*$  and let  $\tau(r)$ , be a function which is 1 for  $r < r_1$ , and zero for  $r > r_2$  and which smoothly interpolates between these values in the interval  $r_1 \leq r \leq r_2$ . Then we see that  $\tilde{F}(x)$  defined to be zero for  $|x| \geq r_2$  and equal to  $\tau(|x|)F(x)$  for  $|x| < r_2$ , has the same Fourier coefficients  $g_e^n$  as  $F$  on  $\Gamma_e$ , and the associated source function  $\tilde{f} = \Delta \tilde{F}$  is supported in the annulus between  $|x| = r_1$  and  $|x| = r_2$ . However, it is not clear whether the Fourier coefficients can be realized as being associated with a Newtonian potential of a source function whose support is located outside the radius  $r_e$  and not surrounding the annulus.

We now impose an additional condition. We assume that  $\{g_e^n\}$  satisfies the following gap property:

GP: There exists a sequence  $\{n_k\}$  with  $|n_1| < |n_2| < \dots$  such that

$$\lim_{k \rightarrow \infty} \rho^{|n_{k+1}|-|n_k|} \frac{|g_e^{n_k}|^2}{|n_k| \rho^{|n_k|}} = \infty.$$

If GP holds, then we immediately see that (65) holds, but the converse is not true. If (65) holds, that is, there is a subsequence  $\{n_k\}$  with  $|n_1| < |n_2| < \dots$  satisfying (66) and the gap  $|n_{k+1}| - |n_k|$  is bounded, then GP holds. In particular, if

$$\lim_{n \rightarrow \infty} \frac{|g_e^n|^2}{|n| \rho^{|n|}} = \infty, \tag{71}$$

then GP holds.

Assume that  $\{g_e^n\}$  satisfies GP and  $\{n_k\}$  is such a sequence. Let  $\delta = 2\rho^\alpha$  for some  $\alpha$  and let  $k(\alpha)$  be the number such that

$$|n_{k(\alpha)}| \leq \alpha < |n_{k(\alpha)+1}|.$$

Then, we have

$$\sum_{0 \neq |n| \leq N_\delta} \frac{\delta |g_\epsilon^n|^2}{|n| \rho^{2|n|}} = \rho^\alpha \sum_{0 \neq |n| \leq \alpha} \frac{|g_\epsilon^n|^2}{|n| \rho^{2|n|}} \geq \rho^{|n_{k(\alpha)+1}| - |n_{k(\alpha)}|} \frac{|g_\epsilon^{n_{k(\alpha)}}|^2}{|n_{k(\alpha)}| \rho^{|n_{k(\alpha)}|}} \rightarrow \infty, \tag{72}$$

as  $\alpha \rightarrow \infty$ .

We obtain the following lemma:

**Lemma 5.2** *If (65) holds, then*

$$\limsup_{\delta \rightarrow 0} E_\delta = \infty. \tag{73}$$

*If  $\{g_\epsilon^n\}$  satisfies the condition GP, then*

$$\lim_{\delta \rightarrow 0} E_\delta = \infty. \tag{74}$$

Suppose that the source function is supported inside the radius  $r_* = \sqrt{r_e^3 r_i^{-1}}$ . Then its Newtonian potential cannot be extended harmonically in  $|x| < r_*$  in general. So, if  $F$  is given by

$$F = c - \sum_{n \neq 0} a_n r^{|n|} e^{in\theta}, \quad r < r_e, \tag{75}$$

then the radius of convergence is less than  $r_*$ . Thus we have

$$\limsup_{|n| \rightarrow \infty} |n| |a_n|^2 r_*^{2|n|} = \infty, \tag{76}$$

that is, (65) holds. The GP condition is equivalent to stating that there exists  $\{n_k\}$  with  $|n_1| < |n_2| < \dots$  such that

$$\lim_{k \rightarrow \infty} \rho^{|n_{k+1}| - |n_k|} |n_k| |a_{n_k}|^2 r_*^{2|n_k|} = \infty. \tag{77}$$

The following is the main theorem of this section.

**Theorem 5.3** *Let  $f$  be a source function supported in  $\mathbb{R}^2 \setminus \overline{B}_e$  and let  $F$  be the Newtonian potential of  $f$ .*

- (i) *If  $F$  does not extend as a harmonic function in  $B_{r_*}$ , then weak CALR occurs, that is,*

$$\limsup_{\delta \rightarrow 0} E_\delta = \infty \tag{78}$$

*and (5) holds with  $a = r_e^2 / r_i$ .*

- (ii) *If the Fourier coefficients of  $F$  satisfy (77), then CALR occurs, that is,*

$$\lim_{\delta \rightarrow 0} E_\delta = \infty \tag{79}$$

*and (5) holds with  $a = r_e^2 / r_i$ .*

(iii) If  $F$  extends as a harmonic function in a neighborhood of  $\overline{B_{r_*}}$ , then CALR does not occur, that is,

$$E_\delta < C \tag{80}$$

for some  $C$  independent of  $\delta$ .

**Proof.** If  $F$  does not extend as a harmonic function in  $B_{r_*}$ , then (65) holds. Thus we have (78). If (77) holds, then (79) holds by Lemma (5.2). Moreover, by (59), we see that

$$\begin{aligned} |V_\delta| &\leq |F| + \sum_{n \neq 0} \left| \frac{2(r_i^{2|n|} - r_e^{2|n|})z_\delta}{|n|r_e^{|n|-1}(4z_\delta^2 - \rho^{2|n|})} \frac{g_e^n}{r^{|n|}} \right| \leq |F| + C \sum_{n \neq 0} \frac{\delta r_e^{|n|}}{(\frac{\delta^2}{4} + \rho^{2|n|})|n|r^{|n|}} \\ &\leq |F| + C \sum_{n \neq 0} \frac{r_e^{2|n|}}{|n|r_i^{|n|}r^{|n|}} < C, \quad \text{if } r = |x| > \frac{r_e^2}{r_i} \end{aligned}$$

for some constants  $C$ , which may differ at each occurrence.

If  $F$  extends as a harmonic function in a neighborhood of  $\overline{B_{r_*}}$ , then the power series of  $F$ , which is given by (55), converges for  $r < r_* + 2\varepsilon$  for some  $\varepsilon > 0$ . Therefore, there exists a constant  $C$  such that

$$\frac{|g_e^n|}{|n|r_e^{|n|-1}} \leq C \frac{1}{(r_* + \varepsilon)^{|n|}}$$

for all  $n$ . It then follows that

$$|g_e^n| \leq C(r_e^2 \rho^{-1} + r_e \varepsilon)^{-|n|/2} r_e^{|n|} \leq (\rho^{-1} + \varepsilon)^{-|n|/2} \tag{81}$$

for all  $n$ . This tells us that

$$\sum_{n \neq 0} \frac{\delta |g_e^n|^2}{|n|(\delta^2 + \rho^{2|n|})} \leq \sum_{n \neq 0} \frac{|g_e^n|^2}{2|n|\rho^{|n|}} \leq \sum_{n \neq 0} \frac{1}{2|n|(1 + \varepsilon \rho)^{|n|}}.$$

This completes the proof.  $\square$

If  $f$  is a dipole in  $B_{r_*} \setminus \overline{B_e}$ , that is,  $f(x) = a \cdot \nabla \delta_y(x)$  for a vector  $a$  and  $y \in B_{r_*} \setminus \overline{B_e}$  where  $\delta_y$  is the Dirac delta function at  $y$ , then  $F(x) = a \cdot \nabla G(x - y)$ . From the expansion of the fundamental solution

$$G(x - y) = \sum_{n=1}^{\infty} \frac{-1}{2\pi n} \left[ \frac{\cos n\theta_y}{r_y^n} r^n \cos n\theta + \frac{\sin n\theta_y}{r_y^n} r^n \sin n\theta \right] + C, \tag{82}$$

we see that the Fourier coefficients of  $F$  have the growth rate  $r_y^{-n}$  and satisfy (77), and hence CALR takes place. Similarly CALR takes place for a sum of dipole sources at different fixed positions in  $B_{r_*} \setminus \overline{B_e}$ . We emphasize that this fact was found in [27].

If  $f$  is a quadrupole, that is,  $f(x) = A : \nabla \nabla \delta_y(x) = \sum_{i,j=1}^2 a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \delta_y(x)$  for a  $2 \times 2$  matrix  $A = (a_{ij})$  and  $y \in B_{r_*} \setminus \overline{B}_e$ , then  $F(x) = \sum_{i,j=1}^2 a_{ij} \frac{\partial^2 G(x-y)}{\partial x_i \partial x_j}$ . Thus CALR takes place. This is in agreement with the numerical result in [34].

If  $f$  is supported in  $\mathbb{R}^2 \setminus \overline{B}_{r_*}$ , then  $F$  is harmonic in a neighborhood of  $\overline{B}_{r_*}$ , and hence CALR does not occur by Theorem 5.3. In fact, we can say more about the behavior of the solution  $V_\delta$  as  $\delta \rightarrow 0$  which is related to the observation in [30,33] that in the limit  $\delta \rightarrow 0$  the annulus itself becomes invisible to sources that are sufficiently far away.

**Theorem 5.4** *If  $f$  is supported in  $\mathbb{R}^2 \setminus \overline{B}_{r_*}$ , then (80) holds (with  $\alpha = 1$  in (3)). Moreover, we have*

$$\sup_{|x| \geq r_*} |V_\delta(x) - F(x)| \rightarrow 0 \quad \text{as } \delta \rightarrow 0. \quad (83)$$

**Proof.** Since  $\text{supp } f \subset \mathbb{R}^2 \setminus \overline{B}_{r_*}$ , the power series of  $F$ , which is given by (55), converges for  $r < r_* + 2\varepsilon$  for some  $\varepsilon > 0$ .

According to (59), if  $r_e < r = |x|$ , then we have

$$V_\delta(x) - F(x) = \sum_{n \neq 0} \frac{2(r_e^{2|n|} - r_i^{2|n|})z_\delta}{|n|r_e^{|n|-1}(\rho^{2|n|} - 4z_\delta^2)} \frac{g_e^n}{r^{|n|}} e^{in\theta}.$$

If  $|x| = r_*$ , then the identity

$$\frac{(r_e^{2|n|} - r_i^{2|n|})z_\delta}{|n|r_e^{|n|-1}(\rho^{2|n|} - 4z_\delta^2)} \frac{g_e^n}{r_*^{|n|}} = \frac{(1 - \rho^{2|n|})z_\delta}{(\rho^{|n|} - 4z_\delta^2 \rho^{-|n|})} \frac{g_e^n r_*^{|n|}}{|n|r_e^{|n|-1}}$$

holds and

$$\begin{aligned} \left| \frac{(1 - \rho^{2|n|})z_\delta}{(\rho^{|n|} - 4z_\delta^2 \rho^{-|n|})} \right| &\leq \left| \frac{1}{(z_\delta^{-1} \rho^{|n|} - z_\delta \rho^{-|n|})} \right| \\ &\leq \left| \frac{1}{\Im(z_\delta^{-1} \rho^{|n|} - z_\delta \rho^{-|n|})} \right| = \left( \frac{\delta}{4 + \delta^2} \rho^{-|n|} + \frac{1}{\delta} \rho^{|n|} \right)^{-1}. \end{aligned}$$

It then follows from (81) that

$$|V_\delta(x) - F(x)| \leq 2 \sum_{n \neq 0} \left( \frac{\delta}{4 + \delta^2} \rho^{-|n|} + \frac{1}{\delta} \rho^{|n|} \right)^{-1} \frac{r_e}{|n|} \left( \frac{\rho^{-1}}{\rho^{-1} + \varepsilon} \right)^{|n|/2},$$

and hence

$$|V_\delta(x) - F(x)| \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

Since  $V_\delta - F$  is harmonic in  $|x| > r_e$  and tends to 0 as  $|x| \rightarrow \infty$ , we obtain (83) by the maximum principle. This completes the proof.  $\square$

Theorem 5.4 shows that any source supported outside  $B_{r_*}$  cannot make the blow-up of the power dissipation happen and is not cloaked. In fact, it is known that we can recover the source  $f$  from its Newtonian potential  $F$  outside  $B_{r_*}$  since  $f$  is supported outside  $\overline{B}_{r_*}$  (see [14]). Therefore we infer from (83) that  $f$  may be recovered approximately by observing  $V_\delta$  outside  $B_{r_*}$ .

## 6. Conclusion

In this paper we have provided for the first time a mathematical justification of cloaking due to anomalous localized resonance in the case of general source terms. In particular, we obtained an explicit necessary and sufficient condition on the source term in order for CALR to take place. In the case of an annulus structure we show that weak CALR takes place for almost any source supported inside the critical radius. We also find a sufficient condition on the Fourier coefficients of the Newtonian potential of the source function for CALR to occur. It would be quite interesting to clarify whether weak CALR implies CALR, or not, for sources whose support does not completely surround the annulus.

The results and techniques of this paper can be immediately extended to the three-dimensional case. The compact operator  $\mathbb{K}^*$  is in the Schatten Von-Neumann class  $\mathcal{C}_p(L^2(\Gamma_i) \times L^2(\Gamma_e))$  for some  $1 \leq p < \infty$ , provided that  $\Omega$  and  $D$  are of class  $\mathcal{C}^{1,\alpha}$  for  $0 < \alpha < 1$ , and consequently, it is symmetrizable.

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