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SCATTERING RESONANCES FOR A TWO-DIMENSIONAL POTENTIAL WELL WITH A THICK BARRIER*

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Abstract. This paper is concerned with the scattering resonances of the Schrödinger operator $-\Delta + V_L$ in $\mathbb{R}^2$. The real valued potential V_L is a low energy well surrounded by a barrier with finite thickness proportional to L. We are interested in the resonances that are close to the nondegenerate bound state frequencies associated with the potential that has infinitely thick barrier. In particular, it is shown that the difference between a resonance and the associated bound state frequency decays exponentially as a function of the barrier thickness with a rate of $L^2 e^{-cL}$. The analysis leads to a perturbative approach for accurately approximating the near bound-state resonances.

Key words. Schrödinger operator, scattering resonances, perturbation analysis

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1. Introduction.

Consider the Schrödinger operator

$$P_L = -\Delta + V_L,$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, and the real valued potential function $V_L(x) \geq 0$ consists of a low energy well surrounded by a barrier of finite thickness. (See Figure 1 for a schematic plot.) More precisely, let B_R and B_L be balls of radii R and L in $\mathbb{R}^2$, respectively, with $R < L$. The potential $V_L(x) \in L^\infty(\mathbb{R}^2)$ is a piecewise continuous function given by

$$V_L(x) = \begin{cases} v(x) \leq V_0, & x \in B_R, \\ V_0, & x \in B_L \backslash \bar{B}_R, \\ 0, & x \in \mathbb{R}^2 \backslash \bar{B}_L. \end{cases}$$

Here, B_R is the smallest ball enclosing the low energy well wherein the potential $V_L(x) < V_0$, and we refer to $(L - R)$ as the thickness of the barrier.

Let the resolvent $R_L(k) := (P_L - k^2)^{-1}$ be defined from $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ with $\text{Im}k < 0$. Denote $G(k; x, y)$ as the outgoing Green's function, and then essentially the resolvent can be expressed with the kernel $G(k; x, y)$ such that

$$R_L(k)\psi(x) = \int_{\mathbb{R}^2} G(k; x, y)\psi(y)dy.$$

The *scattering resonances* are the poles of the resolvent (Green's function) continued meromorphically from the physical half plane to the whole complex plane $\mathbb{C}$ [17]. Equivalently, they are solutions of the following eigenvalue problem:

$$\begin{cases} -\Delta\psi + V_L(x)\psi = k^2\psi & \text{in } \mathbb{R}^2, \\ \psi \text{ is outgoing.} \end{cases} \tag{1.1}$$

The associated eigenfunctions are called *quasi modes (quasi-normal modes)*.

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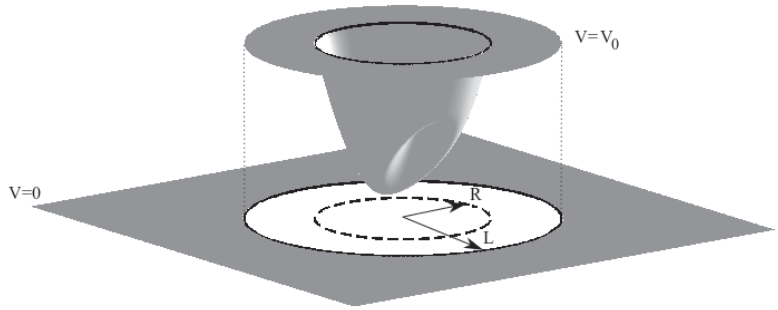

FIG. 1. *The graph of the potential $V_L(x)$. The potential is constant from $R < |x| < L$ and is zero for $|x| > L$. The thickness of the barrier is $(L - R)$. By $V_\infty(x)$, we mean a potential with infinitely thick wall.*

The scattering resonance problem arises, for example, in the study of transient phenomena associated with the wave equation:

$$\begin{aligned} &\frac{\partial^2 u(x,t)}{\partial t^2} = \Delta u(x,t) - V_L(x)u(x,t), \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}^+, \\ &u(x,0) = u_0 \in H^1_{comp}(\mathbb{R}^2), \quad \frac{\partial u}{\partial t}(x,0) = u_1 \in L^2_{comp}(\mathbb{R}^2). \end{aligned} \tag{1.2}$$

Let $u(x,t) = e^{-ikt}\psi(x)$, and then it is seen that $\psi(x)$ solves (1.1). In fact, the eigenvalue problem (1.1) attains an infinite sequence of complex resonances k_j and the corresponding quasi modes $\psi_j(x)$, which are locally integrable. Furthermore, the solution of the wave equation (1.2) can be approximated by the modes $e^{-ik_jt}\psi_j(x)$. This is known as resonance expansion of scattering wave [6, 15, 17].

There has been various work on the study of resonances, particularly their distribution and total number in a bounded region. We refer to [8, 18, 19] and references therein for detailed discussions in both one and higher dimensions. Here, we are focused on the resonances that are close to the bound state frequencies associated with the potential V_∞ that has infinitely thick barrier. For a one-dimensional symmetric potential, it was shown in [7] that the size of the negative imaginary part of the resonance is exponentially small in the barrier thickness. The one-dimensional boundary value problem is transformed into an initial value problem where an explicit solution technique is adopted. A similar result has also been obtained for the related photonic crystal structure with a defect [13]. Readers are also referred to [9, 14] for the studies of lower bounds for the imaginary parts of resonances in one-dimensional structure.

In this paper, we consider the potential well in two dimensions. The extension from one to two dimensions turns out to be nontrivial. Whereas one has initial conditions that can be exploited in one dimension, no such equivalent conditions are available in a higher dimension. In order to analyze a near bound state resonance, we have to devise an entirely different approach. This approach generalizes to dimensions greater than two and to more complicated partial differential equations.

Let us denote by P_∞ the operator $-\Delta + V_\infty$ with the potential V_∞ that has infinite thickness barrier. It is well known that the operator P_∞ consists of a set of discrete energy $k_1^2, k_2^2, \ldots, k_M^2$ located in $(0, V_0)$ and continuous spectrum $[V_0, \infty)$. The eigenfunction associated with the discrete spectrum is called *bound state*, and k_j is referred as *bound state frequency*. The operator P_L with finitely thick potential, on the other hand, attains no bound state. Instead, the spectrum (square of scattering resonances) of P_L lie on the lower complex plane, and there is an infinite number of resonances (Lemma 3.2). Our main result is concerned with the resonances that are close to the nondegenerate bound state frequency of P_∞ (Theorem 3.3). More precisely, we show that the distance between a nearby resonance and a nondegenerate bound state frequency decays with a rate $L^2 e^{-\beta_b L}$, wherein $\beta_b > 0$. Based on this observation, a simple numerical method is also proposed to calculate the near bound-state resonance k accurately and efficiently.

The eigenvalue problems for P_L and that for P_∞ are defined over the whole plane. A key ingredient to obtaining an equivalent nonlinear equation for the resonances (*resonance condition*) is to reformulate both eigenvalue problems in a suitable bounded domain by introducing the Dirichlet-to-Neumann maps. By examining the associated equation for the difference between the bound state and quasi mode, the resonance condition is obtained in an innovative way through the solvability condition in the elliptic theory. Furthermore, the study of resonances, which are solutions of the resonance condition, relies on several key estimates for the Dirichlet-to-Neumann maps that are given section 4.

The rest of the paper is organized as follows. In section 2, the eigenvalue problems are recast in the domain B_R that encloses the low energy well. In section 3 we derive the resonance condition in detail and give the main result for the estimate of resonances. To prove the theorem, several key estimates are established for the Dirichlet-to-Neumann map in section 4, and the proof for the main theorem is given in section 5. Based on the estimate for the resonance, we also propose a simple perturbation method to approximate the near bound-state resonances in section 6.

2. Bound state and scattering resonance problems.

2.1. Preliminaries. Without loss of generality, from now on we assume that the radius $R = 1$ for clarity of exposition. In this section, the bound state and scattering resonances problems are reformulated in a bounded domain by introducing the Dirichlet-to-Neumann maps on its boundary. This is a key step in deriving the resonance condition shown in the next section. We point out that it is also essential to choose this domain to be the smallest ball enclosing the low energy well, namely, B_1, so as to obtain the desired estimates for the near-bound state resonances.

We begin with some standard notation that will be used throughout. Let

$$H^1(B_1) := \{ \, u \mid u \in L^2(B_1), \partial_j u \in L^2(B_1) \, \}$$

be the standard Sobolev space equipped with the inner product

$$(u, w) = \int_{B_1} u(x)\bar{w}(x)dx, \quad (\nabla u, \nabla w) = \int_{B_1} \nabla u(x) \nabla \bar{w}(x)dx,$$

and the norm $\|u\|_{H^1(B_1)} = \sqrt{(u,u) + (\nabla u, \nabla u)}$. Define

$$H^{1/2}(\partial B_1) := \left\{ \, \psi \in L^2(\partial B_1) \, \middle| \, \sum_{n=0}^{\infty} \sqrt{1+n^2} \quad |\psi_n|^2 \; < +\infty \right\}$$

with the norm

$$\|\psi\|_{H^{1/2}(\partial B_1)} = \left(\sum_{n=0}^{\infty} \sqrt{1+n^2} \quad |\psi_n|^2 \right)^{1/2}.$$

Here, ψ_n are Fourier coefficients of the function $\psi(\theta)$ on the circle given by $\psi_n = \frac{1}{2\pi}\int_0^{2\pi}\psi(\theta)e^{-in\theta}d\theta$.

2.2. The bound state problem. We only have to consider positive bound state frequencies. Let $k_b \in (0, \sqrt{V_0})$ be a bound state frequency such that

$$\begin{cases} -\Delta\psi_b + V_\infty\psi_b = k_b^2\psi_b & \text{in } \mathbb{R}^2, \\ \lim_{|x|\to\infty}\psi_b(x) = 0 \end{cases} \tag{2.1}$$

with $\psi_b \not\equiv 0$ and $\|\psi_b\|_{L^2(\mathbb{R}^2)} < \infty$.

We derive the Dirichlet-to-Neumann map on ∂B_1 for the bound state case. For a function $\psi \in H^{1/2}(\partial B_1)$, consider the exterior problem

$$\begin{cases} -\Delta u + V_\infty u = k^2 u & \text{in } \mathbb{R}^2\backslash B_1, \\ u = \psi & \text{on } \partial B_1, \\ \lim_{|x|\to\infty} u(x) = 0. \end{cases}$$

Since $V_\infty(x) = V_0$ for $|x| > 1$, we may expand the solution u in the form of $\sum_{n=0}^\infty u_n(r)e^{in\theta}$ in the polar coordinate. By substituting into the equation, u_n satisfies

$$r^2\frac{\partial^2 u_n}{\partial r^2} + r\frac{\partial u_n}{\partial r} - (n^2 + \beta_b^2 r^2)u_n = 0,$$

where $\beta_b = \sqrt{V_0^2 - k_b^2}$. Equivalently, by letting $\tilde{r} = \beta_b r$, u_n is a solution of Bessel's differential equation

$$\tilde{r}^2\frac{\partial^2 u_n}{\partial \tilde{r}^2} + \tilde{r}\frac{\partial u_n}{\partial \tilde{r}} - (n^2 + \tilde{r}^2)u_n = 0.$$

In light of the decay property for the solution at infinity, we choose $u_n = c_n K_n(\tilde{r}) = c_n K_n(\beta_b r)$. Here, K_n is the modified Bessel's function of the second kind, and c_n is the coefficient which can determined by evaluating u on ∂B_1. As a result, for $r > 1$, u admits the expansion

$$u(r,\theta) = \sum_{n=0}^\infty \frac{K_n(\beta_b r)}{K_n(\beta_b)}\psi_n e^{in\theta}, \quad \text{where } \psi_n = \frac{1}{2\pi}\int_0^{2\pi}\psi(\theta)e^{-in\theta}d\theta.$$

Taking the normal derivative of u on ∂B_1 gives rise to

$$\frac{\partial u}{\partial r}(1,\theta) = \sum_{n=0}^\infty \frac{\beta_b K_n'(\beta_b)}{K_n(\beta_b)}\psi_n e^{in\theta}.$$

We define the Dirichlet-to-Neumann map at the bound state frequency ($\psi_b \to \partial\psi_b/\partial r$) by letting

$$(\mathcal{T}_b\psi_b)(\theta) = \sum_{n=0}^\infty \frac{\beta_b K_n'(\beta_b)}{K_n(\beta_b)}\psi_{b,n}e^{in\theta}, \quad \text{where } \psi_{b,n} = \frac{1}{2\pi}\int_0^{2\pi}\psi_b(1,\theta)e^{-in\theta}d\theta. \tag{2.2}$$

For each bound state ψ_b of (2.1), $\frac{\partial\psi_b}{\partial n} = \mathcal{T}_b\psi_b$ holds on the boundary ∂B_1. Hence, the bounded state ψ_b is a solution of the following boundary value problem:

$$\begin{cases} -\Delta\psi_b + V_\infty\psi_b = k_b^2\psi_b & \text{in } B_1, \\ \dfrac{\partial\psi_b}{\partial n} = \mathcal{T}_b\psi_b & \text{on } \partial B_1. \end{cases} \tag{2.3}$$

2.3. The scattering resonance problem. Let us derive the Dirichlet-to-Neumann map on ∂B_1 for the resonance problem. To do this, for any $\psi \in H^{1/2}(\partial B_1)$, consider the exterior problem

$$\begin{cases} -\Delta u + V_L u = k^2 u & \text{in } \mathbb{R}^2 \backslash B_1, \\ u = \psi & \text{on } \partial B_1, \\ u \text{ is outgoing.} \end{cases}$$

Expand the solution u as

$$u(r,\theta) = \sum_{n=0}^{\infty} u_n(r) e^{in\theta} \tag{2.4}$$

for $1 < r < L$ and $r > L$, respectively. By substituting into the differential equation, u_n satisfies

$$r^2 \frac{\partial^2 u_n}{\partial r^2} + r \frac{\partial u_n}{\partial r} - (n^2 + (V_0^2 - k^2) r^2) u_n = 0 \quad \text{if } 1 < r < L$$

and

$$r^2 \frac{\partial^2 u_n}{\partial r^2} + r \frac{\partial u_n}{\partial r} + (k^2 - n^2) u_n = 0 \quad \text{if } r > L.$$

Therefore,

$$u_n(r) = a_{1,n} I_n(\beta r) + a_{2,n} K_n(\beta r) \quad \text{if } 1 < r < L, \tag{2.5}$$

where $\beta = \sqrt{V_0^2 - k^2}$, I_n and K_n are the modified Bessel's functions of the first and second kind. In addition, the outgoing solution requires that

$$u_n = c_n H_n^{(1)}(kr) / H_n^{(1)}(kL) \quad \text{if } r > L, \tag{2.6}$$

wherein $H_n^{(1)}(kr)$ is the Hankel function of the first kind.

The continuity condition for u on $r = L$ reads

$$u_n(L^-) = u_n(L^+), \quad \frac{\partial u_n}{\partial r}(L^-) = \frac{\partial u_n}{\partial r}(L^+),$$

or more explicitly,

$$\begin{aligned} a_{1,n} I_n(\beta L) + a_{2,n} K_n(\beta L) &= c_n, \\ a_{1,n} \beta I_n'(\beta L) + a_{2,n} \beta K_n'(\beta L) &= c_n k \frac{(H_n^{(1)})'(kL)}{H_n^{(1)}(kL)}. \end{aligned}$$

By the Wronskian formula (B.5): $I_n(\beta L) K_n'(\beta L) - K_n(\beta L) I_n'(\beta L) = -1/(\beta L)$, we solve $a_{1,n}$ and $a_{2,n}$ in terms of c_n:

$$a_{1,n} = \alpha_{1,n}(k) c_n, \quad a_{2,n} = \alpha_{2,n}(k) c_n,$$

where

$$\alpha_{1,n}(k) = L \left(-\beta K_n'(\beta L) + k \frac{\left(H_n^{(1)}\right)'(kL)}{H_n^{(1)}(kL)} K_n(\beta L) \right), \tag{2.7}$$

$$\alpha_{2,n}(k) = L \left(\beta I_n'(\beta L) - k \frac{\left(H_n^{(1)}\right)'(kL)}{H_n^{(1)}(kL)} I_n(\beta L) \right). \tag{2.8}$$

Consequently, by substituting into (2.4)–(2.6), u admits the expansion

$$u(r,\theta)=\sum_{n=0}^{\infty}c_n\big(\alpha_{1,n}(k)I_n(\beta r)+\alpha_{2,n}(k)K_n(\beta r)\big)e^{in\theta}\quad\text{for } 1<r<L.$$

Evaluating u on ∂B_1 gives $u(1,\theta)=\sum_{n=0}^{\infty}c_n(\alpha_{1,n}(k)I_n(\beta)+\alpha_{2,n}(k)K_n(\beta))e^{in\theta}$. We deduce that the coefficient

$$\alpha_{1,n}(k)I_n(\beta)+\alpha_{2,n}(k)K_n(\beta)\neq 0,$$

due to the nonvanishing Fourier coefficients of $u(1,\theta)$ by choosing ψ properly on ∂B_1. In addition, a direct calculation yields

$$c_n=\frac{\psi_n}{(\alpha_{1,n}(k)I_n(\beta)+\alpha_{2,n}(k)K_n(\beta))},\quad\text{where}\quad \psi_n=\frac{1}{2\pi}\int_0^{2\pi}\psi(\theta)e^{-in\theta}d\theta.$$

We calculate the normal derivative of u on ∂B_1, which is given explicitly by

$$\frac{\partial u}{\partial r}(1,\theta)=\sum_{n=0}^{\infty}\frac{\alpha_{1,n}(k)I_n'(\beta)+\alpha_{2,n}(k)K_n'(\beta)}{\alpha_{1,n}(k)I_n(\beta)+\alpha_{2,n}(k)K_n(\beta)}\beta\psi_n e^{in\theta}.$$

Define the Dirichlet-to-Neumann map on ∂B_1 ($\psi\to\partial\psi/\partial r$):

$$(\mathcal{T}(k)\psi)(\theta)=\sum_{n=0}^{\infty}\frac{\alpha_{1,n}(k)I_n'(\beta)+\alpha_{2,n}(k)K_n'(\beta)}{\alpha_{1,n}(k)I_n(\beta)+\alpha_{2,n}(k)K_n(\beta)}\beta\psi_n e^{in\theta},\tag{2.9}$$

where $\alpha_{1,n}$ and $\alpha_{2,n}$ are given by (2.7) and (2.8), respectively. Then for each quasi mode ψ for the scattering resonance problem (1.1), we obtain the boundary condition $\frac{\partial\psi}{\partial n}=\mathcal{T}\psi$ on ∂B_1. In this way, the scattering resonance problem can be formulated in the unit disk B_1 as follows:

$$\begin{cases}-\Delta\psi+V_L\psi=k^2\psi & \text{in } B_1,\\ \dfrac{\partial\psi}{\partial n}=\mathcal{T}\psi & \text{on } \partial B_1.\end{cases}\tag{2.10}$$

LEMMA 2.1. *The Dirichlet-to-Neumann map $\mathcal{T}(k)$ is a bounded operator from $H^{1/2}(\partial B_1)$ to its dual space $(H^{1/2}(\partial B_1))'$ for $k\in\mathbb{C}$. Let $k_b\in(0,\sqrt{V_0})$ be a bound state frequency, and then the operator-valued function $\mathcal{T}(k)$ is analytic with respect to k in the neighborhood of k_b.*

Proof. Let

$$c_n=\frac{\alpha_{1,n}(k)I_n'(\beta)+\alpha_{2,n}(k)K_n'(\beta)}{\alpha_{1,n}(k)I_n(\beta)+\alpha_{2,n}(k)K_n(\beta)}.$$

For any ψ, $w\in H^{1/2}(\partial B_1)$,

$$\langle\mathcal{T}\psi,w\rangle=\int_{\partial B_1}\mathcal{T}\psi\;\bar{w}\;ds=\beta\int_0^{2\pi}\sum_{n=0}^{\infty}c_n\psi_n e^{in\theta}\bar{w}\;d\theta=2\pi\beta\sum_{n=0}^{\infty}c_n\psi_n\bar{w}_n.$$

Here,

$$\psi_n=\frac{1}{2\pi}\int_0^{2\pi}\psi(\theta)e^{-in\theta}d\theta\ \text{ and }\ w_n=\frac{1}{2\pi}\int_0^{2\pi}w(\theta)e^{-in\theta}d\theta.$$

Decompose c_n into three parts as follows:

(2.11)
$$c_n = \frac{K_n'(\beta)}{K_n(\beta)} + \frac{\alpha_{1,n}(k)I_n'(\beta)}{\alpha_{1,n}(k)I_n(\beta)+\alpha_{2,n}(k)K_n(\beta)} - \frac{\alpha_{1,n}(k)I_n(\beta)}{\alpha_{1,n}(k)I_n(\beta)+\alpha_{2,n}(k)K_n(\beta)}\frac{K_n'(\beta)}{K_n(\beta)}.$$

From the recurrence relation of Bessel functions (B.3) and the monotonicity property $|K_{n-1}(\beta)| \le |K_n(\beta)|$ in (B.6), it is obtained that

$$\left|\frac{K_n'(\beta)}{K_n(\beta)}\right| = \left|\frac{-n/\beta\; K_n(\beta) - K_{n-1}(\beta)}{K_n(\beta)}\right| \le \frac{n}{|\beta|} + 1, \quad n \ge 1, \tag{2.12}$$

and

$$\left|\frac{K_0'(\beta)}{K_0(\beta)}\right| = \left|\frac{K_1(\beta)}{K_0(\beta)}\right| \le c_0. \tag{2.13}$$

Here, c_0 is some constant depending on k. On the other hand, from the asymptotic behavior of Bessel's function for large orders given by (A.4) and (B.8), we see that as $n \to \infty$, asymptotically

$$\alpha_{1,n}(k)I_n(\beta) \sim L\left(\frac{e\beta}{2n}\frac{n}{\sqrt{n(n-1)}}\left(\frac{n-1}{nL}\right)^{n-1} + k\frac{e}{2n}\left(\frac{n}{n-1}\right)^{n-1/2}\frac{1}{L^{n-1}}\right),$$
$$\alpha_{2,n}(k)K_n(\beta) \sim L\left(\frac{2}{e}\frac{n}{\sqrt{n(n-1)}}\left(\frac{nL}{n-1}\right)^{n-1} - k\frac{e\beta}{2n^2}\left(\frac{n}{n-1}\right)^{n-1/2}L^{n+1}\right).$$

Note that $L > 1$, and it follows that

$$|\alpha_{1,n}(k)I_n(\beta)| \to 0 \quad \text{and} \quad |\alpha_{2,n}(k)K_n(\beta)| \to \infty \quad \text{as } n \to \infty.$$

Similarly, it can be shown that $|\alpha_{1,n}(k)I_n'(\beta)| \to 0$ as $n \to \infty$. Therefore, there exists a constant $\tilde{c}$ independent of n such that

$$\begin{aligned} &\left|\frac{\alpha_{1,n}(k)I_n'(\beta)}{\alpha_{1,n}(k)I_n(\beta)+\alpha_{2,n}(k)K_n(\beta)}\right| \le \tilde{c} \quad \text{and} \\ &\left|\frac{\alpha_{1,n}(k)I_n(\beta)}{\alpha_{1,n}(k)I_n(\beta)+\alpha_{2,n}(k)K_n(\beta)}\right| \le \tilde{c} \quad \forall n \ge 0. \end{aligned} \tag{2.14}$$

Thus, by combining (2.11)–(2.14) , we obtain that

$$|c_n| \le (1+c_0)(1+2\tilde{c})\left(1+\frac{n}{|\beta|}\right) \le c\sqrt{1+n^2} \tag{2.15}$$

for some positive constant c independent of n.

Now using (2.15), we have

$$\begin{aligned} |\langle \mathcal{T}\psi, w\rangle| \;&\le\; 2\pi c\beta \sum_{n=0}^{\infty} \sqrt{1+n^2}\; |\psi_n \bar{w}_n| \\ &\le 2\pi c\beta \left(\sum_{n=0}^{\infty}\sqrt{1+n^2}\; |\psi_n|^2\right)^{1/2} \left(\sum_{n=0}^{\infty}\sqrt{1+n^2}\; |w_n|^2\right)^{1/2}, \end{aligned}$$

i.e.,

$$|\langle \mathcal{T}\psi, w\rangle| \ \le\ 2\pi c\beta \, \|\varphi\|_{H^{1/2}(\partial B_1)} \, \|\psi\|_{H^{1/2}(\partial B_1)} \,,$$

and $\mathcal{T}$ is bounded from $H^{1/2}(\partial B_1)$ to $(H^{1/2}(\partial B_1))'$.

To show the analyticity of $\mathcal{T}(k)$ with respect to k, let

$$\mathcal{T}(k) = \sum_{n=0}^{\infty} \mathcal{T}_n(k), \quad \text{where} \quad (\mathcal{T}_n(k)\psi)(\theta) = \beta c_n \psi_n e^{in\theta}.$$

It is obvious that each term $\mathcal{T}_n(k)$ is analytic in the neighborhood of k_b. From (2.15), we deduce that the operator-valued series $\sum_{n=0}^{\infty} \mathcal{T}_n(k)$ is absolute convergent, and the analyticity of $\mathcal{T}(k)$ follows. ∎

3. Near bound-sate scattering resonances.

3.1. Resonance condition. In this section, we derive an equivalent formulation for the scattering resonances of (1.1). Let $k_b \in (0, \sqrt{V_0})$ be a nondegenerate bound state frequency of (2.1) with an eigenfunction ψ_b, and (k, ψ) be a resonance pair of (1.1) such that k is a scattering resonance and ψ is the associated quasi mode. Define the difference $\xi = \psi - \psi_b$. By a direct comparison of the associated boundary value problems (2.3) and (2.10), and noting that $V_L = V_\infty$ in B_1, we see that ξ is a solution of

$$\text{(3.1)} \qquad \left\{ \begin{array}{ll} -\Delta\xi + (V_\infty - k_b^2)\xi = (k^2 - k_b^2)\psi & \text{in } B_1, \\ \dfrac{\partial \xi}{\partial n} = \mathcal{T}_b \xi + \tilde{\mathcal{T}}(k)\psi & \text{on } \partial B_1, \end{array} \right.$$

where the operator $\tilde{\mathcal{T}}(k) = \mathcal{T}(k) - \mathcal{T}_b$. The corresponding homogeneous (self-adjoint) problem is given by

$$\text{(3.2)} \qquad \left\{ \begin{array}{ll} -\Delta\psi_b + (V_\infty - k_b^2)\psi_b = 0 & \text{in } B_1, \\ \dfrac{\partial \psi_b}{\partial n} = \mathcal{T}_b \psi_b & \text{on } \partial B_1. \end{array} \right.$$

This is also the boundary value problem for the bound state ψ_b (cf. (2.3)). For a nondegenerate bound-state, it is clear that the solution space $\mathcal{N}$ of the above homogeneous problem is span$\{\psi_b\}$.

We observe that ξ solves the variational problem $a(\xi, \eta) = b(\eta)$, where the bilinear form

$$a(\xi, \eta) = (\nabla\xi, \nabla\eta) + ((V - k_b^2)\xi, \eta) - \langle \mathcal{T}_b \xi, \eta\rangle$$

and the linear functional

$$b(\eta) = (k^2 - k_b^2)(\psi, \eta) + \langle \tilde{\mathcal{T}}\psi, \eta\rangle.$$

Following the standard argument for the second order elliptic equations, the solvability condition for (3.1) reads

$$\text{(3.3)} \qquad (k^2 - k_b^2)(\psi, \psi_b) + \langle \tilde{\mathcal{T}}(k)\psi, \psi_b\rangle = 0.$$

For completeness, we present a brief derivation of (3.3) below.

Let us choose a suitable function $\rho(k) \in H^2(B_1)$ such that the trace

$$\left.\frac{\partial \rho}{\partial n}\right|_{\partial B_1} = \mathcal{T}_b\ (\rho|_{\partial B_1}) + \tilde{\mathcal{T}}(k)\psi \tag{3.4}$$

and

$$||\rho||_{H^2(B_1)} \le C||\tilde{\mathcal{T}}(k)\psi||_{H^{1/2}(\partial B_1)}.$$

By setting $\zeta = \xi - \rho$, (3.1) is reduced into the following boundary value problem:

$$\begin{cases} -\Delta\zeta + (V_\infty - k_b^2)\zeta = (k^2 - k_b^2)\psi + q & \text{in } B_1, \\ \dfrac{\partial \zeta}{\partial n} = \mathcal{T}_b\zeta & \text{on } \partial B_1, \end{cases} \tag{3.5}$$

wherein $q(k) = [\Delta - (V_\infty - k_b^2)]\rho(k)$ and $||q(k)||_{L^2(B_1)} \le C||\tilde{\mathcal{T}}\psi||_{H^{1/2}(B_1)}$. Let us denote

$$g(k) = (k^2 - k_b^2)\psi + q(k). \tag{3.6}$$

It is known that the variational problem $a(\zeta,\eta)+\gamma(\zeta,\eta) = (g,\eta)$ has a unique solution in $H^1(B_1)$ for some sufficiently large constant γ. Hence, the induced operator L_γ^{-1} : $g \to \zeta$ is bounded from $L^2(B_1)$ to $H^1(B_1)$ and compact from $L^2(B_1)$ to $L^2(B_1)$. Therefore, ζ is a weak solution of (3.5) if and only if

$$\zeta - \gamma L_\gamma^{-1}\zeta = L_\gamma^{-1} g. \tag{3.7}$$

The solution of the corresponding homogeneous (self-adjoint) problem (3.2) satisfies

$$\psi_b - \gamma L_\gamma^{-1}\psi_b = 0. \tag{3.8}$$

Employing the Fredholm alternative and integrations-by-parts, we arrive at (3.3). We call (3.3) the *resonance condition*, which is a nonlinear equation of k.

Remark 3.1. For clarity, we may set $\rho(k)$ to be the solution of the elliptic equation $-\Delta\rho + \gamma_0\rho = 0$ with the boundary condition (3.4), wherein γ_0 is a sufficiently large constant. The operator $\mathcal{L}_1 : \tilde{\mathcal{T}}(k)\psi \to \rho(k)$ is bounded from $H^{1/2}(\partial B_1)$ to $H^2(B_1)$ and the operator $\mathcal{L}_2 : \tilde{\mathcal{T}}(k)\psi \to q(k)$ is bounded from $H^{1/2}(\partial B_1)$ to $L^2(B_1)$.

Let $L^2(B_1) = \mathcal{M} \bigoplus \mathcal{N}$, and then it is apparent that (3.8) admits only a trivial solution in $\mathcal{M}$. Accordingly, (3.7) has a unique solution in $\mathcal{M}$. We express such a solution as

$$\hat{\zeta}(k) = (I - \gamma L_\gamma^{-1})^{-1} L_\gamma^{-1} g =: \mathcal{L}_b g. \tag{3.9}$$

The subscript b denotes that the operator depends only on bound state frequency k_b. Define

$$\hat{\xi}(k) = \hat{\zeta}(k) + \rho(k) \quad \text{and} \quad \hat{\psi}_b = \psi - \hat{\xi}(k) \tag{3.10}$$

and substitute into (3.3); then the solvability condition (*resonance condition*) is recast as

$$(k^2 - k_b^2)(\psi, \psi - \hat{\xi}(k)) + \langle \tilde{\mathcal{T}}(k)\psi, \psi - \hat{\xi}(k)\rangle = 0. \tag{3.11}$$

Hence, if (k,ψ) is a resonance pair for (1.1), they satisfy the resonance condition (3.11). Indeed, those k satisfying (3.11) are scattering resonances of (1.1), and hence the nonlinear equation (3.11) is a necessary and sufficient condition for the scattering resonances.

3.2. Main result. We study near bound-state resonances, namely, those k near a bound state frequency $k_b \in (0, \sqrt{V_0})$. First, it is a standard argument that all the resonances lie below the real axis.

LEMMA 3.2. *Let $k \in \mathbb{C}$ be a nonzero scattering resonance of* (1.1), *and then k has a negative imaginary part.*

Proof. For any $L_0 \geq L$, by noting that $V_L(x) = 0$ for $|x| \geq L_0$ and a straightforward derivation, the Dirichlet-to-Neumann map on ∂B_{L_0} for outgoing waves is given by

$$(\mathcal{T}_{L_0}\psi)(\theta) = \sum_{n=0}^{\infty} \frac{k(H_n^{(1)}(kL_0))'}{H_n^{(1)}(kL_0)} \psi_n e^{in\theta},$$

where the Fourier coefficient $\psi_n = \frac{1}{2\pi}\int_0^{2\pi} \psi(L_0, \theta) e^{-in\theta} d\theta$. By using the Dirichlet-to-Neumann map $\mathcal{T}_{L_0}$, the scattering resonance problem (1.1) can be recast in the domain B_{L_0} as

$$\begin{cases} -\Delta\psi + V_L\psi = k^2\psi & \text{in } B_{L_0}, \\ \dfrac{\partial \psi}{\partial n} = \mathcal{T}_{L_0}\psi & \text{on } \partial B_{L_0}. \end{cases}$$

Multiplying the equation by $\bar{\psi}$ and applying Green's formula, we obtain

$$(\nabla\psi, \nabla\psi) + (V_L\psi, \psi) - \langle \mathcal{T}_{L_0}\psi, \psi\rangle - k^2(\psi, \psi) = 0. \tag{3.12}$$

We consider two cases as follows:

(i) $k \in \mathbb{R}$. The imaginary part of the left-hand side for the above equation is

$$\begin{aligned} -\mathrm{Im}\langle \mathcal{T}_{L_0}\psi, \psi\rangle &= -2\pi k L \sum_{n=0}^{\infty} |\psi_n|^2 \frac{J_n(kL_0)Y_n'(kL_0) - Y_n(kL_0)J_n'(kL_0)}{J_n^2(kL_0) + Y_n^2(kL_0)} \\ &= -4\sum_{n=0}^{\infty} \frac{|\psi_n|^2}{J_n^2(kL_0) + Y_n^2(kL_0)}, \end{aligned}$$

where the last equality follows by the Wronskian formula (A.3): $J_n(kL)Y_n'(kL) - Y_n(kL)J_n'(kL) = 2/(\pi kL)$. Thus, $\mathrm{Im}\langle \mathcal{T}_{L_0}\psi, \psi\rangle = 0$ implies that $\psi_n = 0$ for each n. This contradicts the fact that the quasi mode $\psi \not\equiv 0$.

(ii) *k is complex number with Im$k > 0$.* From the asymptotic expansion of the Hankel's functions (A.5) and the expansion for ψ outside B_{L_0}, it is seen that ψ decays exponentially as $L_0 \to \infty$. Thus, $\langle \mathcal{T}_{L_0}\psi, \psi\rangle \to 0$ as $L_0 \to \infty$. Substituting into (3.12), we deduce that

$$(\nabla\psi, \nabla\psi) + (V_L\psi, \psi) - k^2(\psi, \psi) \to 0 \quad \text{as } L_0 \to \infty. \tag{3.13}$$

(a) *Im$k^2 \neq 0$:* An inspection of (3.13) shows that $(\psi, \psi) = 0$ as $L_0 \to \infty$, which leads to $\psi \equiv 0$ and is a contradiction.

(b) *Im$k^2 = 0$:* It is observed that $\mathrm{Im}k^2 = 0$ holds only if $\mathrm{Re}k = 0$ (note that $\mathrm{Im}k > 0$), and thus $k^2 = -(\mathrm{Im}k)^2 < 0$. In this case, (3.12) becomes

$$(\nabla\psi, \nabla\psi) + (V_L\psi, \psi) + (\mathrm{Im}k)^2(\psi, \psi) \to 0 \quad \text{as } L_0 \to \infty,$$

which also leads to $\psi \equiv 0$.

The proof is complete by combining the arguments in (i) and (ii). ∎

TABLE 1
Near bound-state resonances (first column) and the difference between resonances and the bound state frequency for different L (second column).

L	k	$k - k_b$
2	$1.135501965642507 - 0.286732709690725i$	$1.56191 \times 10^{-1} - 2.86733 \times 10^{-1}i$
4	$1.023284198719800 - 0.036528738444848i$	$4.39734 \times 10^{-2} - 3.65287 \times 10^{-2}i$
8	$0.984796308094740 - 0.002858349615355i$	$5.48554 \times 10^{-3} - 2.85835 \times 10^{-3}i$
16	$0.979495236549527 - 0.000080174371694i$	$1.84470 \times 10^{-4} - 8.01744 \times 10^{-5}i$
32	$0.979311056837546 - 0.000000124782683i$	$2.90041 \times 10^{-7} - 1.24783 \times 10^{-7}i$

Our main result regarding near bound-state resonances is stated the following theorem.

THEOREM 3.3. *Let $k_b \in (0, \sqrt{V_0})$ be a nondegenerate bound state frequency. There exists a constant L_0 such that for any $L \geq L_0$, the following estimate holds for the scattering resonance k in the neighborhood of k_b:*

$$| \, k - k_b| < CL^2 e^{-2\beta_b(L-1)},$$

where C is a positive constant independent of L.

Remark 3.4. We have assumed that the low energy well is inside B_1. For a general potential well contained inside $B_R \subset\subset B_L$, the estimate becomes $| \, k - k_b| < CL^2 e^{-2\beta_b(L-R)}$ after a suitable scaling.

We postpone the proof of Theorem 3.3 to the next two sections. To validate the conclusion of Theorem 3.3, let us consider a step potential

$$V_L(x) = \begin{cases} 0, & |x| < 1, \\ 1, & 1 < |x| < L, \\ 0, & |x| > L. \end{cases} \tag{3.14}$$

The bound state and quasi mode can be expressed as

$$\psi_b(x) = \begin{cases} J_n(k_b|x|)e^{in\theta}, & |x| < 1, \\ c_{b,n} K_n(\beta_b|x|)e^{in\theta}, & |x| > 1 \end{cases} \tag{3.15}$$

and

$$\psi(x) = \begin{cases} J_n(k|x|)e^{in\theta}, & |x| < 1, \\ \left(a_{1,n} I_n(\beta|x|) + a_{2,n} K_n(\beta|x|)\right) e^{in\theta}, & 1 < |x| < L, \\ c_n H_n^{(1)}(k|x|)e^{in\theta}, & |x| > L \end{cases} \tag{3.16}$$

for $n = 0, 1, 2, \ldots$. Solving the equations derived from the continuity conditions along the boundary $|x| = 1$ by Newton's method (Appendix C), we obtain the bound state frequency $k_b = 0.979310766796298$ for $n = 0$. The near bound-state scattering resonances can be obtained in a similar way. The disparity between k and k_p for different L is given in the Table 1. From the table and Figure 2, it is seen that the difference between k and k_p decays with an exponential rate with respect to L.

4. Relevant estimates.

4.1. Preliminaries.

LEMMA 4.1. *Let $\alpha_{1,n}(k_b)$ be defined by (2.7). If $k_b \in (0, \sqrt{V_0})$, then*

$$|\alpha_{1,n}(k_b)| < L(\beta_b + k_b)K_n(\beta_b L), \quad n \geq 1,$$

$$|\alpha_{1,0}(k_b)| < L\left(\beta_b K_1(\beta_b L) + k_b K_0(\beta_b L)\left|\frac{H_1^{(1)}(k_b L)}{H_0^{(1)}(k_b L)}\right|\right).$$

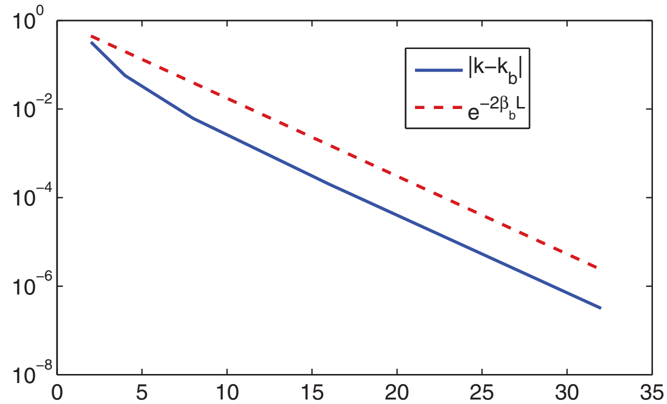

FIG. 2. *Plot of* $|k - k_b|$ *for different* L*'s for the step potential well.*

Proof. For $n \geq 1$, apply the recurrence relations (A.1) and (B.3):

$$K'_{n-1}(\beta_b L) = -\frac{n}{\beta_b L} K_n(\beta_b L) - K_{n-1}(\beta_b L),$$
$$(H_n^{(1)})'(k_b L) = -\frac{n}{k_b L} H_n^{(1)}(k_b L) + H_{n-1}^{(1)}(k_b L);$$

$\alpha_{1,n}(k_b)$ can be recast as

$$\alpha_{1,n}(k_b) = L\left(\beta_b K_{n-1}(\beta_b L) + k_b \frac{H_{n-1}^{(1)}(k_b L)}{H_n^{(1)}(k_b L)} K_n(\beta_b L)\right). \tag{4.1}$$

By the monotonicity property $|H_{n-1}^{(1)}(k_b L)| < |H_n^{(1)}(k_b L)|$ and $|K_{n-1}(k_b L)| < |K_n(k_b L)|$ (cf. (A.7) and (B.6)), it follows that

$$|\alpha_{1,n}(k_b)| < L\left(\beta_b K_{n-1}(\beta_b L) + k_b K_n(\beta_b L)\right) < L(\beta_b + k_b) K_n(\beta_b L).$$

If $n = 0$, the inequality follows from $(H_0^{(1)})'(k_b L) = -H_1^{(1)}(k_b L)$, $K'_0(\beta_b L) = -K_1(\beta_b L)$ and the triangle inequality. ∎

LEMMA 4.2. *Let* $\alpha_{2,n}(k_b)$ *be defined by* (2.8). *If* $k_b \in (0, \sqrt{V_0})$, *then*

$$nI_n(\beta_b L) < |\alpha_{2,n}(k_b)| < (2n + L(\beta_b + k_b)) I_n(\beta_b L), \quad n \geq 1,$$
$$\beta_b L I_1(\beta_b L) < |\alpha_{2,0}(k_b)| < L\left(\beta_b I_1(\beta_b L) + k_b I_0(\beta_b L)\left|\frac{H_1^{(1)}(k_b L)}{H_0^{(1)}(k_b L)}\right|\right).$$

Proof. We first prove the estimate for the upper bound. Similar to Lemma 4.1, an application of the recurrence relations (A.1) and (B.2) for H_n and I_n yields

$$\alpha_{2,n}(k_b) = L\left(\beta_b I_{n-1}(\beta_b L) - k_b \frac{H_{n-1}^{(1)}(k_b L)}{H_n^{(1)}(k_b L)} I_n(\beta_b L)\right) \quad \text{for } n \geq 1. \tag{4.2}$$

Taking into account the recurrence relation (B.1), $I_{n-1}(\beta_b L) - I_{n+1}(\beta_b L) = \frac{2n}{\beta_b L} I_n(\beta_b L)$, we obtain

$$|\alpha_{2,n}(k_b)| = L\left|\frac{2n}{L} I_n(\beta_b L) + \beta_b I_{n+1}(\beta_b L) - k_b \frac{H_{n-1}^{(1)}(k_b L)}{H_n^{(1)}(k_b L)} I_n(\beta_b L)\right|$$
$$< (2n + L(\beta_b + k_b)) I_n(\beta_b L),$$

where the monotonicity properties (A.7) and (B.6) have been used for the last inequality. If $n = 0$, the inequality follows by $(H_0^{(1)})'(k_bL) = -H_1^{(1)}(k_bL)$, $I_0'(\beta_bL) = -I_1(\beta_bL)$ (cf. (A.2) and (B.4)).

To obtain the lower bounds, note that $H_n^{(1)} = J_n + i\,Y_n$, where J_n and Y_n are the first and second kind Bessel functions, respectively, and the modulus of each Hankel function is a decreasing function (Proposition A.7); we have

$$\mathrm{Re}\left\{\frac{\left(H_n^{(1)}\right)'(k_bL)}{H_n^{(1)}(k_bL)}\right\} = \frac{J_n(k_bL)J_n'(k_bL) + Y_n(k_bL)Y_n'(k_bL)}{J_n^2(k_bL) + Y_n^2(k_bL)}$$
$$= \frac{1}{2}\frac{(J_n^2)'(k_bL) + (Y_n^2)'(k_bL)}{J_n^2(k_bL) + Y_n^2(k_bL)} < 0.$$

Consequently, by noting that both $I_n'(\beta_bL)$ and $I_n(\beta L)$ are real and positive,

$$\mathrm{Re}\{\alpha_{2,n}(k_b)\} = \beta_bLI_n'(\beta_bL) - k_bLI_n(\beta L)\mathrm{Re}\left\{\frac{\left(H_n^{(1)}\right)'(k_bL)}{H_n^{(1)}(k_bL)}\right\} > \beta_bLI_n'(\beta_bL). \tag{4.3}$$

Using the recurrence relations (B.2) and (B.4), namely,

$$I_n'(\beta_bL) = \frac{n}{\beta_bL}I_n(\beta_bL) + I_{n+1}(\beta_bL) \quad \text{for } n \geq 1 \quad \text{and} \quad I_0'(\beta_bL) = I_1(\beta_bL),$$

the inequalities

$$|\alpha_{2,n}(k_b)| \geq \mathrm{Re}\{\alpha_{2,n}(k_b)\} > nI_n(\beta_bL)$$

and

$$|\alpha_{2,0}(k_b)| \geq \mathrm{Re}\{\alpha_{2,0}(k_b)\} > \beta_bLI_1(\beta_bL)$$

follow. ∎

Lemma 4.3. *Let $\alpha_{1,n}(k_b)$ be defined by (2.7). If $k_b \in (0, \sqrt{V_0})$, then*

$$|\alpha_{1,n}'(k_b)| < C(n+L)LK_n(\beta_bL), \quad n \geq 1.$$
$$|\alpha_{1,0}'(k_b)| < CL\left(1 + L + (1+k_b)L\left|\frac{H_1^{(1)}(k_bL)}{H_0^{(1)}(k_bL)}\right| + k_bL\left|\frac{H_1^{(1)}(k_bL)}{H_0^{(1)}(k_bL)}\right|^2\right)(K_0(\beta_bL) + K_1(\beta_bL)).$$

Proof. If $n \geq 1$, an application of the chain rule and the recurrence relation for (4.1) leads to

$$\begin{aligned}\alpha'_{1,n}(k_b) = L\Bigg\{&\left(n\beta'_b - L\beta'_b k_b \frac{H^{(1)}_{n-1}(k_bL)}{H^{(1)}_n(k_bL)}\right) K_{n-1}(\beta_b L)\\ &+\Bigg(-L\beta_b\beta'_b + \left(1 - \frac{nk_b\beta'_b}{\beta_b}\right)\frac{H^{(1)}_{n-1}(k_bL)}{H^{(1)}_n(k_bL)}\\ &+ k_bL\left(\frac{H^{(1)}_{n-1}(k_bL)}{H^{(1)}_n(k_bL)}\right)'\Bigg) K_n(\beta_b L)\Bigg\},\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}&\left(\frac{H^{(1)}_{n-1}(k_bL)}{H^{(1)}_n(k_bL)}\right)'\\ &\quad= \frac{(2n-1)/(kL)H^{(1)}_{n-1}(k_bL)H^{(1)}_n(k_bL) - (H^{(1)}_{n-1}(k_bL))^2 - (H^{(1)}_n(k_bL))^2}{(H^{(1)}_n(k_bL))^2}.\end{aligned}$$

By using the monotonicity property $|H^{(1)}_{n-1}(k_bL)| < |H^{(1)}_n(k_bL)|$ (cf.(A.7)) for the above equation, it is observed that

$$\left|\left(\frac{H^{(1)}_{n-1}(k_bL)}{H^{(1)}_n(k_bL)}\right)'\right| < 2\left(1 + \frac{n}{k_bL}\right).$$

Substitute into (4.4) and use the monotonicity inequalities for $H^{(1)}_n$ and K_n (cf. (A.7) and (B.6)); it follows that

$$|\alpha'_{1,n}(k_b)| < C(n+L)LK_n(\beta_b L)$$

for some positive constant C depending only on k_b. The proof for $n = 0$ follows similarly by the recurrence relation and the triangle inequality. ∎

LEMMA 4.4. *Let $\alpha_{2,n}(k_b)$ be defined by (2.8). If $k_b \in (0, \infty)$, then*

$$|\alpha'_{2,n}(k_b)| < C\left(\frac{n^2}{L} + n + L\right) LI_n(\beta_b L), \quad n \geq 1.$$

$$\begin{aligned}|\alpha'_{2,0}(k_b)| < CL\Bigg(&1 + L + (1+k_b)L\left|\frac{H^{(1)}_1(k_bL)}{H^{(1)}_0(k_bL)}\right|\\ &+ k_bL\left|\frac{H^{(1)}_1(k_bL)}{H^{(1)}_0(k_bL)}\right|^2\Bigg)(I_0(\beta_b L) + I_1(\beta_b L)).\end{aligned}$$

Proof. A similar calculation in Lemma 4.3 for (4.2) yields

$$\begin{aligned}\alpha'_{2,n}(k_b) = L\Bigg\{&\left(n\beta'_b - L\beta'_b k_b \frac{H^{(1)}_{n-1}(k_b L)}{H^{(1)}_n(k_b L)}\right) I_{n-1}(\beta_b L)\\ &+\left(L\beta_b\beta'_b + \left(\frac{nk_b\beta'_b}{\beta_b} - 1\right)\frac{H^{(1)}_{n-1}(k_b L)}{H^{(1)}_n(k_b L)}\right.\\ &\left.- k_b L\left(\frac{H^{(1)}_{n-1}(k_b L)}{H^{(1)}_n(k_b L)}\right)'\right) I_n(\beta_b L)\Bigg\}.\end{aligned} \tag{4.5}$$

Apply the recurrence relation (B.1), $I_{n-1}(\beta_b L) = I_{n+1}(\beta_b L) + \frac{2n}{\beta_b L} I_n(\beta_b L)$, and the monotonicity inequalities for $H^{(1)}_n$ and I_n and it follows that

$$|\alpha'_{2,n}(k_b)| < C\left(\frac{n^2}{L} + n + L\right) L I_n(\beta_b L).$$

The case for $n = 0$ can be shown in a similar fashion. ∎

LEMMA 4.5. *Let $n \geq 0$, $x, y \in (0, \infty)$; then*

$$\frac{K_n(x)}{K_n(y)} > e^{y-x} \quad \textit{and} \quad \frac{I_n(y)}{I_n(x)} > e^{y-x}\frac{x}{y} \quad \textit{if} \quad x < y.$$

Lemma 4.5 plays an important role in the estimate of resonances. For completeness, we give a sketch of the proof in the following. Readers are referred to [3, 11] and references therein for a more comprehensive study of the inequalities for modified Bessel functions.

Proof. For fixed n, let $g(x) = e^x K_n(x)$. By the integral representation of the modified Bessel's function $K_n(x)$,

$$g(x) = \int_0^\infty e^{(1-\cosh t)x} \cosh nt \, dt.$$

It is apparent that $g'(x) < 0$. Hence, $g(x)$ is decreasing on $(0, \infty)$, and the inequality for $K_n(x)$ follows.

It is shown in [5] that $xI'_n(x)/I_n(x) > x - 1$ holds for positive x when $n \geq 0$. For fixed n, define a function $h(x) = xe^{-x} I_n(x)$; we shall show that $h(x)$ is strictly increasing on $(0, \infty)$. Indeed,

$$\frac{h'(x)}{e^{-x} I_n(x)} = x\frac{I'_n(x)}{I_n(x)} + 1 - x > 0.$$

Hence, $h'(x) > 0$, i.e., $h(x)$ is increasing and the inequality for $I_n(x)$ follows. ∎

LEMMA 4.6. *If $x \in (0, \infty)$, the function $F(x) = \frac{|H^{(1)}_1(x)|}{|H^{(1)}_0(x)|}$ is a decreasing function with respect to x.*

Proof. We show that $F^2(x)$ is decreasing. By Nicholson's integral formula (A.6),

$$|H^{(1)}_n(x)|^2 = J^2_n(x) + Y^2_n(x) = \frac{8}{\pi^2}\int_0^\infty K_0(2x \sinh t) \cosh 2nt \, dt. \tag{4.6}$$

Applying integration by parts, we observe that

$$\begin{aligned}\frac{d}{dx}|H_0^{(1)}(x)|^2 &= \frac{8}{\pi^2}\int_0^\infty K_0'(2x\sinh t)2\sinh t\;dt\\ &= \frac{8}{\pi^2 x}\left(K_0(2x\sinh t)\tanh t|_0^\infty - \int_0^\infty K_0(2x\sinh t)\text{sech}^2 t\;dt\right)\\ &= -\frac{8}{\pi^2 x}\int_0^\infty K_0(2x\sinh t)\text{sech}^2 t\;dt.\end{aligned} \tag{4.7}$$

The last equality follows by noting that $K_0(z) \sim -\ln(z/2)$ as $z \to 0$, and $K_0(z) \to 0$ as $z \to \infty$. Similarly, it can be shown that

$$\begin{aligned}\frac{d}{dx}|H_1^{(1)}(x)|^2 &= \frac{8}{\pi^2}\int_0^\infty K_0'(2x\sinh t)2\sinh t\cosh 2t\;dt\\ &= -\frac{8}{\pi^2 x}\int_0^\infty K_0(2x\sinh t)\frac{d}{dt}(\cosh 2t\tanh t)\;dt.\end{aligned} \tag{4.8}$$

Note that

$$\frac{d}{dx}F^2(x) = \frac{|H_0^{(1)}(x)|^2\frac{d}{dx}|H_1^{(1)}(x)|^2 - |H_1^{(1)}(x)|^2\frac{d}{dx}|H_0^{(1)}(x)|^2}{|H_0^{(1)}(x)|^4} =: \frac{N(x)}{D(x)}.$$

We only need to show that the numerator $N(x) < 0$. Indeed, a direct calculation and an application of the formula $\cosh 2t = \cosh^2 t + \sinh^2 t$ leads to

$$\frac{d}{dt}(\cosh 2t\tanh t) = 2\sinh 2t\tanh t + \cosh 2t\;\text{sech}^2 t = 1 + 4\sinh^2 t + \tanh^2 t.$$

In addition,

$$\cosh 2t = 1 + 2\sinh^2 t, \quad \text{sech}^2 t = 1 - \tanh^2 t.$$

Substitute the above equalities into (4.6)–(4.8) and note that K_0 is positive; it follows that

$$\begin{aligned}N(x) = &-\frac{64}{\pi^4 x}\int_0^\infty K_0(2x\sinh t)\;dt\int_0^\infty K_0(2x\sinh t)(1+4\sinh^2 t+\tanh^2 t)\;dt\\ &+\frac{64}{\pi^4 x}\int_0^\infty K_0(2x\sinh t)(1+2\sinh^2 t)\;dt\int_0^\infty K_0(2x\sinh t)(1-\tanh^2 t)\;dt\\ = &-\frac{64}{\pi^4 x}\Bigg(\int_0^\infty K_0(2x\sinh t)\;dt\int_0^\infty K_0(2x\sinh t)(2\sinh^2 t+2\tanh^2 t)\;dt\\ &+\int_0^\infty K_0(2x\sinh t)2\sinh^2 t\;dt\int_0^\infty K_0(2x\sinh t)\tanh^2 t\;dt\Bigg) < 0.\end{aligned}$$

The proof is complete. ◻

4.2. Estimate for $\tilde{\mathcal{T}}(k_b)$. Let $\mathcal{T}(k)$ and $\mathcal{T}_b$ be given by (2.9) and (2.2), respectively, and $\tilde{\mathcal{T}}(k) = \mathcal{T}(k) - \mathcal{T}_b$. We rewrite the Dirichlet-to-Neumann map $\mathcal{T}(k)$ as

$$\mathcal{T}(k)\psi(1,\theta) = \sum_{n=0}^{\infty}\left[c_{0,n}(k) + c_{1,n}(k) + c_{2,n}(k)\right]\beta\psi_n e^{in\theta}, \tag{4.9}$$

where

$$c_{0,n}(k) = \frac{K_n'(\beta)}{K_n(\beta)}, \tag{4.10}$$

$$c_{1,n}(k) = \frac{\alpha_{1,n}(k) I_n'(\beta)}{\alpha_{1,n}(k) I_n(\beta) + \alpha_{2,n}(k) K_n(\beta)}, \tag{4.11}$$

and

$$c_{2,n}(k) = -\frac{\alpha_{1,n}(k) I_n(\beta)}{\alpha_{1,n}(k) I_n(\beta) + \alpha_{2,n}(k) K_n(\beta)} \frac{K_n'(\beta)}{K_n(\beta)}. \tag{4.12}$$

In the rest of the paper, C denotes some generic constant that depends on k_b only.

THEOREM 4.7. *There exists $L_0 \in (1, \infty)$ such that if $L > L_0$, then*

$$\left| \langle \tilde{\mathcal{T}}(k_b)\psi, w \rangle \right| < C L^2 e^{-2\beta_b (L-1)} ||\psi||_{H^{1/2}(\partial B_1)} ||w||_{H^{1/2}(\partial B_1)}$$

for any ψ, $w \in H^{1/2}(\partial B_1)$.

Proof. A direct calculation yields $\tilde{\mathcal{T}}(k_b)\psi(1,\theta) = \tilde{\mathcal{T}}_1(k_b)\psi(1,\theta) + \tilde{\mathcal{T}}_2(k_b)\psi(1,\theta)$, where

$$\tilde{\mathcal{T}}_1(k_b)\psi(1,\theta) = \sum_{n=0}^{\infty} c_{1,n}(k_b)\beta\psi_n e^{in\theta} \quad \text{and} \quad \tilde{\mathcal{T}}_2(k_b)\psi(1,\theta) = \sum_{n=0}^{\infty} c_{2,n}(k_b)\beta\psi_n e^{in\theta}$$

and the coefficients $c_{1,n}$, $c_{2,n}$ are given by (4.11) and (4.12), respectively.

We first estimate $\langle \tilde{\mathcal{T}}_1(k_b)\psi, w \rangle$. To this end, an upper bound for the coefficients $c_{1,n}$ needs to be derived. Let us begin with two key inequalities,

$$K_n(\beta_b L) < e^{\beta_b(1-L)} K_n(\beta_b) \quad \text{and} \quad I_n(\beta_b L) > \frac{e^{\beta_b(L-1)}}{L} I_n(\beta_b), \tag{4.13}$$

which follow by an application of Lemma 4.5. The inequalities give explicit dependence of Bessel's functions $K_n(\beta_b L)$ and $I_n(\beta_b L)$ on the thickness parameter L.

(i) $n \geq 1$: By Lemmas 4.1–4.2 and the inequalities (4.13), it is obtained that

$$|\alpha_{2,n}(k_b) K_n(\beta_b)| > n I_n(\beta_b L) K_n(\beta_b) > \frac{n e^{\beta_b(L-1)}}{L} I_n(\beta_b) K_n(\beta_b) \tag{4.14}$$

and

$$|\alpha_{1,n}(k_b) I_n(\beta_b)| < L(\beta_b + k_b) K_n(\beta_b L) I_n(\beta_b) < e^{\beta_b(1-L)} L(\beta_b + k_b) K_n(\beta_b) I_n(\beta_b). \tag{4.15}$$

Hence, if L is sufficiently large, then $|\alpha_{2,n}(k_b) K_n(\beta_b)| > |\alpha_{1,n}(k_b) I_n(\beta_b)|$. In particular, if we choose $L_1 \in [1/\beta_b, +\infty)$ such that

$$(\beta_b + k_b) L_1^2 e^{2\beta_b(1-L_1)} = 1/2, \tag{4.16}$$

then when $L > L_1$,

$$\begin{aligned} &|\alpha_{2,n}(k_b) K_n(\beta_b)| - |\alpha_{1,n}(k_b) I_n(\beta_b)| \\ &\quad > \frac{e^{\beta_b(L-1)}}{L} \left(n - (\beta_b + k_b) L^2 e^{2\beta_b(1-L)} \right) K_n(\beta_b) I_n(\beta_b) \\ &\quad > \frac{e^{\beta_b(L-1)}}{L} \left(n - \frac{1}{2} \right) K_n(\beta_b) I_n(\beta_b). \end{aligned} \tag{4.17}$$

On the other hand, by Lemma 4.1 and the recurrence relation (B.2),

$$\begin{aligned}|\alpha_{1,n}(k_b)I_n'(\beta_b)| &< L(\beta_b+k_b)K_n(\beta_b L)\left(\frac{n}{\beta_b}I_n(\beta_b)+I_{n+1}(\beta_b)\right)\\ &< L(\beta_b+k_b)\left(\frac{n}{\beta_b}+1\right)K_n(\beta_b L)I_n(\beta_b),\end{aligned}$$

where the last inequality follows by noting the monotonicity property (B.6) for I_n. Employing the inequality (4.13) one more time, it follows that

$$|\alpha_{1,n}(k_b)I_n'(\beta_b)| < L(\beta_b+k_b)\left(\frac{n}{\beta_b}+1\right)K_n(\beta_b)I_n(\beta_b)e^{\beta_b(1-L)}. \tag{4.18}$$

Now in light of (4.17) and (4.18), if $L > L_1$,

$$\begin{aligned}|c_{1,n}| &= \frac{|\alpha_{1,n}(k_b)I_n'(\beta_b)|}{|\alpha_{1,n}(k_b)I_n(\beta_b)+\alpha_{2,n}(k_b)K_n(\beta_b)|}\\ &\le \frac{|\alpha_{1,n}(k_b)I_n'(\beta_b)|}{|\,|\alpha_{2,n}(k_b)K_n(\beta_b)|-|\alpha_{1,n}(k_b)I_n(\beta_b)|\,|}\\ &< \frac{L(\beta_b+k_b)\left(\frac{n}{\beta_b}+1\right)K_n(\beta_b)I_n(\beta_b)e^{\beta_b(1-L)}}{\frac{e^{\beta_b(L-1)}}{L}\left(n-\frac{1}{2}\right)K_n(\beta_b)I_n(\beta_b)}\\ &\le \frac{2}{\beta_b}(1+\beta_b)(\beta_b+k_b)L^2e^{-2\beta_b(L-1)}. \end{aligned} \tag{4.19}$$

(ii) $n=0$: By Lemmas 4.1–4.2 and the inequalities (4.13), it is seen that

$$|\alpha_{2,0}(k_b)K_0(\beta_b)| > \beta_b e^{\beta_b(L-1)}K_0(\beta_b)I_1(\beta_b) \tag{4.20}$$

and

$$|\alpha_{1,0}(k_b)I_0(\beta_b)| < e^{\beta_b(1-L)}L\left(\beta_b K_1(\beta_b)I_0(\beta_b)+k_bK_0(\beta_b)I_0(\beta_b)\left|\frac{H_1^{(1)}(k_bL)}{H_0^{(1)}(k_bL)}\right|\right).$$

Note that an application of the inequalities in (B.7) for $n=1$ gives rise to

$$K_1(\beta_b) < \frac{1+\sqrt{\beta_b^2+1}}{\beta_b}K_0(\beta_b) < \frac{2+\beta_b}{\beta_b}K_0(\beta_b)$$

and

$$I_0(\beta_b) < \frac{\beta_b}{\sqrt{\beta_b^2+1}-1}I_1(\beta_b) < \frac{2+\beta_b}{\beta_b}I_1(\beta_b).$$

Therefore,

$$\begin{aligned}&|\alpha_{1,0}(k_b)I_0(\beta_b)|\\ &\quad< e^{\beta_b(1-L)}L\left(\frac{(\beta_b+2)^2}{\beta_b}+\frac{k_b(\beta_b+2)}{\beta_b}\left|\frac{H_1^{(1)}(k_bL)}{H_0^{(1)}(k_bL)}\right|\right)K_0(\beta_b)I_1(\beta_b).\end{aligned}$$

If we choose $\tilde{L}_2$ such that $k_b|\frac{H_1^{(1)}(k_b\tilde{L}_2)}{H_0^{(1)}(k_b\tilde{L}_2)}| \leq 2(1+k_b)$, then from the monotonicity of $|\frac{H_1^{(1)}(x)}{H_0^{(1)}(x)}|$ in Lemma 4.6, we see that such an inequality holds for all $L \geq \tilde{L}_2$. In this way,

$$|\alpha_{1,0}(k_b)I_0(\beta_b)| < e^{\beta_b(1-L)}L\left(\frac{(\beta_b+2)^2}{\beta_b} + \frac{2(\beta_b+2)(1+k_b)}{\beta_b}\right)K_0(\beta_b)I_1(\beta_b). \tag{4.21}$$

Now let $L_2 \in [1/\beta_b, +\infty) \cap [\tilde{L}_2, +\infty)$ such that

$$L_2 e^{2\beta_b(1-L_2)}\left(\frac{(\beta_b+2)^2}{\beta_b} + \frac{2(\beta_b+2)(1+k_b)}{\beta_b}\right) = \frac{\beta_b}{2}. \tag{4.22}$$

Then if $L > L_2$, from (4.20)–(4.21) we obtain that

$$\begin{aligned} &|\alpha_{2,0}(k_b)K_0(\beta_b)| - |\alpha_{1,0}(k_b)I_0(\beta_b)| \\ &\quad > e^{\beta_b(L-1)}\left\{\beta_b - L_2 e^{2\beta_b(1-L_2)}\left(\frac{(\beta_b+2)^2}{\beta_b}\right.\right. \\ &\qquad\qquad \left.\left. + \frac{2(\beta_b+2)(1+k_b)}{\beta_b}\right)\right\}K_0(\beta_b)I_1(\beta_b) \\ &\quad > e^{\beta_b(L-1)}\frac{\beta_b}{2}K_0(\beta_b)I_1(\beta_b). \end{aligned} \tag{4.23}$$

On the other hand, following the same argument as in (4.21), it can be shown that

$$|\alpha_{1,0}(k_b)I_0'(\beta_b)| < Le^{\beta_b(1-L)}(\beta_b + k_b + 4)K_0(\beta_b)I_1(\beta_b), \tag{4.24}$$

provided that $L > L_2$. Now, combine (4.23) and (4.24), and it is derived that

$$\begin{aligned} |c_{1,0}| &\leq \frac{|\alpha_{1,0}(k_b)I_0'(\beta_b)|}{|\,|\alpha_{2,0}(k_b)K_0(\beta_b)| - |\alpha_{1,0}(k_b)I_0(\beta_b)|\,|} \\ &< \frac{2}{\beta_b}(\beta_b + k_b + 4)Le^{-2\beta_b(L-1)}. \end{aligned} \tag{4.25}$$

By combining (4.19) and (4.25), we see that there exist a positive constant C depending on k_b such that

$$|2\pi\beta_b c_{1,n}| \leq CL^2 e^{-2\beta_b(L-1)} \qquad \forall\, n \geq 0, \tag{4.26}$$

provided that $L > \max\{L_1, L_2\}$. We obtain the desired estimate for $\tilde{\mathcal{T}}_1$:

$$\left|\langle\tilde{\mathcal{T}}_1\psi, w\rangle\right| \;\leq\; CL^2 e^{-2\beta_b(L-1)}\sum_{n=0}^{\infty}\,|\psi_n\bar{w}_n| \;\leq CL^2 e^{-2\beta_b(L-1)}||\psi||_{L^2(\partial B_1)}||w||_{L^2(\partial B_1)}. \tag{4.27}$$

The estimate for $\langle\tilde{\mathcal{T}}_2(k_b)\psi, w\rangle$ follows a similar spirit. Indeed, from the recurrence relation (B.3) and the monotonicity property (B.6), $|K_{n-1}(\beta_b)| \leq |K_n(\beta_b)|$,

$$\left|\frac{K_n'(\beta_b)}{K_n(\beta_b)}\right| = \left|\frac{-n/\beta_b\; K_n(\beta_b) - K_{n-1}(\beta_b)}{K_n(\beta_b)}\right| \leq \frac{n}{\beta_b} + 1.$$

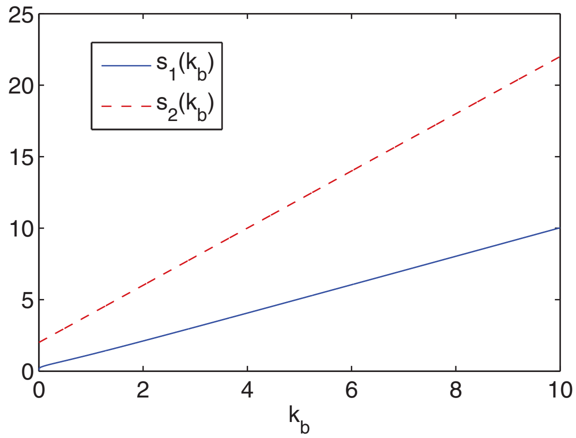

FIG. 3. *Comparison of $s_1(k_b)$ and $s_1(k_b)$.*

For $n \geq 1$, by (4.15) and (4.17) it follows that if $L > L_0$, then

$$|c_{2,n}| \leq \frac{|\alpha_{1,n}(k_b) I_n(\beta_b)|}{|\,|\alpha_{2,n}(k_b) K_n(\beta_b)| - |\alpha_{1,n}(k_b) I_n(\beta_b)|\,|} \left| \frac{K_n'(\beta_b)}{K_n(\beta)} \right|$$

$$\leq \frac{L(\beta_b + k_b)\left(\dfrac{n}{\beta_b} + 1\right) K_n(\beta_b) I_n(\beta_b) e^{\beta_b(1-L)}}{\dfrac{e^{\beta_b(L-1)}}{L}\left(n - \dfrac{1}{2}\right) K_n(\beta_b) I_n(\beta_b)}$$

$$\leq \frac{2}{\beta_b}(1+\beta_b)(\beta_b + k_b) L^2 e^{-2\beta_b(L-1)}. \tag{4.28}$$

The estimate for the case of $n = 0$ can be shown in a similar fashion. The proof is omitted for conciseness of exposition. Consequently,

$$\left|\langle \tilde{\mathcal{T}}_2 \psi, w\rangle\right| \;\leq\; C L^2 e^{-2\beta_b(L-1)} ||\psi||_{L^2(\partial B_1)} ||w||_{L^2(\partial B_1)}. \tag{4.29}$$

The proof is completed by combining (4.27) and (4.29). ∎

Remark 4.8. In fact, in the proof the inequality $k_b|\frac{H_1^{(1)}(k_b \tilde{L}_2)}{H_0^{(1)}(k_b \tilde{L}_2)}| \leq 2(1+k_b)$ holds for any $\tilde{L}_2 \geq 1$. Figure 3 is a plot for the comparison of two functions $s_1(k_b) = k_b|\frac{H_1^{(1)}(k_b)}{H_0^{(1)}(k_b)}|$ and $s_2(k_b) = 2(1+k_b)$. From the plot it is clear that $s_1 < s_2$. Hence, the inequalities hold whenever $\tilde{L}_2 \geq 1$ by the monotonicity of $|\frac{H_1^{(1)}(x)}{H_0^{(1)}(x)}|$ in Lemma 4.6.

An observation of the estimates (4.26) and (4.28) for the coefficients also leads to the following proposition.

PROPOSITION 4.9. *There exist $L_0 \in (1, \infty)$ such that if $L > L_0$, then $\tilde{\mathcal{T}}(k_b)$ is a bounded operator on $H^{1/2}(\partial B_1)$. In addition, the operator norm $||\tilde{\mathcal{T}}(k_b)|| < CL^2 e^{-2\beta_b(L-1)}$ for some positive constant C.*

4.3. Estimate for $\tilde{\mathcal{T}}'(k_b)$.

DEFINITION 4.10. *An operator $\tilde{A}$ from $H^{1/2}(\partial B_1)$ to $H^{-1/2}(\partial B_1)$ is positive if $\langle \tilde{A} w, w\rangle\ > 0$ for any $w \in H^{1/2}(\partial B_1)$.*

THEOREM 4.11. *Let $\tilde{\mathcal{T}}'(k_b)$ be the derivative of the operator-valued function $\tilde{\mathcal{T}}(k)$ at $k = k_b$; then $\tilde{\mathcal{T}}'(k_b)$ can be decomposed as $\tilde{\mathcal{T}}'(k_b) = \tilde{\mathcal{T}}_p'(k_b) + \tilde{\mathcal{T}}_s'(k_b)$, where* *the following hold:*

(a) *The operator* $\tilde{\mathcal{T}}_p'(k_b)$ *is independent of* L*. In addition,* $\tilde{\mathcal{T}}_p'(k_b)$ *is positive and bounded if* $k_b > 0$*.*

(b) $\tilde{\mathcal{T}}_s'(k_b)$ *is bounded such that*

$$\left|\langle \tilde{\mathcal{T}}_s'(k_b)\psi, w\rangle\right| < CL^4 e^{-2\beta_b(L-1)}||\psi||_{H^{1/2}(\partial B_1)}||w||_{H^{1/2}(\partial B_1)}$$

for any $\psi,\ w \in H^{1/2}(\partial B_1)$*, provided that* L *is sufficiently large.*

Proof. Let

$$\tilde{\mathcal{T}}_p'(k_b)\psi = \sum_{n=0}^{\infty}(\beta c_{0,n})'(k_b)\ \psi_n e^{in\theta} \tag{4.30}$$

and

$$\tilde{\mathcal{T}}_s'(k_b)\psi = \sum_{n=0}^{\infty}\left[(\beta c_{1,n})'(k_b) + (\beta c_{2,n})'(k_b)\right]\psi_n e^{in\theta}, \tag{4.31}$$

where $c_{0,n}$, $c_{1,n}$, and $c_{2,n}$ are given by (4.10)–(4.12). Then from a direct calculation, it is apparent that $\tilde{\mathcal{T}}'(k_b) = \tilde{\mathcal{T}}_p'(k_b) + \tilde{\mathcal{T}}_s'(k_b)$ and that $\tilde{\mathcal{T}}'(k_b)$ is independent of L. The statements (a) and (b) follow from Theorems 4.12 and 4.13, respectively, which are given in the following. ∎

THEOREM 4.12. *Let the operator* $\tilde{\mathcal{T}}_p'(k_b)$ *be defined by* (4.30)*; then* $\tilde{\mathcal{T}}_p'(k_b)$ *is positive and bounded if* $k_b > 0$*.*

Proof. Fist, a direct calculation yields

$$\begin{aligned}\langle \tilde{\mathcal{T}}_p'(k_b)\psi, w\rangle &\\ &= \sum_{n=0}^{\infty} 2\pi(\beta c_{0,n})'(k_b)\psi_n w_n \\ &= \sum_{n=0}^{\infty} 2\pi\beta_b'\left(\frac{K_n'(\beta_b)}{K_n(\beta_b)} + \beta_b\frac{K_n''(\beta_b)K_n(\beta_b) - (K_n'(\beta_b))^2}{K_n^2(\beta_b)}\right)\psi_n w_n,\end{aligned} \tag{4.32}$$

where $\beta_b' = -k_b/\beta_b$.

When $n \geq 1$, from the recurrence relations $K_n'(\beta_b) = \frac{n}{\beta_b}K_n(\beta_b) - K_{n+1}(\beta_b)$ and $K_n'(\beta_b) = -\frac{n}{\beta_b}K_n(\beta_b) - K_{n-1}(\beta_b)$, it follows that

$$\begin{aligned}K_n''(\beta_b) &= -\frac{n}{\beta_b^2}K_n(\beta_b) + \frac{n}{\beta_b}K_n'(\beta_b) - K_{n+1}'(\beta_b) \\ &= \left(\frac{n^2}{\beta_b^2} - \frac{n}{\beta_b^2} + 1\right)K_n(\beta_b) + \frac{1}{\beta_b}K_{n+1}(\beta_b).\end{aligned}$$

Therefore, by substituting into (4.32), we obtain that

$$\begin{aligned}(\beta c_{0,n})'(k_b) &= \frac{\beta_b'}{K_n^2(\beta_b)}\left(\beta_b K_n^2(\beta_b) - \beta_b K_{n+1}^2(\beta_b) + 2nK_n(\beta_b)K_{n+1}(\beta_b)\right) \\ &= \beta_b\beta_b'\left(1 - \frac{K_{n-1}(\beta_b)K_{n+1}(\beta_b)}{K_n^2(\beta_b)}\right),\end{aligned}$$

where the last equality follows by the recurrence relation (B.1). Since $\beta_b\beta_b' < 0$, apply the Turán type inequality (B.11), namely, $K_n^2(\beta_b) - K_{n-1}(\beta_b)K_{n+1}(\beta_b) < 0$, and we arrive at

$$(\beta c_{0,n})'(k_b) > 0, \quad n \geq 1. \tag{4.33}$$

The case for $n = 0$ can be proved in a similar way. Indeed, apply the recurrence relation (B.4),

$$(\beta c_{0,0})'(k_b) = \beta_b \beta_b' \left(1 - \frac{K_1^2(\beta_b)}{K_0^2(\beta_b)}\right) > 0. \tag{4.34}$$

As a result, by substituting (4.33) and (4.34) into (4.32), it follows that

$$\langle \tilde{\mathcal{T}}_p'(k_b)\psi, \psi\rangle \ > 0.$$

The boundedness of $\tilde{\mathcal{T}}'(k_b)$ follows again by the Turán inequalities (B.10). More precisely, from the Turán inequalities, it is easily seen that

$$|(\beta c_{0,n})'(k_b)| < 2\beta_b|\beta_b'|, \quad n \geq 1.$$

Hence,

$$|\langle \tilde{\mathcal{T}}_p'(k_b)\psi, w\rangle| < C\,\|\psi\|_{H^{1/2}(\partial B_1)}\,\|w\|_{H^{1/2}(\partial B_1)}$$

for some positive constant C. ∎

THEOREM 4.13. *Let the operator $\tilde{\mathcal{T}}_s'(k_b)$ be defined by (4.31); there exist $L_0 \in (1,\infty)$ such that*

$$\left|\langle \tilde{\mathcal{T}}_s'(k_b)\psi, w\rangle\right| < CL^4 e^{-2\beta_b(L-1)}||\psi||_{H^{1/2}(\partial B_1)}||w||_{H^{1/2}(\partial B_1)} \tag{4.35}$$

for any ψ, $w \in H^{1/2}(\partial B_1)$ if $L > L_0$.

Proof. Let $\mathcal{T}_s'(k_b) = \mathcal{T}_{s,1}'(k_b) + \mathcal{T}_{s,2}'(k_b)$, where

$$\tilde{\mathcal{T}}_{s,i}'(k_b)\psi = \sum_{n=0}^{\infty} (\beta c_{i,n})'(k_b)\psi_n e^{in\theta}, \quad i = 1, 2.$$

We only need to establish (4.35) for $\tilde{\mathcal{T}}_{s,1}'$; the estimate for $\tilde{\mathcal{T}}_{s,2}'$ follows by a parallel argument.

To begin with, note that

$$\langle \tilde{\mathcal{T}}_{s,1}'(k_b)\psi, w\rangle = \sum_{n=0}^{\infty} 2\pi(\beta_b' c_{1,n}(k_b) + \beta_b c_{1,n}'(k_b))\ \psi_n w_n. \tag{4.36}$$

From (4.19) and (4.25) in Theorem 4.7, we see that

$$\sum_{n=0}^{\infty} 2\pi|\beta_b' c_{1,n}\psi_n w_n| < CL^2 e^{-2\beta_b(L-1)}||\psi||_{L^2(\partial B_1)}||w||_{L^2(\partial B_1)}. \tag{4.37}$$

Therefore, we only need to establish the estimate (4.35) for the second part in (4.36).

We rewrite $c_{1,n}'$ as

$$\begin{aligned} c_{1,n}' &= \frac{\big(\alpha_{1,n}(k)I_n'(\beta)\big)'}{\alpha_{1,n}(k)I_n(\beta) + \alpha_{2,n}(k)K_n(\beta)} - \frac{\alpha_{1,n}(k)I_n'(\beta)\big(\alpha_{1,n}(k)I_n(\beta) + \alpha_{2,n}(k)K_n(\beta)\big)'}{(\alpha_{1,n}(k)I_n(\beta) + \alpha_{2,n}(k)K_n(\beta))^2} \\ &=: c_{1,n,1}' + c_{1,n,2}'. \end{aligned}$$

First, let us focus on the case with $n \geq 1$. To estimate $c'_{1,n,1}$, apply the recurrence relations

$$I'_n(\beta_b) = \frac{n}{\beta_b} I_n(\beta_b) + I_{n+1}(\beta_b),$$
$$I''_n(\beta_b) = \left(\frac{n^2}{\beta_b^2} - \frac{n}{\beta_b^2} + 1\right) I_n(\beta_b) - \frac{1}{\beta_b} I_{n+1}(\beta_b).$$

From Lemmas 4.1 and 4.3, it follows that for $L > \max(\beta_b, 1)$,

$$\begin{aligned}\left|\big(\alpha_{1,n}(k) I'_n(\beta_b)\big)'\right| &= \left|\big(\alpha_{1,n}(k)\big)' I'_n(\beta_b) + \beta'_b \alpha_{1,n}(k) I''_n(\beta_b)\right| \\ &\leq C(n+L)^2 L K_n(\beta_b L) I_n(\beta_b) \\ &\leq C(n+1)^2 L^3 K_n(\beta_b L) I_n(\beta_b).\end{aligned}$$

In light of Lemma 4.5, we arrive at

$$\left|\big(\alpha_{1,n}(k) I'_n(\beta_b)\big)'\right| < C(n+1)^2 L^3 e^{\beta_b(1-L)} K_n(\beta_b) I_n(\beta_b).$$

On the other hand, from (4.17) in Theorem 4.7,

$$|\alpha_{2,n}(k_b) K_n(\beta_b)| - |\alpha_{1,n}(k_b) I_n(\beta_b)| > \frac{e^{\beta_b(L-1)}}{L} \left(n - \frac{1}{2}\right) K_n(\beta_b) I_n(\beta_b),$$

provided that $L > L_1$, where L_1 is given by (4.16) . Therefore,

$$|c'_{1,n,1}| < C(n+1) L^4 e^{-2\beta_b(L-1)}. \tag{4.38}$$

To estimate $c'_{1,n,2}$, by Lemmas 4.1–4.4 and the recurrence relations, it is seen that, if L is sufficiently large,

$$\begin{aligned}&\left|\frac{\big(\alpha_{1,n}(k) I_n(\beta) + \alpha_{2,n}(k) K_n(\beta)\big)'}{\alpha_{1,n}(k) I_n(\beta) + \alpha_{2,n}(k) K_n(\beta)}\right| \\ &\quad< \frac{C\left|(n+1)L^2 K_n(\beta_b L) I_n(\beta) + (n+1)^2 L^2 I_n(\beta_b L) K_n(\beta)\right|}{|n I_n(\beta_b L) K_n(\beta_b) - L(\beta_b + k_b) K_n(\beta_b L) I_n(\beta_b)|} \\ &\quad= \frac{C\left|(n+1)L^2 \dfrac{K_n(\beta_b L) I_n(\beta)}{K_n(\beta) I_n(\beta_b L)} + (n+1)^2 L^2\right|}{\left|n - L(\beta_b + k_b)\dfrac{K_n(\beta_b L) I_n(\beta_b)}{K_n(\beta) I_n(\beta_b L)}\right|}.\end{aligned}$$

Now applying Lemma 4.5 again,

$$L(\beta_b + k_b) \frac{K_n(\beta_b L) I_n(\beta_b)}{K_n(\beta) I_n(\beta_b L)} < (\beta_b + k_b) L^2 e^{-2\beta_b(L-1)} < \frac{1}{2}$$

if L is chosen appropriately. Thus,

$$\left|\frac{\big(\alpha_{1,n}(k) I_n(\beta) + \alpha_{2,n}(k) K_n(\beta)\big)'}{\alpha_{1,n}(k) I_n(\beta) + \alpha_{2,n}(k) K_n(\beta)}\right| < C(n+1) L^2.$$

Combining the above inequality with the estimate (4.19) for $|c_{1,n}|$ in Theorem 4.7, we obtain that

$$|c'_{1,n,2}| \leq |c_{1,n}| \left| \frac{\big(\alpha_{1,n}(k) I_n(\beta) + \alpha_{2,n}(k) K_n(\beta)\big)'}{\alpha_{1,n}(k) I_n(\beta) + \alpha_{2,n}(k) K_n(\beta)} \right| < C(n+1) L^4 e^{-2\beta_b(L-1)}. \tag{4.39}$$

The case for $n = 0$ can be shown in a similar fashion, which is given briefly as follows. An application of the recurrence relation, inequalities (B.7), and Lemmas 4.1–4.5 gives

$$\left| \big(\alpha_{1,0}(k) I'_0(\beta_b)\big)' \right| < CLe^{-\beta_b(L-1)} M(k_b, L) K_0(\beta_b) I_1(\beta_b)),$$

where

$$M(k_b, L) = 1 + L + (1 + k_b) L \left| \frac{H_1^{(1)}(k_b L)}{H_0^{(1)}(k_b L)} \right| + k_b L \left| \frac{H_1^{(1)}(k_b L)}{H_0^{(1)}(k_b L)} \right|^2 .$$

Let $L > L_2$, where L_2 is given by (4.22) in Theorem 4.7 such that $k_b |\frac{H_1^{(1)}(k_b \tilde{L}_2)}{H_0^{(1)}(k_b \tilde{L}_2)}| \leq 2(1 + k_b)$, and then

$$\left| \big(\alpha_{0,n}(k) I'_n(\beta_b)\big)' \right| < CL^2 e^{\beta_b(1-L)} K_0(\beta_b) I_1(\beta_b).$$

On the other hand, if $L > L_2$, we observe that

$$|\alpha_{2,0}(k_b) K_0(\beta_b)| - |\alpha_{1,0}(k_b) I_0(\beta_b)| > e^{\beta_b(L-1)} \frac{\beta_b}{2} K_0(\beta_b) I_1(\beta_b)$$

by (4.23). Therefore,

$$|c'_{1,0,1}| < C(n+1) L^2 e^{-2\beta_b(L-1)}. \tag{4.40}$$

In addition, by a similar calculation it can be shown that

$$\left| \frac{\big(\alpha_{1,0}(k) I_0(\beta) + \alpha_{2,0}(k) K_0(\beta)\big)'}{\alpha_{1,0}(k) I_0(\beta) + \alpha_{2,0}(k) K_0(\beta)} \right| < CL^2,$$

provided that $L > L_2$. As a result, combine this inequality with the estimate (4.25) in Theorem 4.7,

$$|c'_{1,0,2}| < CL^3 e^{-2\beta_b(L-1)}. \tag{4.41}$$

Finally, by virtue of (4.38) and (4.41), we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} 2\pi\beta_b |c'_{1,n}(k_b)\ \psi_n w_n| \\ &\quad \leq \sum_{n=0}^{\infty} 2\pi\beta_b C L^4 e^{-2\beta_b(L-1)} (1+n)\ |\psi_n|\, |w_n| \\ &\quad \leq 4\pi\beta_b C L^4 e^{-2\beta_b(L-1)} \left(\sum_{n=0}^{\infty} \sqrt{1+n^2}\ |\psi_n|^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} \sqrt{1+n^2}\ |w_n|^2 \right)^{1/2} \\ &\quad = 4\pi\beta_b C L^4 e^{-2\beta_b(L-1)} \|\varphi\|_{H^{1/2}(\partial B_1)} \|\psi\|_{H^{1/2}(\partial B_1)} . \end{aligned} \tag{4.42}$$

The proof is complete by combining (4.36), (4.37), and (4.42). $\square$

5. Proof of Theorem 3.3. We rewrite the resonance condition (3.11) as

$$F(k) = (k^2 - k_b^2)(\psi, \psi) - (k^2 - k_b^2)(\psi, \hat{\xi}) + \langle \tilde{\mathcal{T}}(k)\psi, \psi\rangle - \langle \tilde{\mathcal{T}}(k)\psi, \hat{\xi}\rangle = 0. \tag{5.1}$$

Let $\kappa = k - k_b$. An expansion at the bound state frequency k_b gives

$$\langle \tilde{\mathcal{T}}(k)\psi, \psi\rangle = \langle \tilde{\mathcal{T}}(k_b)\psi, \psi\rangle + \kappa\langle \tilde{\mathcal{T}}'(k_b)\psi, \psi\rangle + \langle \tilde{\mathcal{T}}_h(k)\psi, \psi\rangle \tag{5.2}$$

and

$$\langle \tilde{\mathcal{T}}\psi, \hat{\xi}\rangle = \langle \tilde{\mathcal{T}}(k_b)\psi, \hat{\xi}\rangle + \kappa\langle \tilde{\mathcal{T}}'(k_b)\psi, \hat{\xi}\rangle + \langle \tilde{\mathcal{T}}_h(k)\psi, \hat{\xi}\rangle, \tag{5.3}$$

where $\tilde{\mathcal{T}}_h(k)$ represents the higher order term.

Let $\rho(k)$ and $q(k)$ be the functions as defined in section 3.1. Similarly, we may expand $\rho(k)$ and $q(k)$ as

$$\rho(k) = \rho_0 + \kappa\rho_1 + \rho_h(k),$$
$$q(k) = q_0 + \kappa q_1 + q_h(k).$$

It follows from a direction calculation that

$$\rho_0 = \mathcal{L}_1(\tilde{\mathcal{T}}(k_b)\psi), \quad \rho_1 = \mathcal{L}_1(\tilde{\mathcal{T}}'(k_b)\psi), \quad \rho_h = \mathcal{L}_1(\tilde{\mathcal{T}}_h(k_b)\psi),$$
$$q_0 = \mathcal{L}_2(\tilde{\mathcal{T}}(k_b)\psi), \quad q_1 = \mathcal{L}_2(\tilde{\mathcal{T}}'(k_b)\psi), \quad q_h = \mathcal{L}_2(\tilde{\mathcal{T}}_h(k_b)\psi).$$

Here, the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are bounded from $H^{1/2}(\partial B_1)$ to $H^2(B_1)$ and from $H^{1/2}(\partial B_1)$ to $L^2(B_1)$, respectively (cf. section 3.1). From (3.9)–(3.10), we see that $\hat{\xi}$ can be written as

$$\hat{\xi}(k) = \hat{\xi}_0 + \kappa\hat{\xi}_1 + \hat{\xi}_h(k), \tag{5.4}$$

wherein $\hat{\xi}_0 = \mathcal{L}_b q_0 + \rho_0$, $\hat{\xi}_1 = \mathcal{L}_b[2k_b\psi + q_1] + \rho_1$, and $\hat{\xi}_h = \mathcal{L}_b[\kappa^2\psi + q_h(k)] + \rho_h$. A standard energy estimate yields

$$\begin{aligned} ||\hat{\xi}_0||_{H^1(B_1)} &\le C\left(||q_0||_{L^2(B_1)} + ||\rho_0||_{H^1(B_1)}\right) \le C||\tilde{\mathcal{T}}(k_b)\psi||_{H^{1/2}(\partial B_1)}, \\ ||\hat{\xi}_1||_{H^1(B_1)} &\le C\Big(||\psi||_{L^2(B_1)} + ||q_1||_{L^2(B_1)} + ||\rho_1||_{H^1(B_1)}) \\ &\le C(||\psi||_{L^2(B_1)} + ||\tilde{\mathcal{T}}'(k_b)\psi||_{H^{1/2}(\partial B_1)}\Big). \end{aligned} \tag{5.5}$$

Substituting the expansions (5.2)–(5.4) into (5.1), it follows that

$$F(k_b) + F'(k_b)\kappa + F_h(k) = 0. \tag{5.6}$$

Here,

$$\begin{aligned} F(k_b) &= \langle \tilde{\mathcal{T}}(k_b)\psi, \psi\rangle - \langle \tilde{\mathcal{T}}(k_b)\psi, \hat{\xi}_0\rangle, \\ F'(k_b) &= 2k_b||\psi||^2_{L^2(B_1)} - 2k_b(\psi, \hat{\xi}_0) + \langle \tilde{\mathcal{T}}'(k_b)\psi, \psi\rangle - \langle \tilde{\mathcal{T}}'(k_b)\psi, \hat{\xi}_0\rangle - \langle \tilde{\mathcal{T}}(k_b)\psi, \hat{\xi}_1\rangle, \\ F_h(k) &= \kappa^2||\psi||^2_{L^2(B_1)} - (2k_b\kappa^2\langle\psi, \hat{\xi}_1\rangle + 2k_b\kappa\langle\psi, \hat{\xi}_h\rangle) - \kappa^2(\psi, \hat{\xi}) + \langle \tilde{\mathcal{T}}_h(k)\psi, \psi\rangle \\ &\quad - \Big(\langle \tilde{\mathcal{T}}(k_b)\psi, \hat{\xi}_h\rangle + \kappa^2\langle \tilde{\mathcal{T}}'(k_b)\psi, \hat{\xi}_1\rangle + \kappa\langle \tilde{\mathcal{T}}'(k_b)\psi, \hat{\xi}_h\rangle + \langle \tilde{\mathcal{T}}_h(k)\psi, \hat{\xi}\rangle\Big). \end{aligned}$$

By employing Theorem 4.7, there exist L_0 such that when $L > L_0$,

$$\begin{aligned} |F(k_b)| &\le |\langle \tilde{\mathcal{T}}(k_b)\psi, \psi\rangle| + |\langle \tilde{\mathcal{T}}(k_b)\psi, \hat{\xi}_0\rangle| \\ &< CL^2 e^{-2\beta_b(L-1)}\left(||\psi||^2_{H^{1/2}(\partial B_1)} + ||\psi||_{H^{1/2}(\partial B_1)}||\hat{\xi}_0||_{H^{1/2}(\partial B_1)}\right). \end{aligned}$$

Applying Proposition 4.9 to the regularity estimate (5.5) and substituting into the above inequality, we observe that

$$\begin{aligned} |F(k_b)| &< CL^2 e^{-2\beta_b(L-1)} \left(||\psi||^2_{H^{1/2}(\partial B_1)} + L^2 e^{-2\beta_b(L-1)} ||\psi||^2_{H^{1/2}(\partial B_1)} \right) \\ &< 2CL^2 e^{-2\beta_b(L-1)} ||\psi||^2_{H^{1/2}(\partial B_1)} \end{aligned} \tag{5.7}$$

if $L^2 e^{-2\beta_b(L-1)} \le 1$.

From Theorem 4.11, $\tilde{\mathcal{T}}'(k_b)$ can be decomposed as $\tilde{\mathcal{T}}'(k_b) = \tilde{\mathcal{T}}_p'(k_b) + \tilde{\mathcal{T}}_s'(k_b)$, where $\tilde{\mathcal{T}}_p'(k_b)$ is positive. Therefore,

$$\begin{aligned} F'(k_b) &> 2k_b ||\psi||^2_{L^2(B_1)} \\ &\quad - \left(2k_b |(\psi, \hat{\xi}_0)| + |\langle \tilde{\mathcal{T}}_s'(k_b)\psi, \psi\rangle| + |\langle \tilde{\mathcal{T}}'(k_b)\psi, \hat{\xi}_0\rangle| + |\langle \tilde{\mathcal{T}}(k_b)\psi, \hat{\xi}_1\rangle| \right). \end{aligned}$$

Since for sufficiently large L,

$$\begin{aligned} |(\psi, \hat{\xi}_0)| &< ||\psi||_{L^2(B_1)} ||\hat{\xi}_0||_{L^2(B_1)} < CL^2 e^{-2\beta_b(L-1)} ||\psi||^2_{H^1(B_1)}, \\ |\langle \tilde{\mathcal{T}_s}'(k_b)\psi, \psi\rangle| &< CL^4 e^{-2\beta_b(L-1)} ||\psi||^2_{H^1(B_1)}, \\ |\langle \tilde{\mathcal{T}}'(k_b)\psi, \hat{\xi}_0\rangle| &< C||\psi||_{H^{1/2}(\partial B_1)} ||\hat{\xi}_0||_{H^{1/2}(\partial B_1)} < CL^2 e^{-2\beta_b(L-1)} ||\psi||^2_{H^1(B_1)}, \\ |\langle \tilde{\mathcal{T}}(k_b)\psi, \hat{\xi}_1\rangle| &< CL^2 e^{-2\beta_b(L-1)} ||\psi||_{H^{1/2}(\partial B_1)} ||\hat{\xi}_1||_{H^{1/2}(\partial B_1)} \\ &< CL^2 e^{-2\beta_b(L-1)} ||\psi||^2_{H^2(B_1)}, \end{aligned}$$

we deduce that there exits L_1 such that when $L > L_1$,

$$F'(k_b) > k_b ||\psi||^2_{L^2(B_1)} > 0. \tag{5.8}$$

On the other hand, $F(k)$ is analytic in the neighborhood of k_b since $\mathcal{T}(k)$ is analytic (Lemma 2.1). By Taylor's theorem [2], there exists an analytic function $\tilde{F}(k)$ such that

$$F(k) = F(k_b) + \tilde{F}(k)(k - k_b) \tag{5.9}$$

and

$$\tilde{F}(k_b) = F'(k_b).$$

In addition, from (5.8) it follows that

$$|\tilde{F}(k)| \ge \frac{k_b}{2} ||\psi||^2_{L^2(B_1)} \text{ in the neighborhood of } k_b.$$

Therefore, by combining (5.7) and (5.9), the following estimate holds:

$$|k - k_b| < \frac{4C}{k_b} \frac{||\psi||^2_{H^{1/2}(\partial B_1)}}{||\psi||^2_{L^2(B_1)}} L^2 e^{-2\beta_b(L-1)}. \qquad \square$$

Remark 5.1. The existence of solution for the nonlinear equation $F(k) = 0$ on the complex plane is not given here. In fact, rewriting (5.6) as $\kappa = -(F(k_b) + F_h(k))/F'(k_b) =: \sigma(\kappa)$, the existence of the solution may be argued by showing that $\kappa \to \sigma(\kappa)$ is a contraction mapping, and the iteration $\kappa_{n+1} = \sigma(\kappa_n)$ converges to the solution. Equivalently, one proves that $|F_h(k_1) - F_h(k_2)|/F'(k_b) < c|k_1 - k_2|$ for some constant $c < 1$. To accomplish this, we need to employ inequalities similar to Lemma 4.5 for the modified Bessel's function in the case of a complex variable. More precisely,

$$\frac{|I_n(z_1)|}{|I_n(z_2)|} < \left|e^{z_1-z_2}\right| \left|\frac{z_2}{z_1}\right| \quad \text{and} \quad \frac{|K_n(z_1)|}{|K_n(z_2)|} > \left|e^{z_2-z_1}\right| \quad \text{if} \quad |z_1| < |z_2|, \tag{5.10}$$

where $z_1, z_2 \in \mathbb{C}$ such that Arg z_1 = Arg $z_2 = \theta < \pi/2$. We conjecture that such inequalities hold. However, its proof remains completely open.

6. A perturbation approach to calculate near-bound state resonances. We propose a simple perturbation approach to calculate the near-bound state resonances based on Theorem 3.3. In fact, since $|k-k_b| \sim O(L^2 e^{-2\beta_b(L-1)})$ for sufficiently thick L, if one knows the bound state frequency k_b and corresponding eigenfunction ψ_b, the resonance k may be calculated by a linear approximation of the resonance condition (3.3). More precisely, let $\kappa = k - k_b$; by noting that $||\hat{\xi}_0||_{H^1(B_1)} \sim O(L^2 e^{-2\beta_b(L-1)})$, (3.3) can be reduced to

$$2k_b\kappa(\psi_b, \psi_b) + \langle \tilde{T}(k_b)\psi_b, \psi_b\rangle + \kappa\langle \tilde{T}'(k_b)\psi_b, \psi_b\rangle = 0$$

if the high order terms are neglected. In this way, κ is computed via the following approximation formula:

$$\kappa_{approx} = \frac{-\langle \tilde{T}(k_b)\psi_b, \psi_b\rangle}{2k_b(\psi_b, \psi_b) + \langle \tilde{T}'(k_b)\psi_b, \psi_b\rangle}. \tag{6.1}$$

To test the accuracy of such an approximation, let us consider the potential

$$V_L(x) = \begin{cases} 0, & |x| < 1, \\ 10, & 1 < |x| < L, \\ 0, & |x| > L. \end{cases}$$

The quasi mode takes the form

$$\psi(x) = \begin{cases} J_n(k|x|)e^{in\theta}, & |x| < 1, \\ (a_{1,n}I_n(\beta|x|) + a_{2,n}K_n(\beta|x|))\, e^{in\theta}, & 1 < |x| < L, \\ c_n H_n^{(1)}(k|x|)e^{in\theta}, & |x| > L. \end{cases}$$

In Table 2 we compare the resonance obtained by solving directly the equations arising from the continuity conditions of ψ at $|x| = 1$ and $|x| = L$ by Newton's method (denoted as κ_N) and by the approximation formula (6.1). The modes for $n = 0$ and

Table 2

Comparison of κ calculated by Newton's method (second column) and the perturbation approach (third column) for different L.

$n = 0$		
L	κ_N	κ_{approx}
2	$-1.33312 \times 10^{-3} - 3.61219 \times 10^{-3}i$	$-1.32972 \times 10^{-3} - 3.60054 \times 10^{-3}i$
4	$-4.16857 \times 10^{-8} - 1.11464 \times 10^{-7}i$	$-4.16879 \times 10^{-8} - 1.11470 \times 10^{-7}i$
6	$-1.27831 \times 10^{-12} - 3.39833 \times 10^{-12}i$	$-1.27800 \times 10^{-12} - 3.39846 \times 10^{-12}i$

$n = 1$		
L	κ_N	κ_{approx}
2	$1.05615 \times 10^{-2} - 2.09036 \times 10^{-2}i$	$1.05988 \times 10^{-2} - 1.97834 \times 10^{-2}i$
4	$2.43365 \times 10^{-5} - 3.87015 \times 10^{-5}i$	$2.43426 \times 10^{-5} - 3.86849 \times 10^{-5}i$
6	$5.55640 \times 10^{-8} - 8.70365 \times 10^{-8}i$	$5.55656 \times 10^{-8} - 8.70386 \times 10^{-8}i$

$n = 1$ are shown. It is seen that the resonance is approximated accurately by the proposed formula.

Appendix A. Bessel's functions. For completeness, we collect some basic facts about the Bessel functions. Readers are referred to [1, 4, 10, 12, 16] for a comprehensive exposition. m, n below denote some nonnegative integers.

Two linear independent solutions of Bessel's equation

$$z^2\frac{d^2W}{dz^2} + z\frac{dW}{dz} + (z^2 - n^2)W = 0$$

are called the Bessel functions of the first kind $J_n(z)$ and second kind $Y_n(z)$ $(n \geq 0)$. They can be expressed by power series as follows:

$$J_n(z) = \left(\frac{z}{2}\right)^n \sum_{m=0}^{\infty} \frac{\left(-z^2/4\right)^m}{m!\Gamma(m+k+1)}$$

and

$$Y_n(z) = -\frac{1}{\pi}\left(\frac{z}{2}\right)^{-n} \sum_{m=0}^{n-1} \frac{(n-m-1)}{m!}\left(\frac{z^2}{4}\right)^m + \frac{2}{\pi}\ln\left(\frac{z}{2}\right) J_n(z)$$
$$- \frac{1}{\pi}\left(\frac{z}{2}\right)^n \sum_{m=0}^{\infty} (\psi(m+1) + \psi(n+m+1)) \frac{(-z^2/4)}{m!(n+m)!}.$$

Both $J_n(z)$ and $Y_n(z)$ are analytic in $\mathbb{C}\backslash(-\infty, 0]$. The Hankel functions of the first kind $H_n^1(z)$ and second kind $H_n^2(z)$ are defined by

$$H_n^{(1)}(z) = J_n(x) + iY_n(x),$$
$$H_n^{(2)}(z) = J_n(x) - iY_n(x).$$

Next, we collect some properties of the Bessel's functions that are used in the paper.

Proposition A.1 (recurrence relations: [1, sections 9.1.27, 9.1.28]). *Let $\mathcal{F}_n$ denote J_n, Y_n, $H_n^{(1)}$, and $H_n^{(2)}$, and then the following recurrence relations hold for the derivate of the Bessel functions:*

$$\text{(A.1)} \quad \mathcal{F}_n'(z) = \mathcal{F}_{n-1}(z) - \frac{n}{z}\mathcal{F}_n(z), \quad \mathcal{F}_n'(z) = \frac{n}{z}\mathcal{F}_n(z) - \mathcal{F}_{n+1}(z), \quad n \geq 1,$$
$$\text{(A.2)} \quad \mathcal{F}_0'(z) = -\mathcal{F}_1(z).$$

Proposition A.2 (Wronskian: [1, section 9.1.16]).

$$\text{(A.3)} \qquad J_n(z)Y_n'(z) - Y_n(z)J_n'(z) = \frac{2}{\pi z} \quad \textit{for } n \geq 0.$$

Proposition A.3 (asymptotic expansions for large order: [1, section 9.3.1]). *If $n \to +\infty$ with nonzero z fixed, then*

$$\text{(A.4)} \qquad -iH_n^{(1)}(z) \sim -iH_n^{(2)}(z) \sim \sqrt{\frac{2}{n\pi}}\left(\frac{ez}{2n}\right)^{-n}.$$

Proposition A.4 (asymptotic expansions for large argument: [1, section 9.2.3]). *If $|z| \to \infty$ with n fixed, then*

$$\text{(A.5)} \qquad H_n^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}e^{i(z-n\pi/2-pi/4)} \quad \textit{and} \quad H_n^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}e^{-i(z-n\pi/2-pi/4)}.$$

PROPOSITION A.5 (Nicholson's integral: [16, section 13.73]). *If $Rez > 0$, then*

$$|H_n^{(1)}(z)|^2 = J_n^2(z) + Y_n^2(z) = \frac{8}{\pi^2}\int_0^\infty K_0(2z\sinh t)\cosh 2nt\,dt \quad \text{for } n \geq 0. \tag{A.6}$$

Here, $K_0(z)$ is the modified Bessel's function of zero order defined in Appendix B.

A direct consequence of the Nicholson's integral is the monotonicity of modulus for the Hankel's function.

PROPOSITION A.6 (monotonicity). *If $0 \leq m < n$, then*

$$|H_m^{(i)}(x)| < |H_n^{(i)}(x)| \quad \text{for } x \in (0,\infty), \quad i = 1, 2. \tag{A.7}$$

PROPOSITION A.7 (modulus of Hankel functions). *If $x \in (0,\infty)$, then the modulus of the Hankel function $|H_n^{(1)}(x)|$ is a decreasing function with respect to x.*

Proof. From Nicholson's integral formula, we observe that

$$\begin{aligned}\frac{d}{dx}|H_n^{(1)}(x)|^2 &= \frac{8}{\pi^2}\int_0^\infty K_0'(2x\sinh t)2\sinh t\cosh 2nt\,dt\\ &= -\frac{8}{\pi^2}\int_0^\infty K_1(2x\sinh t)2\sinh t\cosh 2nt\,dt < 0,\end{aligned}$$

since K_1 is positive. Hence, the modulus $|H_n^{(1)}(x)|$ is a decreasing function. □

Appendix B. Modified Bessel's functions. The modified Bessel's functions $I_n(z)$ and $K_n(z)$ ($n \geq 0$) are two linearly independent solutions of the modified Bessel's equation:

$$z^2\frac{d^2W}{dz^2} + z\frac{dW}{dz} - (z^2+n^2)W = 0.$$

They are related by regular Bessel functions as follows:

$$\begin{aligned}I_n(z) &= e^{-in\pi/2}J_n(ze^{i\pi/2}) \quad (-\pi < Argz < \pi/2),\\ I_n(z) &= e^{i3n\pi/2}J_n(ze^{-i3\pi/2}) \quad (\pi/2 < Argz \leq \pi),\\ K_n(z) &= \frac{i\pi}{2}e^{in\pi/2}H_n^{(1)}(ze^{i\pi/2}) \quad (-\pi < Argz < \pi/2),\\ K_n(z) &= -\frac{i\pi}{2}e^{-in\pi/2}H_n^{(2)}(ze^{-i\pi/2}) \quad (\pi/2 < Argz \leq \pi).\end{aligned}$$

$I_n(z)$ and $K_n(z)$ are analytic in $\mathbb{C}\backslash(-\infty,0]$. If z is real and $z > 0$, $I_n(z)$ and $K_n(z)$ are real and positive. (See Figure 4 for the plots of $K_0(z)$ and $I_0(z)$ on the real line.)

PROPOSITION B.1 (recurrence relations: [1, sections 9.6.26, 9.6.27]). *The following recurrence relations hold for the modified Bessel's functions:*

$$I_{n-1}(z) - I_{n+1}(z) = \frac{2n}{z}I_n(z), \quad K_{n+1}(z) - K_{n-1}(z) = \frac{2n}{z}K_n(z) \quad n \geq 0. \tag{B.1}$$

In addition, for the derivative of Bessel's functions,

$$I_n'(z) = I_{n-1}(z) - \frac{n}{z}I_n(z), \quad I_n'(z) = \frac{n}{z}I_n(z) + I_{n+1}(z), \quad n \geq 1, \tag{B.2}$$

$$K_n'(z) = -K_{n-1}(z) - \frac{n}{z}K_n(z), \quad K_n'(z) = \frac{n}{z}K_n(z) - K_{n+1}(z), \quad n \geq 1, \tag{B.3}$$

$$I_0'(z) = I_1(z), \quad K_0'(z) = -K_1(z). \tag{B.4}$$

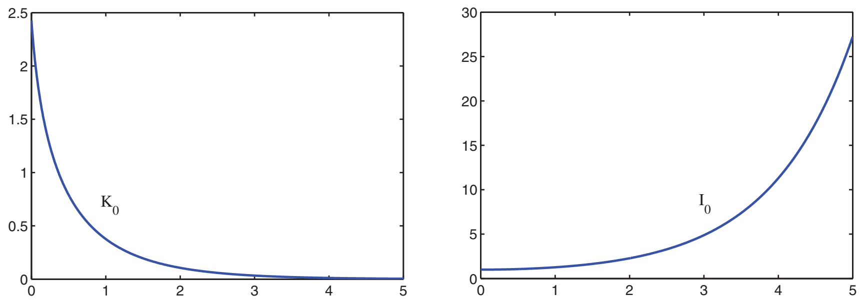

FIG. 4. *Graphs of K_0 and I_0 on the real line.*

PROPOSITION B.2 (Wronskian: [1, section 9.6.15]).

$$I_n(z)K_n'(z) - K_n(z)I_n'(z) = -\frac{1}{z} \quad \textit{for } n \geq 0. \tag{B.5}$$

PROPOSITION B.3 (monotonicity: [12, Theorems 1.1 and 1.2]). *If $x \in (0, \infty)$, then*

$$0 < K_m(x) < K_n(x) \quad \textit{and} \quad I_m(x) > I_n(x) > 0 \quad \textit{if} \quad 0 \leq m < n. \tag{B.6}$$

PROPOSITION B.4 (reverse inequality: [12, Theorems 1.1 and 1.2]). *If $x \in (0, \infty)$, then*

$$\frac{K_n(x)}{K_{n-1}(x)} < \frac{n + \sqrt{x^2+n^2}}{x}, \quad \frac{I_n(x)}{I_{n-1}(x)} > \frac{-n + \sqrt{x^2+n^2}}{x} \quad \textit{if} \quad n \geq 0. \tag{B.7}$$

PROPOSITION B.5 (asymptotic expansions for large order: [1, sections 9.7.7, 9.7.8]). *If $n \to +\infty$ with nonzero z fixed, then*

$$I_n(z) \sim \frac{1}{\sqrt{2\pi n}}\left(\frac{ez}{2n}\right)^n, \quad K_n(z) \sim \sqrt{\frac{\pi}{2n}}\left(\frac{ez}{2n}\right)^{-n}. \tag{B.8}$$

PROPOSITION B.6 (Turán type inequalities: [4, Theorems 2.1 and 3.1]; [10, equation (1.7)]). *If $x \in (0, \infty)$, then*

$$0 < I_n^2(x) - I_{n-1}(x)I_{n+1}(x) < \frac{I_n^2(x)}{n+1}, \quad n \geq 0, \tag{B.9}$$

$$\frac{n}{1-n}K_n^2(x) < K_n^2(x) - K_{n-1}(x)K_{n+1}(x), \quad n > 1, \tag{B.10}$$

$$K_n^2(x) - K_{n-1}(x)K_{n+1}(x) < 0, \quad n \geq 0. \tag{B.11}$$

Appendix C. Calculation of bound state frequencies and resonances by Newton's method. For the step potential (3.14) considered in section 3, the bound state and quasi mode may be expressed explicitly as (3.15) and (3.16), respectively. To calculate bound state frequencies, we impose continuities conditions for ψ_b and $\frac{\partial \psi_b}{\partial n}$ over the circle $|x| = 1$ and obtain a nonlinear equation

$$G_b(k_b) := k_b J_n'(k_b)K_n(\beta_b) - \beta_b J_n(k_b)K_n'(\beta_b) = 0.$$

The iterative formula for Newton's method to approximate the solution of the above equation is given by

$$k_b^{(m+1)} = k_b^{(m)} - \frac{G_b'(k_b)}{G_b(k_b)}, \qquad m = 0, 1, 2, \ldots .$$

The resonances can be obtained in a similar fashion. More precisely, the continuity of the solution over $|x| = 1$ and $|x| = L$ with the explicit expression (3.16) leads to a nonlinear equation

$$G(k) := kJ_n'(k)\left[\alpha_{1,n}(k)I_n(\beta) + \alpha_{2,n}(k)K_n(\beta)\right] - \beta\left[\alpha_{1,n}(k)I_n'(\beta) + \alpha_{2,n}(k)K_n'(\beta)\right] = 0,$$

wherein $\alpha_{1,n}$ and $\alpha_{2,n}$ are given in (2.7) and (2.8). The Newton iteration formula is employed to evaluate the resonances.

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