

AN EXTENSION OF YAMAMOTO’S THEOREM ON THE EIGENVALUES AND SINGULAR VALUES OF A MATRIX

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ABSTRACT. We extend, in the context of real semisimple Lie group, a result of T. Yamamoto which asserts that $\lim_{m \rightarrow \infty} [s_i(X^m)]^{1/m} = |\lambda_i(X)|$, $i = 1, \dots, n$, where $s_1(X) \geq \dots \geq s_n(X)$ are the singular values, and $\lambda_1(X), \dots, \lambda_n(X)$ are the eigenvalues of the $n \times n$ matrix X , in which $|\lambda_1(X)| \geq \dots \geq |\lambda_n(X)|$.

1. INTRODUCTION

Let $X \in \mathbb{C}_{n \times n}$. It is well known [9, p.70] that

$$(1.1) \quad \lim_{m \rightarrow \infty} \|X^m\|^{1/m} = r(X),$$

where $r(X)$ denotes the spectral radius of X and $\|X\|$ denotes the spectral norm of X . Indeed the sequence $\{\|X^m\|^{1/m}\}$ is monotonically decreasing and converges to $r(X)$ [10, p.174].

Suppose that the singular values $s_1(X), \dots, s_n(X)$ of X and the eigenvalues $\lambda_1(X), \dots, \lambda_n(X)$ of X are arranged in nonincreasing order

$$(1.2) \quad s_1(X) \geq s_2(X) \geq \dots \geq s_n(X), \quad |\lambda_1(X)| \geq |\lambda_2(X)| \geq \dots \geq |\lambda_n(X)|.$$

Since $\|X\| = s_1(X)$ and $r(X) := |\lambda_1(X)|$, the following result of Yamamoto [10] is a direct generalization of (1.1):

$$(1.3) \quad \lim_{m \rightarrow \infty} [s_i(X^m)]^{1/m} = |\lambda_i(X)|, \quad i = 1, \dots, n.$$

Loesener [6] rediscovered (1.3). We remark that (1.1) remains true for Hilbert space operators [1, p.45]. Also see [3, 4, 8] for some generalizations of Yamamoto’s theorem.

Equation (1.3) relates the two important sets of scalars of X in (1.2) in a very nice asymptotic way. It may be interpreted as a relation between the singular value decomposition and the complete multiplicative Jordan decomposition of a nonsingular matrix. Since $\text{GL}_n(\mathbb{C})$ is dense in $\mathbb{C}_{n \times n}$ and the eigenvalues of a matrix are continuous functions of its entries [9, p.44], it is sufficient to consider $X \in \text{GL}_n(\mathbb{C})$ or $\text{SL}_n(\mathbb{C})$ when we study (1.3). Let $A_+ \subset \text{GL}_n(\mathbb{C})$ denote the set of all positive diagonal matrices with diagonal entries in nonincreasing order. Recall that the singular value decomposition of $X \in \text{GL}_n(\mathbb{C})$ asserts [9, p.129] that there exist unitary matrices U, V such that

$$(1.4) \quad X = U a_+(X) V,$$

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where $a_+(X) = \text{diag}(s_1(X), \dots, s_n(X)) \in A_+$. Though U and V in the decomposition (1.4) are not unique, $a_+(X) \in A_+$ is uniquely defined. The complete multiplicative Jordan decomposition [2, p.430-431] of $X \in \text{GL}_n(\mathbb{C})$ asserts that $X = ehv$ where e is diagonalizable with eigenvalues of modulus 1, h is diagonalizable over \mathbb{R} with positive eigenvalues and $v = \exp \ell$ where ℓ is nilpotent [2]. The eigenvalues of h are the moduli of the eigenvalues of X , counting multiplicities. The elements e, h, v commute with each others and are uniquely defined. Moreover h is conjugate to a unique element in A_+ , namely, $b(X) = \text{diag}(|\lambda_1(X)|, \dots, |\lambda_n(X)|) \in A_+$. Thus (1.3) may be rewritten as

$$(1.5) \quad \lim_{m \rightarrow \infty} a_+(X^m)^{1/m} = b(X), \quad X \in \text{GL}_n(\mathbb{C}).$$

Our main result is to establish (1.5) in the context of real semisimple Lie groups.

2. EXTENSION OF YAMAMOTO'S THEOREM

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a fixed Cartan decomposition of a real semisimple Lie algebra \mathfrak{g} and let G be any connected Lie group having \mathfrak{g} as its Lie algebra. Let $K \subset G$ be the subgroup with Lie algebra \mathfrak{k} . Then K is connected and closed and that $\text{Ad}_G(K)$ is compact [2, p.252-253]. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace. Fix a *closed* Weyl chamber \mathfrak{a}_+ in \mathfrak{a} and set $A_+ := \exp \mathfrak{a}_+$. The Cartan decomposition [5, p.434], [2, p.402] asserts that

$$G = KA_+K.$$

Though $k_1, k_2 \in K$ are not unique in $g = k_1ak_2$ ($g \in G$, $k_1, k_2 \in K$, $a \in A_+$), the element $a = a_+(g) \in A_+$ is unique.

An element $h \in G$ is called *hyperbolic* if $h = \exp(X)$ where $X \in \mathfrak{g}$ is real semisimple, that is, $\text{ad } X \in \text{End}(\mathfrak{g})$ is diagonalizable over \mathbb{R} . An element $u \in G$ is called *unipotent* if $u = \exp(N)$ where $N \in \mathfrak{g}$ is nilpotent, that is, $\text{ad } N \in \text{End}(\mathfrak{g})$ is nilpotent. An element $e \in G$ is *elliptic* if $\text{Ad}(e) \in \text{Aut}(\mathfrak{g})$ is diagonalizable over \mathbb{C} with eigenvalues of modulus 1. The complete multiplicative Jordan decomposition [5, Proposition 2.1] for G asserts that each $g \in G$ can be uniquely written as

$$g = ehv,$$

where e is elliptic, h is hyperbolic and v is unipotent and the three elements e, h, v commute. We write $g = e(g)h(g)v(g)$.

Remark 2.1. By [5, Proposition 3.4] and its proof, if π is a representation of G , then $\pi(g) = \pi(e)\pi(h)\pi(v) \in \text{SL}(V_\pi)$ is the complete multiplicative Jordan decomposition of $\pi(g)$ in $\text{SL}(V_\pi)$, where V_π is the representation space.

It turns out that $h \in G$ is hyperbolic if and only if it is conjugate to a unique element $b(h) \in A_+$ [5, Proposition 2.4]. Denote

$$b(g) := b(h(g)).$$

Since $\exp : \mathfrak{a} \rightarrow A$ is bijective, $\log b(g)$ is well-defined. We denote

$$(2.1) \quad A(g) := \exp(\text{conv}(W \log b(g))),$$

where $\text{conv } Wx$ denotes the convex hull of the orbit of $x \in \mathfrak{a}$ under the action of the Weyl group of $(\mathfrak{g}, \mathfrak{a})$, which may be defined as the quotient of the normalizer of A in K modulo the centralizer of A in K .

Given any $g \in G$, we consider the two sequences $\{(a_+(g^m))\}^{1/m}$ and $\{(b(g^m))\}^{1/m}$. The latter is simply a constant sequence since if $g = ehu$, then $h(g^m) = h(g)^m$ (because e, h, u commute) so that $b(g^m) = b(g)^m$ and thus $(b(g^m))^{1/m} = b(g)$ for all $m \in \mathbb{N}$. The following is an extension of Yamamoto's theorem (1.3).

Theorem 2.2. *Given $g \in G$, let $b(g) \in A_+$ be the unique element in A_+ conjugate to the hyperbolic part $h(g)$ of g . Then the sequence of compact sets $\{A((a_+(g^m))^{1/m})\}$ is monotonically decreasing and converges to $A(b(g))$ with respect to set inclusion. Moreover*

$$\lim_{m \rightarrow \infty} [a_+(g^m)]^{1/m} = b(g).$$

Proof. Denote by $I(G)$ the set of irreducible representations of G , V_π the representation space of $\pi \in I(G)$, $r(X)$ the spectral radius of the endomorphism X . For each $\pi \in I(G)$, there is an inner product structure [5, p.435] such that

- (1) $\pi(k)$ is unitary for all $k \in K$,
- (2) $\pi(a)$ is positive definite for all $a \in A_+$.

We will assume that V_π is endowed with this inner product. Thus for any $g \in G$, if $g = k_1 a_+(g) k_2$, where $a_+(g) \in A_+$, $k_1, k_2 \in K$, then

$$\begin{aligned} \|\pi(g)\| &= \|\pi(k_1 a_+(g) k_2)\| = \|\pi(k_1) \pi(a_+(g)) \pi(k_2)\| \\ (2.2) \quad &= \|\pi(a_+(g))\| = r(\pi(a_+(g))), \end{aligned}$$

since the spectral norm $\|\cdot\|$ is invariant under unitary equivalence, and $\|X\| = r(X)$ for every positive definite matrix X .

Now the sequence $\{\|\pi(g^m)\|^{1/m}\}$ is monotonically decreasing and converges to $r(\pi(g))$ by (1.1). According to (2.2),

$$(2.3) \quad \|\pi(g^m)\|^{1/m} = r(\pi(a_+(g^m)))^{1/m} = r(\pi(a_+(g^m))^{1/m}) = r(\pi(a_+(g^m))^{1/m}).$$

So $\{r(\pi(a_+(g^m))^{1/m})\}$ is monotonically decreasing and converges to $r(\pi(g))$. A result of Kostant [5, Theorem 3.1] asserts that if $f, g \in G$, then $A(f) \supset A(g)$ if and only if $r(\pi(f)) \geq r(\pi(g))$ for all $\pi \in I(G)$, where $A(g)$ is defined in (2.1). Thus

$$(2.4) \quad A(a_+(g)) \supset A(a_+(g^m))^{1/m} \supset A(a_+(g^{m+1}))^{1/(m+1)} \supset A(g).$$

From (2.1) the set $A(g)$, for all $g \in G$, is compact in G since the Weyl group W is finite. Thus $A((a_+(g^m))^{1/m})$ is a sequence of monotonically decreasing compact sets. Let $\ell \in A(a_+(g)) \cap A_+$ be any limit point of the sequence $\{a_+(g^m)^{1/m}\} \subset A(a_+(g)) \cap A_+$ (by (2.4)), namely

$$\lim_{i \rightarrow \infty} a_+(g^{m_i})^{1/m_i} = \ell,$$

for some natural number sequence $m_1 < m_2 < \dots$. Since r and π are continuous,

$$r(\pi(a_+(g^{m_i})^{1/m_i})) \rightarrow r(\pi(\ell)).$$

So $r(\pi(\ell)) = r(\pi(g))$ for all $\pi \in I(G)$, which implies that $A(\ell) = A(g) = A(b(g))$ by the result of Kostant [5, Theorem 3.1] again. Both ℓ and $b(g)$ are in A_+ . Thus $\ell = b(g)$ and

$$\lim_{m \rightarrow \infty} a_+(g^m)^{1/m} = b(g).$$

□

Remark 2.3. When $G = \mathrm{SL}_n(\mathbb{C})$, Theorem 2.2 is reduced to Yamamoto's theorem whose proofs in [7, 10] for (1.3) use compound matrices which are corresponding to the fundamental representations on the exterior spaces.

Example 2.4. The following example exhibits that a subgroup $G \subset G'$ may not have the same $a_+(g)$ component in the KA_+K decomposition when $g \in G$ is viewed as an element of G' . But the complete multiplicative Jordan decomposition remains the same (See Remark 2.1). Let $G = \mathrm{SO}_{2n}(\mathbb{C}) := \{g \in \mathrm{SL}_{2n}(\mathbb{C}) : g^T g = 1\}$ be a connected group [2, p.449] whose Lie algebra is

$$\mathfrak{g} := \mathfrak{so}_{2n}(\mathbb{C}) = \{X \in \mathbb{C}_{2n \times 2n} : X^T = -X\}.$$

Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where

$$\mathfrak{k} = \{X \in \mathbb{R}_{2n \times 2n} : X^T = -X\}, \quad \mathfrak{p} = i\mathfrak{k},$$

that is, the corresponding Cartan involution is $\theta(Y) = -Y^*$, $Y \in \mathfrak{g}$. So $K = \mathrm{SO}(2n)$. Pick

$$\mathfrak{a} = \left\{ \left(\begin{array}{cc} 0 & it_1 \\ -it_1 & 0 \end{array} \right) \oplus \left(\begin{array}{cc} 0 & it_2 \\ -it_2 & 0 \end{array} \right) \oplus \cdots \oplus \left(\begin{array}{cc} 0 & it_n \\ -it_n & 0 \end{array} \right) : t_1, \dots, t_n \in \mathbb{R} \right\},$$

and

$$\mathfrak{a}_+ = \left\{ \left(\begin{array}{cc} 0 & it_1 \\ -it_1 & 0 \end{array} \right) \oplus \left(\begin{array}{cc} 0 & it_2 \\ -it_2 & 0 \end{array} \right) \oplus \cdots \oplus \left(\begin{array}{cc} 0 & it_n \\ -it_n & 0 \end{array} \right) : t_1 \geq \cdots \geq t_{n-1} \geq |t_n| \right\},$$

Thus

$$A_+ = \left\{ \left(\begin{array}{cc} \cosh t_1 & i \sinh t_1 \\ -i \sinh t_1 & \cosh t_1 \end{array} \right) \oplus \left(\begin{array}{cc} \cosh t_2 & i \sinh t_2 \\ -i \sinh t_2 & \cosh t_2 \end{array} \right) \oplus \cdots \oplus \left(\begin{array}{cc} \cosh t_n & i \sinh t_n \\ -i \sinh t_n & \cosh t_n \end{array} \right) : t_1 \geq \cdots \geq t_{n-1} \geq |t_n| \right\}.$$

According to the Cartan decomposition, each $g \in \mathrm{SO}_{2n}(\mathbb{C})$ may be written as $g = k_1 a k_2$ where $a \in A_+$ and $k_1, k_2 \in \mathrm{SO}(2n)$ (Notice that it is different from the singular value decomposition of g in $\mathrm{SL}_{2n}(\mathbb{C})$). By Remark 2.1 we may view $g \in \mathrm{SO}_{2n}(\mathbb{C}) \subset \mathrm{SL}_{2n}(\mathbb{C})$ as an element in $\mathrm{SL}_{2n}(\mathbb{C})$ while computing its complete multiplicative Jordan decomposition $g = ehv$. Of course h is conjugate to $a_+(h) \in A_+$ via some element in $\mathrm{SO}_{2n}(\mathbb{C})$. Notice that if $t_2 \neq 0$, then

$$h_1 = \left(\begin{array}{cc} \cosh t_1 & i \sinh t_1 \\ -i \sinh t_1 & \cosh t_1 \end{array} \right) \oplus \left(\begin{array}{cc} \cosh t_2 & i \sinh t_2 \\ -i \sinh t_2 & \cosh t_2 \end{array} \right)$$

and

$$h_2 = \left(\begin{array}{cc} \cosh t_1 & i \sinh t_1 \\ -i \sinh t_1 & \cosh t_1 \end{array} \right) \oplus \left(\begin{array}{cc} \cosh t_2 & -i \sinh t_2 \\ i \sinh t_2 & \cosh t_2 \end{array} \right)$$

are not in the same coset in $\mathrm{SO}(4) \backslash \mathrm{SO}_4(\mathbb{C}) / \mathrm{SO}(4)$. If $g_1 := y h_1 y^{-1}$ and $g_2 := y h_2 y^{-1}$, where $y \in \mathrm{SO}_{2n}(\mathbb{C})$, then

$$\lim_{m \rightarrow \infty} (a_+(g_1^m))^{1/m} = h_1, \quad \lim_{m \rightarrow \infty} (a_+(g_2^m))^{1/m} = h_2.$$

But in $\mathrm{SL}_4(\mathbb{C})$, if we pick $A_+ \subset \mathrm{SL}_4(\mathbb{C})$ to be the group of positive diagonal matrices and $K = \mathrm{SU}(4)$, then

$$\lim_{m \rightarrow \infty} (a_+(g_1^m))^{1/m} = \lim_{m \rightarrow \infty} (a_+(g_2^m))^{1/m} = \mathrm{diag}(e^{t_1}, e^{t_2}, e^{-t_2}, e^{-t_1}),$$

if $t_1 \geq t_2 \geq 0$, since g_1, g_2 have the same set of singular values.

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