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AN ASYMPTOTIC RESULT ON THE A-COMPONENT IN IWASAWA DECOMPOSITION

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Abstract. Let G be a real connected semisimple Lie group. For each $v', v, g \in G$, we prove that

$$\lim_{m\to\infty}[a(v'g^m v)]^{1/m} = s^{-1} \cdot b(g),$$

where $a(g)$ denotes the a-component in the Iwasawa decomposition of $g = kan$ and $b(g) \in A_+$ denotes the unique element that conjugate to the hyperbolic component in the complete multiplicative Jordan decomposition of $g = ehu$. The element s in the Weyl group of (G, A) is determined by $yv \in G$ (not unique in general) in such a way that $yv \in N^- m_s MAN$, where $yhy^{-1} = b(g)$ and $G = \cup_{s\in W} N^- m_s MAN$ is the Bruhat decomposition of G.

1. Introduction

Given $X \in \mathrm{GL}_n(\mathbb{C})$, the well-known QR decomposition asserts that

$$X = QR,$$

where Q is unitary and R is upper triangular with positive diagonal entries. The decomposition is unique. Let $a(X) := \operatorname{diag} R$. Very recently it is known that [2] given $A, B \in \mathrm{GL}_n(\mathbb{C})$, the following limit

$$\lim_{m\to\infty}[a(AX^m B)]^{1/m}$$

exists and the limit is related to the eigenvalue moduli of X. More precisely,

Theorem 1.1. [2] *Let $A, B, X \in \mathrm{GL}_n(\mathbb{C})$. Let $X = Y^{-1}JY$ be the Jordan decomposition of X, where J is the Jordan form of X, $\operatorname{diag} J = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ satisfying $|\lambda_1| \ge \cdots \ge |\lambda_n|$. Then*

$$\lim_{m\to\infty}[a(AX^m B)]^{1/m} = \operatorname{diag}(|\lambda_{\omega(1)}|, \ldots, |\lambda_{\omega(n)}|),$$

where the permutation ω is uniquely determined by the $L\omega U$ decomposition of $YB = L\omega U$, where L is lower triangular and U is unit upper triangular.

The $L\omega U$ decomposition is also known as Gelfand-Naimark decomposition [1, p.434].

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The above asymptotic result relates three decompositions, namely, QR decomposition of X, Jordan decomposition of X, and Gelfand-Naimark decomposition of YB. Indeed the matrix Y (not unique in general) can be viewed from the standpoint of complete multiplicative Jordan decomposition (CMJD) of X [1]. Write $J = D + B$ where $D := \mathrm{diag}\, J$ is diagonal and B is the nilpotent part in Jordan form J. Then

$$X = Y^{-1}JY = Y^{-1}[D(1 + D^{-1}B)]Y,$$

where $1 + D^{-1}B$ is unipotent. Decompose the diagonal

$$D = \mathrm{diag}\,(e^{i\theta_1}|\lambda_1|, \dots, e^{i\theta_n}|\lambda_n|) = EH,$$

where

$$E := \mathrm{diag}\,(e^{i\theta_1}, \dots, e^{i\theta_n}), \qquad H := \mathrm{diag}\,(|\lambda_1|, \dots, |\lambda_n|).$$

Now we have the CMJD

$$X = ehu,$$

where

$$e = Y^{-1}EY, \quad h = Y^{-1}HY, \quad u = Y^{-1}(1 + D^{-1}B)Y.$$

Notice that the diagonalizable e has eigenvalue moduli 1, and the diagonalizable h has positive eigenvalues and u is unipotent. They commute with each other and such decomposition is unique. Now Y is an element which via conjugation turns h into a positive diagonal matrix with descending diagonal entries.

Our goal is to extend Theorem 1.1 in the context of real connected semisimple Lie group G. The three decompositions have their counterparts, namely Iwasawa decomposition, complete multiplicative Jordan decomposition (CMJD) and Bruhat decomposition. Motivated by Theorem 1.1, for any given $v', v, g \in G$, we study the sequence $\{[a(v'g^m v)]^{1/m}\}_{m\in\mathbb{N}}$ in which the a-component of a matrix would be played by the a-component $a(g)$ of g, where

$$g = kan$$

with respect to the Iwasawa decomposition $G = KAN$. The eigenvalue moduli $|\lambda|$ in descending order is replaced by the element $b(g) \in A_+$ that is conjugate to the hyperbolic element h in the CMJD of g. Here $A_+ := \exp \mathfrak{a}_+$ in which $\mathfrak{a}_+$ is a (closed) fundamental chamber. Finally the permutation ω would be provided by s in the Bruhat decomposition of $yv \in N^- m_s MAN$ such that $yhy^{-1} = b(g)$.

2. CMJD, Iwasawa decomposition, Bruhat decomposition

Let G be a connected semisimple Lie group having $\mathfrak{g}$ as its Lie algebra. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a fixed Cartan decomposition. Let $K \subset G$ be the connected subgroup with Lie algebra $\mathfrak{k}$. Then K is closed and $\mathrm{Ad}_G(K)$ is compact [1, p.252-253]. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace. Fix a *closed* Weyl chamber $\mathfrak{a}_+$ in $\mathfrak{a}$ so that the positive roots and thus the simple roots are fixed. Set $A_+ := \exp \mathfrak{a}_+$.

Following the terminology in [4, p.419], an element $h \in G$ is called *hyperbolic* if $h = \exp(X)$ where $X \in \mathfrak{g}$ is real semisimple, that is, $\mathrm{ad}\, X \in \mathrm{End}\,(\mathfrak{g})$ is diagonalizable over $\mathbb{R}$. An element $u \in G$ is called *unipotent* if $u = \exp(N)$ where $N \in \mathfrak{g}$ is nilpotent, that is, $\mathrm{ad}\, N \in \mathrm{End}\,(\mathfrak{g})$ is nilpotent. An element $e \in G$ is *elliptic* if $\mathrm{Ad}\,(e) \in \mathrm{Aut}\,(\mathfrak{g})$ is diagonalizable over $\mathbb{C}$ with eigenvalues of modulus 1. The complete multiplicative Jordan decomposition (CMJD) [4, Proposition 2.1] for G asserts that each $g \in G$ can be uniquely written as

$$g = ehu,$$

where e is elliptic, h is hyperbolic and u is unipotent and the three elements e, h, u commute. We write $g = e(g)h(g)u(g)$.

A hyperbolic $h \in G$ is conjugate to a unique element $b(h) \in A_+$ [4, Proposition 2.4]. Denote

$$b(g) := b(h(g)).$$

The group $A := \exp \mathfrak{a}$ is simply connected [3, p.317] and abelian so that the map $\mathfrak{a} \to A$ defined by exp is a diffeomorphsim [3, p.63]. Thus $\log a \in \mathfrak{a}$ is well defined for any $a \in A$. The Weyl group W of $(\mathfrak{g}, \mathfrak{a})$ is defined as the quotient of M' (the normalizer of $\mathfrak{a}$ in K) modulo M (the centralizer of $\mathfrak{a}$ in K). The Weyl group W acts on A and $\mathfrak{a}$ in the following manner: for each $s \in W$,

$$s \cdot a = \sigma\, a\, \sigma^{-1}, \qquad a \in A,\ \sigma \in s,$$
$$s \cdot H = \mathrm{Ad}\,(\sigma) H, \qquad H \in \mathfrak{a},\ \sigma \in s.$$

So W acts on A and $\mathfrak{a}$ in a way that the isomorphism $\mathfrak{a} \to A$ defined by exp is a W-isomorphism since

$$\exp(\mathrm{Ad}\,(\sigma)X) = \sigma(\exp X)\sigma^{-1}, \qquad \sigma \in G,\ X \in \mathfrak{g}.$$

Thus we may view W as the normalizer of A in K modulo the centralizer of A in K.

Let

$$\mathfrak{n} := \sum_{\alpha > 0} \mathfrak{g}_\alpha$$

be the sum of all positive root spaces. Set $N := \exp \mathfrak{n}$. Similarly let $\mathfrak{n}_- := \sum_{\alpha<0} \mathfrak{g}_\alpha$ and set $N^- := \exp \mathfrak{n}_-$. Let $G = KAN$ be the Iwasawa decomposition of G [3, p.317]. If $g \in G$, we write

$$g = kan,$$

where $k \in K$, $a \in A$, $n \in N$ are uniquely defined. For the semisimple $G = \mathrm{SL}_n(\mathbb{C})$, the QR decomposition for a nonsingular matrix $X = QR$ may be viewed as the Iwasawa decomposition $X = kan$ where $an = R$ and a is the diagonal part of R. We assume that G is noncompact otherwise the Iwasawa decomposition is trivial, that is, $a = n$ are simply the identity element.

The Bruhat decomposition of G [5, p.117] [1, p.403-407] asserts that

$$G = \cup_{s \in W} N^- m_s MAN$$

is a disjoint union, where $m_s \in W$. Moreover N^-MAN is an open submanifold of G and other terms are submanifolds of lower dimensions.

3. Asymptotic behavior of the Iwasawa component

Let G be a real connected semisimple Lie group. Let $a(g)$ be the a-component of $g \in G$ with respect to the Iwasawa decomposition

$$g = k(g)a(g)n(g).$$

Given $v', v, g \in G$, we now prove the following main theorem concerning the asymptotic behavior of the sequence $\{[a(v'g^m v)]^{1/m}\}_{m\in\mathbb{N}}$.

Theorem 3.1. *Let $v', v, g \in G$. Let $g = ehu$ be the complete multiplicative Jordan decomposition of g. Let $h = y^{-1}b(g)y$ for some $y \in G$, and $yv \in N^-m_sMAN$ in the Bruhat decomposition, where $s \in W$ is determined by $m_s \in M'$. Then*

$$\lim_{m\to\infty} [a(v'g^m v)]^{1/m} = s^{-1} \cdot b(g) = m_s^{-1} b(g) m_s, \tag{3.1}$$

where the limit is independent of the choice of $m_s \in M'$ such that $s = m_sM$.

Proof. We will make use of Theorem 1.1 by considering $\mathrm{Ad}\, G$ which can be viewed as a matrix group by choosing an appropriate orthonormal basis of $\mathfrak{g}$ with respect to the inner product

$$B_\theta(X, Y) = -B(X, \theta Y),$$

where $\theta \in \mathrm{Aut}\,(\mathfrak{g})$ is the Cartan involution

$$\theta(X + Y) = X - Y, \quad X \in \mathfrak{k},\ Y \in \mathfrak{p}$$

with respect to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $B(\cdot,\cdot)$ is the Killing form on $\mathfrak{g}$.

It is known that [1, p.261] there is an orthonormal basis of $\mathfrak{g}$,

$$\mathcal{X} = \{X_i : i = 1, \dots, \dim \mathfrak{g}\} \tag{3.2}$$

compatible with the (restricted) root space decomposition of $\mathfrak{g}$ [3, p.313]

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha$$

such that $X_i \in \mathfrak{g}_{\alpha_i}$ and $X_j \in \mathfrak{g}_{\alpha_j}$ with $i < j$ implies $\alpha_i \geq \alpha_j$ (by the lexicographic order $\mathcal{L}$ over the coordinates induced by pre-ordering the simple roots). Moreover, since $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$ is an orthogonal sum [3, p.313], we can select $\mathcal{X}$ in a way that $\mathfrak{a}$ is spanned by some $\{X_i, X_{i+1}, \cdots, X_{i+\dim\mathfrak{a}-1}\} \subseteq \mathcal{X}$. With the above basis, we view the elements in $\mathrm{GL}(\mathfrak{g})$ as matrices. The matrices $\mathrm{Ad}\,(K)$, $\mathrm{Ad}\,(A)$, $\mathrm{Ad}\,(N)$, and $\mathrm{Ad}\,(N^-)$ are orthogonal, positive diagonal, real unit upper triangular, and real unit lower triangular, respectively [3, p.317]. Because $\mathrm{Ad} : G \to \mathrm{Aut}\,(\mathfrak{g})$ is a representation of G, we may view the elements $\mathrm{Ad}\, g \in \mathrm{SL}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$ as nonsingular matrices. Thus we have the following Iwasawa decomposition for $\mathrm{Ad}\, g$, that is, QR decomposition:

$$\mathrm{Ad}\, g = \mathrm{Ad}\, k\, \mathrm{Ad}\, a\, \mathrm{Ad}\, n.$$

Therefore,

$$a(\mathrm{Ad}\,(g)) = \mathrm{Ad}\,(a(g)), \tag{3.3}$$

for all $g \in G$, where $a(\mathrm{Ad}\,(g))$ is the diagonal part of the matrix R in the QR decomposition $\mathrm{Ad}\, g = QR$. By (3.3)

$$\mathrm{Ad}\,[a(v'g^m v)]^{1/m} = [a((\mathrm{Ad}\, v')(\mathrm{Ad}\, g)^m(\mathrm{Ad}\, v))]^{1/m}, \quad m \in \mathbb{N}. \tag{3.4}$$

Since the kernel of Ad is the center Z of G and $A \cap Z$ is trivial [3, p.305], the map $A \to \mathrm{Ad}\,(A)$ induced by Ad is a diffeomorphism. Because of (3.4), to prove (3.1), it suffices to show

$$\lim_{m\to\infty}[a((\mathrm{Ad}\, v')(\mathrm{Ad}\, g)^m(\mathrm{Ad}\, v))]^{1/m} = \mathrm{Ad}\,(s^{-1}\cdot b(g)), \tag{3.5}$$

where

$$yv = n^- m_s man$$

is the Bruhat decomposition of yv, and $n^- \in N^-$, $m_s \in M'$, $m \in M$, $a \in A$ and $n \in N$. Since M is the centralizer of A in K and $m_s^{-1}b(g)m_s \in A$ so that

$$\begin{aligned} \mathrm{Ad}\,(s^{-1}\cdot b(g)) &= \mathrm{Ad}\,(m_s^{-1}b(g)m_s) = \mathrm{Ad}\,(m^{-1}m_s^{-1}b(g)m_s m) \\ &= (\mathrm{Ad}\,(m_s m))^{-1}\mathrm{Ad}\,(b(g))(\mathrm{Ad}\,(m_s m)). \end{aligned} \tag{3.6}$$

Now (3.5) is equivalent to the following by (3.6)

$$\lim_{m\to\infty}[a((\mathrm{Ad}\, v')(\mathrm{Ad}\, g)^m(\mathrm{Ad}\, v))]^{1/m} = (\mathrm{Ad}\,(m_s m))^{-1}\mathrm{Ad}\,(b(g))(\mathrm{Ad}\,(m_s m)). \tag{3.7}$$

In order to establish (3.7), we need several lemmas.

For each $H \in \mathfrak{a}_+$, $\mathrm{ad}\, H$ is a diagonal matrix

$$\mathrm{ad}\, H = \mathrm{diag}\,(h_1, \cdots, h_{\dim \mathfrak{g}}).$$

The diagonal entries may *not* be in descending order so it is not readily to apply Theorem 1.1.

Lemma 3.2. *Let $H \in \mathfrak{a}_+$ and write $\mathrm{ad}\, H = diag\,(h_1, \cdots, h_{\dim \mathfrak{g}})$. If $h_i > h_j$ for certain $i > j$, then the (i, j) entry of each element of $\mathrm{ad}\,\mathfrak{n}^-$ is always zero, where $\mathfrak{n}^- = \sum_{\alpha<0} \mathfrak{g}_\alpha$.*

Proof. For each $X \in \mathfrak{g}_\alpha$, where α is a negative root, $\mathrm{ad}\, X = (x_{ij})_{\dim \mathfrak{g}\times\dim \mathfrak{g}} \in \mathrm{ad}\,\mathfrak{g}_\alpha \subseteq \mathrm{ad}\,\mathfrak{n}_-$ is a strictly lower triangular matrix. Since ad respects brackets,

$$\begin{aligned} \alpha(H)\,(x_{ij})_{\dim \mathfrak{g}\times\dim \mathfrak{g}} &= \mathrm{ad}\,(\alpha(H)X) = \mathrm{ad}\,[H, X] \\ &= [\mathrm{ad}\, H, \mathrm{ad}\, X] = ((h_i - h_j)x_{ij})_{\dim \mathfrak{g}\times\dim \mathfrak{g}}, \end{aligned} \tag{3.8}$$

where the last equality is obtained by matrix Lie bracket operation. Since α is a negative root and $H \in \mathfrak{a}_+$, $\alpha(H) \le 0$. If $x_{ij} \neq 0$ for some $i > j$, then $h_i \le h_j$. □

If we relabel the roots in $\Sigma \cup \{0\}$ as $\mu_1 > \cdots > \mu_{\frac{k+1}{2}} (= 0) > \cdots > \mu_k$ according to the lexicographic order $\mathcal{L}$, and let $n_i = \dim \mathfrak{g}_{\mu_i}$, then we get a partition η of $\dim \mathfrak{g}$

$$\eta := (n_1, n_2, \cdots, n_k). \tag{3.9}$$

The partition η is symmetric ($n_t = n_{k+1-t}$ for $1 \le t \le k$) since $\theta \mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}$ [3, p.313].

Lemma 3.3. *For $H \in \mathfrak{a}$ and $a \in A$, $\operatorname{ad} H$ and $\operatorname{Ad}(a)$ are block diagonal matrices. More precisely,*

$$\operatorname{ad} H = \mathit{diag}\,(\mu_1(H) I_{n_1}, \ldots, \mu_k(H) I_{n_k}) \tag{3.10}$$

$$\operatorname{Ad} a = \mathit{diag}\,(e^{\mu_1(\log a)} I_{n_1}, \cdots, e^{\mu_k(\log a)} I_{n_k}). \tag{3.11}$$

Proof. Since $\operatorname{ad} H(X) = \alpha(H)X$ if $X \in \mathfrak{g}_\alpha$, and the lexicographic order $\mathcal{L}$ places the positive roots first, then the zero root, and then the negative roots, $\operatorname{ad} H$ is a block diagonal matrix and each block is a scalar multiple of identity according to the partition $\eta = (n_1, \cdots, n_k)$, in which all the positive entries are the first and the negative entries are the last. Then use $\operatorname{Ad} a = e^{\operatorname{ad}(\log a)}$ to obtain (3.11). □

Let $e_i \in \mathbb{R}^{\dim \mathfrak{g}}$ be the standard vector taking 1 at the i-th position and 0 elsewhere. We associate a permutation $\omega \in S_{\dim \mathfrak{g}}$ ($S_{\dim \mathfrak{g}}$ is the full symmetric group on $\{1, \ldots, \dim \mathfrak{g}\}$) with the permutation matrix:

$$P_\omega = [e_{\omega(1)}\, e_{\omega(2)} \, \cdots \, e_{\omega(\dim \mathfrak{g})}].$$

Then

$$P_\omega^{-1} (x_{ij})_{\dim \mathfrak{g} \times \dim \mathfrak{g}} P_\omega = (x_{\omega(i)\omega(j)})_{\dim \mathfrak{g} \times \dim \mathfrak{g}},$$

and in particular

$$P_\omega^{-1} \operatorname{diag}(h_1, \ldots, h_{\dim \mathfrak{g}}) P_\omega = \operatorname{diag}(h_{\omega(1)}, \ldots, h_{\omega(\dim \mathfrak{g})}). \tag{3.12}$$

From now on, let $H := \log b(g) \in \mathfrak{a}_+$ and write

$$\operatorname{ad} H := \operatorname{diag}(h_1, \cdots, h_{\dim \mathfrak{g}}) = \operatorname{diag}(\mu_1(H) I_{n_1}, \cdots, \mu_k(H) I_{n_k}).$$

Lemma 3.4. *There is $\omega \in S_{\dim \mathfrak{g}}$ uniquely determined by $b(g)$ that*

(1) *The diagonal entries of*

$$P_\omega^{-1} (\operatorname{ad} H) P_\omega = \mathit{diag}\,(h_{\omega(1)}, \cdots, h_{\omega(\dim \mathfrak{g})})$$

are in descending order.

(2) *If $h_{\omega(i)} = h_{\omega(j)}$ for $\omega(i) > \omega(j)$, then $i > j$.*

In other words, ω is the unique permutation which has the smallest number of transpositions in its factorization, such that the diagonal entries of $P_\omega^{-1} (\operatorname{ad} H) P_\omega$ are in descending order. The permutation ω satisfies the following further properties:

(3) *P_ω acts as identity on $\mathfrak{g}_0 \supseteq \mathfrak{a} \supseteq \mathfrak{a}_+$.*

(4) P_ω *is a block permutation matrix according to the row partition* $\eta = (n_1, \cdots, n_k)$ *in* (3.9). *Precisely, there is* $\gamma \in S_k$ *such that*

$$P_\omega^{-1} diag\,(x_1 I_{n_1}, \cdots, x_k I_{n_k}) P_\omega = diag\,(x_{\gamma(1)} I_{n_{\gamma(1)}}, \cdots, x_{\gamma(k)} I_{n_{\gamma(k)}})$$

for the free variables $x_1, \cdots, x_k$. *If we partition the rows of* P_ω *by* η, *and partition the columns of* P_ω *by* $\gamma(\eta) := (n_{\gamma(1)}, \cdots, n_{\gamma(k)})$, *then the* $(i, \gamma^{-1}(i))$ *block of* P_ω *is* I_{n_i} *for* $1 \le i \le k$, *and the other blocks of* P_ω *are zero blocks.*

Proof. Let $\omega \in S_{\dim \mathfrak{g}}$ be the unique permutation acting on the sequence $\{(-h_1, 1), (-h_2, 2), \cdots, (-h_{\dim \mathfrak{g}}, \dim \mathfrak{g})\}$ in the way that the resulting sequence is increasing in lexicographic order:

$$\left\{(-h_{\omega(1)}, \omega(1)) < (-h_{\omega(2)}, \omega(2)) < \cdots < (-h_{\omega(\dim \mathfrak{g})}, \omega(\dim \mathfrak{g}))\right\}.$$

Then ω is the desired permutation satisfying (1) and (2).

The matrix $\operatorname{ad} H$ is symmetric about the anti-diagonal. So is P_ω. Therefore, P_ω acts as identity on $\mathfrak{g}_0$ and thus (3) holds. Finally, if $h_j = h_{j+1}$, then by (1) and (2), it is impossible to have $1 \le t \le k$ such that $\omega(t)$ is a number between $\omega(j)$ and $\omega(j+1)$. So $\omega(j) + 1 = \omega(j+1)$. This implies that P_ω is a block permutation matrix and (4) follows. □

For each $X \in \mathfrak{n}^-$, write $\operatorname{ad} X = \left(x_{ij}\right)_{\dim \mathfrak{g} \times \dim \mathfrak{g}}$ which is strictly lower triangular.

Lemma 3.5. *Let* ω *be determined by* $b(g)$ *as in Lemma 3.4. Then for all* $X \in \mathfrak{n}^-$,

$$P_\omega^{-1} (\operatorname{ad} X) P_\omega = \left(x_{\omega(i)\omega(j)}\right)_{\dim \mathfrak{g} \times \dim \mathfrak{g}}$$

remains strictly lower triangular.

Proof. Clearly the diagonal entries of $P_\omega^{-1}(\operatorname{ad} X)P_\omega$ are 0. The (i,j) entry of $P_\omega^{-1}(\operatorname{ad} X)P_\omega$ is $x_{\omega(i)\omega(j)}$. Suppose on the contrary, $x_{\omega(i)\omega(j)} \ne 0$ for some $i < j$. Then $\omega(i) > \omega(j)$ since $\operatorname{ad} X$ is strictly lower triangular. Also $h_{\omega(i)} \ge h_{\omega(j)}$ by Lemma 3.4 (1). But $h_{\omega(i)} = h_{\omega(j)}$ contradicts Lemma 3.4 (2) since $\omega(i) > \omega(j)$ and $i < j$. On the other hand, if $h_{\omega(i)} > h_{\omega(j)}$, then it contradicts Lemma 3.2 since $\omega(i) > \omega(j)$ but the $(\omega(i), \omega(j))$ entry of $\operatorname{ad} X$ is $x_{\omega(i)\omega(j)} \ne 0$. This proves that $P_\omega^{-1}(\operatorname{ad} X)P_\omega$ is strictly lower triangular for every $X \in \mathfrak{n}^-$. □

Let us now prove (3.7). Taking exponentials, from Lemma 3.4 (1)

$$\begin{aligned} P_\omega^{-1}(\operatorname{Ad} b(g))P_\omega &= P_\omega^{-1} \exp(\operatorname{ad} \log b(g)) P_\omega \\ &= \exp(\operatorname{diag}(h_{\omega(1)}, \cdots, h_{\omega(\dim \mathfrak{g})})) \end{aligned} \tag{3.13}$$

is a positive diagonal matrix with descending diagonal entries, and from Lemma 3.5 the elements of $P_\omega^{-1}(\mathrm{Ad}\,N^-)P_\omega$ are unit lower triangular matrices. Now

$$\begin{aligned} \mathrm{Ad}\,g &= (\mathrm{Ad}\,e)(\mathrm{Ad}\,h)(\mathrm{Ad}\,u), && (3.14)\\ \mathrm{Ad}\,h &= (\mathrm{Ad}\,y)^{-1}(\mathrm{Ad}\,b(g))(\mathrm{Ad}\,y)\\ &= (\mathrm{Ad}\,y)^{-1}P_\omega[P_\omega^{-1}(\mathrm{Ad}\,b(g))P_\omega]P_\omega^{-1}(\mathrm{Ad}\,y), && (3.15)\end{aligned}$$

where $y \in G$ such that $h = yb(g)y^{-1}$. Recall

$$yv = n^- m_s m a n.$$

Since $n^- = \exp X$ for some $X \in \mathfrak{n}^-$, $P_\omega^{-1}(\mathrm{ad}\,X)P_\omega$ is strictly lower triangular so that

$$L := P_\omega^{-1}(\mathrm{Ad}\,n^-)P_\omega = P_\omega^{-1}\mathrm{Ad}\,(\exp X)P_\omega = P_\omega^{-1}e^{\mathrm{ad}\,X}P_\omega = e^{P_\omega^{-1}(\mathrm{ad}\,X)P_\omega}$$

is unit lower triangular by Lemma 3.5. We have

$$\begin{aligned} P_\omega^{-1}\mathrm{Ad}\,(yv) &= P_\omega^{-1}(\mathrm{Ad}\,n^-)(\mathrm{Ad}\,(m_s m))(\mathrm{Ad}\,(an))\\ &= L(P_\omega^{-1}\mathrm{Ad}\,(m_s m))(\mathrm{Ad}\,(an)) && (3.16)\end{aligned}$$

in which L is unit lower triangular and $\mathrm{Ad}\,(an)$ is upper triangular.

Let us examine the matrix $P_\omega^{-1}\mathrm{Ad}\,(m_s m)$. On one hand, for each $m' \in M'$, $\mathrm{Ad}\,m'$ permutes the root spaces of the same dimensions [1, p.406] and $\mathrm{Ad}\,m' \in \mathrm{Ad}\,K$ is an orthogonal matrix. Since $m_s m \in M'$,

$$\mathrm{Ad}\,(m_s m) = P_\sigma D, \qquad (3.17)$$

for some block permutation matrix P_σ and some block diagonal matrix D in which each diagonal block is an orthogonal matrix, all are in accordance with the partition η. On the other hand, $P_\omega^{-1} = P_\omega^T$ is a block permutation matrix with respect to the row partition $\gamma(\eta) = (n_{\gamma(1)}, \cdots, n_{\gamma(k)})$ and the column partition $\eta = (n_1, \cdots, n_k)$.

Let

$$D = L_D \Omega_D U_D$$

be the Gelfand-Naimark decomposition of D, where L_D, Ω_D, and U_D are block diagonal; L_D is unit lower triangular, Ω_D is a permutation, and U_D is upper triangular; all are in accordance with the partition η. Then

$$P_\omega^{-1}\mathrm{Ad}\,(m_s m) = P_\omega^{-1}P_\sigma L_D \Omega_D U_D = L' P_\omega^{-1} P_\sigma \Omega_D U_D$$

where $L' = P_\omega^{-1}P_\sigma L_D P_\sigma^{-1}P_\omega$ is a unit lower triangular matrix, and L' is block diagonal according to the row and column partition $\gamma(\eta) = (n_{\gamma(1)}, \cdots, n_{\gamma(k)})$ (since $P_\sigma^{-1}P_\omega$ is a block permutation according to the row partition η and the column partition $\gamma(\eta)$). By (3.16), $P_\omega^{-1}(\mathrm{Ad}\,y)(\mathrm{Ad}\,v)$ has the Gelfand-Naimark decomposition

$$\begin{aligned} P_\omega^{-1}(\mathrm{Ad}\,y)(\mathrm{Ad}\,v) &= P_\omega^{-1}\mathrm{Ad}\,(yv)\\ &= L(P_\omega^{-1}\mathrm{Ad}\,(m_s m))(\mathrm{Ad}\,(an))\\ &= (LL')(P_\omega^{-1}P_\sigma \Omega_D)(U_D \mathrm{Ad}\,(an)), && (3.18)\end{aligned}$$

where $P_\omega^{-1} P_\sigma \Omega_D$ is the desired permutation matrix. Notice that the diagonal entries of the diagonal matrix $P_\omega^{-1}(\mathrm{Ad}\, b(g))P_\omega$ are in descending order. Applying Theorem 1.1 on $\mathrm{Ad}\, g$ and taking (3.13), (3.14), (3.15) and (3.18) into account,

$$\begin{aligned} & \lim_{m\to\infty} [a((\mathrm{Ad}\, v')(\mathrm{Ad}\, g)^m(\mathrm{Ad}\, v))]^{1/m} \\ = \ & (P_\omega^{-1} P_\sigma \Omega_D)^{-1} [P_\omega^{-1}(\mathrm{Ad}\, b(g))P_\omega](P_\omega^{-1} P_\sigma \Omega_D) \\ = \ & \Omega_D^{-1} P_\sigma^{-1} (\mathrm{Ad}\, b(g)) P_\sigma \Omega_D \\ = \ & D^{-1} P_\sigma^{-1} (\mathrm{Ad}\, b(g)) P_\sigma D \qquad (3.19) \\ = \ & (\mathrm{Ad}\,(m_s m))^{-1} (\mathrm{Ad}\, b(g)) (\mathrm{Ad}\,(m_s m)) \quad \text{by (3.18)}. \end{aligned}$$

Equality (3.19) holds because both the permutation matrix Ω_D and the orthogonal matrix $D = L_D \Omega_D U_D$ are block diagonal, and $P_\sigma^{-1}(\mathrm{Ad}\, b(g))P_\sigma$ is block diagonal in which each diagonal block is a scalar multiple of an identity matrix, all are in accordance with the partition η. So we prove (3.7) and thus (3.1). □

Corollary 3.6. Let $g = ehu$ be the complete multiplicative Jordan decomposition of $g \in G$. Let $h = y^{-1} b(g) y$ for some $y \in G$, and $y \in N^- m_s MAN$ in the Bruhat decomposition, where $s \in W$ is determined by $m_s \in M'$. Then

$$\lim_{m\to\infty} [a(g^m)]^{1/m} = s^{-1} \cdot b(g) = m_s^{-1} b(g) m_s,$$

where the limit is independent of the choice of $m_s \in M'$ such that $s = m_s M$.

Proof. (1) follows immediately from Theorem 3.1 by setting v, v' to be the identity element. □

By Corollary 3.6 the map $L : G \to A$ where $L(g) := \lim_{m\to\infty} [a(g^m)]^{1/m}$ is well defined. We can see that the map L is not continuous by constructing a sequence $g_k = y_k d y_k^{-1}$ in the open dense set $N^- MAN$ such that $d \in A_+$ and y_k converges to some $m_s \in M'$ but s is not the identity and $s \cdot d \neq d$. For example, $g_k = y_k d y_k^{-1}$, where $d := \mathrm{diag}\,(d_1, \ldots, d_n)$ with $d_1 > d_2 \geq d_3 \geq \cdots \geq d_n > 0$, $y_k := I_n/k + \omega \in \mathrm{GL}_n(\mathbb{R})$ where ω is the transposition $(1, 2)$. So $L(g_k) = d$ but $L(\lim_{k\to\infty} g_k) = \mathrm{diag}\,(d_2, d_1, d_3, \ldots, d_n)$.

The next result asserts that the images under L of the orbits $O_G(g) := \{vgv^{-1} : v \in G\}$ and $O_K(g) := \{vgv^{-1} : v \in K\}$ of $g \in G$ are equal to $W\, b(g)$.

Corollary 3.7. Let $g = ehu$ be the complete multiplicative Jordan decomposition of $g \in G$. Let $h = y^{-1} b(g) y$ for some $y \in G$, and $y \in N^- m_s MAN$ in the Bruhat decomposition, where $s \in W$ is determined by $m_s \in M'$. Then

$$L(O_K(g)) = L(O_G(g)) = W\, b(g),$$

where $W\, b(g)$ is the orbit of $b(g)$ under the action of the Weyl group W.

Proof. Clearly $L(O_K(g)) \subset L(O_G(g)) \subset W\, b(g)$ because $b(vgv^{-1}) = b(g)$ for all $v \in G$. It suffices to show that $W\, b(g) \subset L(O_K(g))$.

Let $y \in G$. Denote by $s(g) \in W$ such that $g \in N^- m_s MAN$. Notice that $(vgv^{-1})^m = vg^m v^{-1}$ for $v \in G$ so that from Theorem 3.1

$$L(vgv^{-1}) = \lim_{m\to\infty} [a(vg^m v^{-1})]^{1/m} = [s(yv^{-1})]^{-1} \cdot b(g).$$

Since $G = G^{-1} = NAK$, one has $G = \theta(G) = N^- AK$, where $\theta : G \to G$ is the Cartan involution of G. So $y = n^- ak$ where $n^- \in N^-$, $a \in A$ and $k \in K$. Thus

$$s(yK) = s(n^- aK) = s(aK) \supset s(aM').$$

Since $M' \subset K$ normalizes A, for each m_s, there exists $a' \in A$ such that $m_s a' = am_s \in aM'$. So $s(am_s) = s$ for all $s \in W$. Hence $s(yK) = W$ and thus $W\, b(g) \subset L(O_K(g))$. ☐

4. Some remarks

Remark 4.1. Iwasawa decomposition may be expressed in the form $G = NAK$ in which we write $g = n'a'k' = kan$, $g \in G$, $n, n' \in N$, $a, a' \in A$ and $k, k' \in K$. Since $g^{-1} = n^{-1}a^{-1}k^{-1}$, by the uniqueness of Iwasawa decomposition $a'(g) = [a(g^{-1})]^{-1}$ so that

$$\begin{aligned}\lim_{m\to\infty} [a'(v'g^m v)]^{1/m} &= \lim_{m\to\infty} [a(v^{-1}(g^{-1})^m v'^{-1})]^{-1/m} \\ &= (\lim_{m\to\infty} [a(v^{-1}(g^{-1})^m v'^{-1})]^{1/m})^{-1}.\end{aligned}$$

Now the CMJD $g = ehu$ and $yhy^{-1} = b(g)$ imply $g^{-1} = e^{-1}h^{-1}u^{-1}$ and $yh^{-1}y^{-1} = (b(g))^{-1} \in A_+^{-1}$ respectively. So $b(g^{-1}) = m_\ell (b(g))^{-1} m_\ell^{-1} = \ell \cdot (b(g))^{-1} \in A_+$, where $\ell \in W$ is the longest element [1, p.406] which maps A_+ to A_+^{-1}. By Theorem 3.1

$$\lim_{m\to\infty} [a'(v'g^m v)]^{1/m} = [s^{-1} \cdot \ell \cdot (b(g))^{-1}]^{-1} = (s^{-1}\ell) \cdot b(g),$$

where $m_\ell y v'^{-1} \in N^- m_s MAN$.

The situation for the two variants $G = KNA$ and $G = ANK$ is identical to KAN and NAK, respectively, since A normalizes N (in general $G = AKN$ and $G = NKA$ are not true).

Remark 4.2. Suppose G_1 is a semisimple subgroup of G. The Iwasawa decomposition of $g \in G_1$ is not necessarily equal to that of g when g is viewed as an element of G. For example, if $G_1 := \mathrm{Sp}_n(\mathbb{R})$ and $G := \mathrm{SL}_{2n}(\mathbb{R})$ [1, p.445], where

$$Sp_n(\mathbb{R}) = \{g \in \mathrm{GL}_{2n}(\mathbb{R}) : g^T J_n g = J_n\}, \qquad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

The Iwasawa decomposition of $G_1 = K_1 A_1 N_1$ is given by [6, p.491]

$$K_1 = \{\begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A + iB \in U(n)\},$$

$$\begin{aligned} A_1 &= \{\mathrm{diag}\,(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}) : a_1, \ldots, a_n > 0\}, \\ N_1 &= \{\begin{pmatrix} A & B \\ 0 & (A^{-1})^T \end{pmatrix} : A \text{ upper triangular, 1's on the diagonal, } AB^T = B^T A\}. \end{aligned}$$

If we consider

$$X = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \in N_1 \subset \mathrm{Sp}_n(\mathbb{R}),$$

then clearly $a_{G_1}(X) = I_4$ but

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and thus $a_G(X) = \mathrm{diag}\,(1, 1, \sqrt{2}, 1/\sqrt{2})$. Clearly $a_G(X)$ and $a_{G_1}(X)$ are not conjugate in G. But all Iwasawa decompositions of a connected semisimple Lie group are conjugate [1, p.435]. So $a_G(X)$ cannot be $a_{G_1}(X)$ for any Iwasawa decomposition of G_1.

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