

ASYMPTOTIC BEHAVIOR OF GELFAND-NAIMARK DECOMPOSITION

HUAJUN HUANG

ABSTRACT. Let $X = L\sigma U$ be the Gelfand-Naimark decomposition of $X \in \mathrm{GL}_n(\mathbb{C})$, where L is unit lower triangular, σ is a permutation matrix, and U is upper triangular. Call $u(X) := \mathrm{diag} U$ the u -component of X . We show that in a Zariski dense open subset of the ω -orbit of certain Bruhat decomposition,

$$\lim_{m \rightarrow \infty} |u(X^m)|^{1/m} = \mathrm{diag} (|\lambda_{\omega(1)}|, \dots, |\lambda_{\omega(n)}|).$$

The other situations where $\lim_{m \rightarrow \infty} |u(X^m)|^{1/m}$ converge to different limits or diverge are also discussed.

1. INTRODUCTION

Gelfand-Naimark decomposition asserts that each $X \in \mathrm{GL}_n(\mathbb{C})$ can be decomposed as $X = L\sigma U$, where L is unit lower triangular, σ is a permutation matrix, and U is upper triangular. Though Gelfand-Naimark decomposition is not unique, σ and $\mathrm{diag} U$ are uniquely determined by X . We denote

$$u(X) := (u_1(X), \dots, u_n(X)) = \mathrm{diag} U \in \mathbb{C}^n$$

the u -component of X .

Suppose that X has eigenvalues

$$\lambda(X) := (\lambda_1(X), \dots, \lambda_n(X)) \in \mathbb{C}^n$$

with ascending moduli: $|\lambda_1| \leq |\lambda_2| \leq \dots \leq |\lambda_n|$. Let

$$s(X) := (s_1(X), \dots, s_n(X)) \in \mathbb{R}_+^n$$

be the singular values of X in ascending order: $s_1(X) \leq \dots \leq s_n(X)$. Let $X = QR$ be the QR decomposition of X and denote

$$a(X) := \mathrm{diag} R = (a_1(X), \dots, a_n(X)) \in \mathbb{R}_+^n.$$

2000 Mathematics Subject Classification: Primary 15A23, 15A42.

Key Words: Gelfand-Naimark decomposition, Bruhat decomposition, u -component, eigenvalues.

We view $\lambda(X)$, $s(X)$, $a(X)$ and $u(X)$ as diagonal matrices. Relations between those four quantities were very recently studied [5].

Yamamoto proved [9] that

$$(1.1) \quad \lim_{m \rightarrow \infty} [s(X^m)]^{1/m} = \text{diag}(|\lambda_1|, \dots, |\lambda_n|).$$

Huang and Tam proved [7, Theorem 2.7] (also see [4]) that

$$(1.2) \quad \lim_{m \rightarrow \infty} [a(X^m)]^{1/m} = \text{diag}(|\lambda_{\omega(1)}|, \dots, |\lambda_{\omega(n)}|).$$

Here ω is a permutation obtained as follow: Let $X = C^{-1}TC$ in which $T = [t_{ij}] \in \text{GL}_n(\mathbb{C})$ is an upper triangular matrix with $t_{ii} = \lambda_i$ (so $|t_{11}| \leq \dots \leq |t_{nn}|$). Then ω comes from the permutation matrix (also denoted by ω) in a Bruhat decomposition $C = V'\omega U'$ of C , for certain upper triangular matrices V' and U' . This result can be applied to both the Jordan decomposition and the Schur triangularization of X . Note that the Gelfand-Naimark decomposition is a variation of the Bruhat decomposition [8].

When X has distinct eigenvalue moduli: $|\lambda_1| < |\lambda_2| < \dots < |\lambda_n|$, consider the QR iteration:

$$X_1 = X = Q_1R_1; \quad X_m := R_{m-1}Q_{m-1} = Q_mR_m, \quad m = 2, 3, \dots$$

where $X_m = Q_mR_m$ is the QR decomposition of X_m for $m = 1, 2, \dots$. In [8, Theorem 3.1], Tyrtshnikov essentially proved that the lower triangular part of $\{X_m\}_{m=1}^\infty$ converges to 0 and

$$(1.3) \quad \lim_{m \rightarrow \infty} \text{diag}(X_m) = \text{diag}(\lambda_{\omega(1)}, \lambda_{\omega(2)}, \dots, \lambda_{\omega(n)}),$$

where ω is the same as in (1.2). Indeed, Tyrtshnikov proved this result for GR iterations with no shifts. A detailed description on the asymptotic behavior of $\{X_m\}_{m=1}^\infty$ for the above QR iteration is given in [6, Theorem 5.1].

The main purpose of this paper is to discuss the asymptotic behavior of $|u(X^m)|^{1/m}$ as $m \rightarrow \infty$. Unlike $s(X^m)^{1/m}$ and $a(X^m)^{1/m}$, the sequence $\{|u(X^m)|^{1/m}\}_{m=1}^\infty$ may not converge. However, as we will see, the permutation ω obtained from the Bruhat decomposition of C continues to play a significant role for the asymptotic behavior of $|u(X^m)|^{1/m}$. We provide some sufficient conditions for the convergence of the sequence. In particular, when X is positive definite (it has Cholesky decomposition [3]) [7, Theorem 3.1]

$$\lim_{m \rightarrow \infty} |u(X^m)|^{1/m} = \text{diag}(|\lambda_{\omega(1)}|, \dots, |\lambda_{\omega(n)}|).$$

In Section 2, we give an upper bound on the asymptotic behavior of $|u_1(X^m) \cdots u_k(X^m)|^{1/m}$ in terms of ω . Then we prove that

$\lim_{m \rightarrow \infty} |u_1(X^m) \cdots u_k(X^m)|^{1/m}$ converges to a product of k eigenvalue moduli of X if the k -compound matrix of X has distinct eigenvalue moduli. Moreover, when C goes through a Zariski dense open subset of the ω -orbit in the Bruhat decomposition $C = V'\omega U'$, the matrix $X^m = C^{-1}\lambda(X)^m C$ has LU -decomposition for sufficiently large m , and

$$\lim_{m \rightarrow \infty} |u(X^m)|^{1/m} = \text{diag}(|\lambda_{\omega(1)}|, \dots, |\lambda_{\omega(n)}|).$$

In particular, $\lim_{m \rightarrow \infty} |u(X^m)|^{1/m} = \text{diag}(|\lambda_n|, |\lambda_{n-1}|, \dots, |\lambda_1|)$ when X is in a dense open subset (in Euclidean topology) of $\text{GL}_n(\mathbb{C})$.

In Section 3, we use examples to illustrate the theoretical results given in Section 2. In particular, we show that when the eigenvalue moduli of the k -compound of X are not distinct for some k , $\lim_{m \rightarrow \infty} |u(X^m)|^{1/m}$ may diverge.

2. THE ASYMPTOTIC BEHAVIOR OF $|u(X^m)|^{1/m}$

Obviously, $u_1(X)$ in the Gelfand-Naimark decomposition $X = L\sigma U$ is the first nonzero entry in the first column of X , and $a_1(X)$ in the QR decomposition $X = QR$ is the norm of the first column of X . Therefore, $|u_1(X)| \leq a_1(X)$. Using compound matrix technique, we get the following relationship between $u(X)$ and $a(X)$.

Theorem 2.1. [5, Theorem 3.1] *The scalars $u(X) \in \mathbb{C}^n$ and $a(X) \in \mathbb{R}_+^n$ satisfy that*

$$\begin{aligned} |u_1(X) \cdots u_k(X)| &\leq a_1(X) \cdots a_k(X), & 1 \leq k \leq n-1, \\ u_1(X) \cdots u_n(X) &= \pm a_1(X) \cdots a_n(X), \end{aligned}$$

where the sign on the last equality depends on whether σ is even (+) or σ is odd (-).

Suppose that $X = C^{-1}TC \in \text{GL}_n(\mathbb{C})$ where $T = [t_{ij}]$ is an upper triangular matrix with $t_{ii} = \lambda_i$ (so $|t_{11}| \leq |t_{22}| \leq \cdots \leq |t_{nn}|$), and C has a Bruhat decomposition $C = V'\omega U'$ where V' and U' are upper triangular, and ω is a permutation matrix. By Theorem 2.1 and (1.2), we get the following result:

Theorem 2.2. *The asymptotic behavior of $|u(X^m)|^{1/m}$ is bounded by*

$$\begin{aligned} \overline{\lim}_{m \rightarrow \infty} |u_1(X^m) \cdots u_k(X^m)|^{1/m} &\leq |\lambda_{\omega(1)} \cdots \lambda_{\omega(k)}|, & 1 \leq k \leq n-1; \\ |u_1(X^m) \cdots u_n(X^m)|^{1/m} &= |\lambda_{\omega(1)} \cdots \lambda_{\omega(n)}| = |\det(X)|. \end{aligned}$$

Denote

$$C^{-1} = [c'_{ij}], \quad C = [c_{ij}], \quad X^m = [x_{ij}^{(m)}], \quad \text{and} \quad T^m = [t_{ij}^{(m)}].$$

The coefficients of X^m are related to those of T^m by $X^m = C^{-1}T^mC$. To study X^m , we first investigate the coefficients of T^m .

A subsequence $s := \{s_0, s_1, \dots, s_k\}$ of $\{1, 2, \dots, n\}$ is called a T -path if $t_{s_i s_{i+1}} \neq 0$ for $i = 0, \dots, k-1$. Clearly $s_0 < s_1 < \dots < s_k$. We call s_0 the *initial point*, s_k the *terminal point*, and $|s| := k$ the *length*, of the T -path s respectively. Let S_{ij} denote the set of all T -paths of with the initial point i and the terminal point j . Then $S_{ij} = \emptyset$ whenever $i > j$, since T is upper triangular.

Let

$$T(s) := (t_{s_0 s_0}, \dots, t_{s_k s_k})$$

be the $(k+1)$ -tuple of the diagonal entries of T corresponding to the T -path s . Define the polynomial

$$(2.1) \quad M_p(T(s)) := M_p(t_{s_0 s_0}, \dots, t_{s_k s_k}) := \sum_{\substack{a_0, \dots, a_k \geq 0 \\ a_0 + \dots + a_k = p}} t_{s_0 s_0}^{a_0} \dots t_{s_k s_k}^{a_k}$$

the sum of all degree p monomials for the variables $t_{s_0 s_0}, \dots, t_{s_k s_k}$. Then $M_p(T(s)) = 0$ whenever $p < 0$. Define the polynomial

$$(2.2) \quad f_s(T) := \begin{cases} t_{s_0 s_1} t_{s_1 s_2} \dots t_{s_{k-1} s_k}, & |s| \geq 1; \\ 1, & |s| = 0. \end{cases}$$

Lemma 2.3. *The entries of $T^m = [t_{ij}^{(m)}]$ for an upper triangular matrix $T = [t_{ij}]$ are given by*

$$(2.3) \quad t_{ij}^{(m)} = \sum_{s \in S_{ij}} f_s(T) M_{m-|s|}(T(s)).$$

Proof. By direct computation,

$$\begin{aligned} t_{ij}^{(m)} &= \sum_{i=i_0 \leq i_1 \leq \dots \leq i_m=j} \left(\prod_{p=0}^{m-1} t_{i_p i_{p+1}} \right) \\ &= \sum_{\substack{i=j_0 < j_1 < \dots < j_k=j \\ a_0 \geq 0, \dots, a_k \geq 0 \\ a_0 + \dots + a_k = m-k}} \left(\prod_{p=0}^{k-1} t_{j_p j_{p+1}} \cdot \prod_{p=0}^k t_{j_p j_p}^{a_p} \right) \\ &= \sum_{i=j_0 < j_1 < \dots < j_k=j} \left(\prod_{p=0}^{k-1} t_{j_p j_{p+1}} \right) \left[\sum_{\substack{a_0 \geq 0, \dots, a_k \geq 0 \\ a_0 + \dots + a_k = m-k}} \left(\prod_{p=0}^k t_{j_p j_p}^{a_p} \right) \right] \\ &= \sum_{s \in S_{ij}} f_s(T) M_{m-|s|}(T(s)). \end{aligned}$$

This proves (2.3). \square

Formula (2.3) implies the following asymptotic result when the diagonal entries of T have strictly ascending moduli.

Theorem 2.4. *Suppose that the upper triangular matrix $T \in \text{GL}_n(\mathbb{C})$ has strictly ascending moduli diagonal entries, that is, $|t_{11}| < |t_{22}| < \dots < |t_{nn}|$. Let $\text{diag}(T)$ be the diagonal matrix of T . Then $\lim_{m \rightarrow \infty} T^m \text{diag}(T)^{-m}$ converges to an upper triangular matrix. Precisely, $t_{jj}^{(m)}/t_{jj}^m = 1$ for $j = 1, 2, \dots, n$, and*

$$(2.4) \quad \lim_{m \rightarrow \infty} \frac{t_{ij}^{(m)}}{t_{jj}^m} = \sum_{(s_0, \dots, s_k) \in S_{ij}} \frac{\prod_{p=0}^{k-1} t_{s_p s_{p+1}}}{\prod_{p=0}^{k-1} (t_{jj} - t_{s_p s_p})} \quad \text{for } 1 \leq i < j \leq n.$$

Remark 2.5. Therefore, $\lim_{m \rightarrow \infty} |t_{ij}^{(m)}|^{1/m} = 0$ or $|t_{jj}|$, for $1 \leq i \leq j \leq n$.

Proof of Lemma 2.4. According to (2.3), for $i < j$ we have

$$(2.5) \quad \frac{t_{ij}^{(m)}}{t_{jj}^m} = \sum_{s \in S_{ij}} \left[\frac{f_s(T)}{t_{jj}^{|s|}} \cdot \frac{M_{m-|s|}(T(s))}{t_{jj}^{m-|s|}} \right].$$

Let us discuss the asymptotic behavior of $\frac{M_\ell(T(s))}{t_{jj}^\ell}$ when $\ell \rightarrow \infty$. Rewrite $T(s) := (y_0, \dots, y_k)$ where $|y_0| < \dots < |y_k|$. Then

$$(2.6) \quad \begin{aligned} \frac{M_\ell(y_0, \dots, y_k)}{y_k^\ell} &= \sum_{\substack{a_0 \geq 0, \dots, a_k \geq 0 \\ a_0 + \dots + a_k = \ell}} \frac{y_0^{a_0} \dots y_k^{a_k}}{y_k^{a_0 + \dots + a_k}} \\ &= 1 + M_1 \left(\frac{y_0}{y_k}, \dots, \frac{y_{k-1}}{y_k} \right) + M_2 \left(\frac{y_0}{y_k}, \dots, \frac{y_{k-1}}{y_k} \right) \\ &\quad + \dots + M_\ell \left(\frac{y_0}{y_k}, \dots, \frac{y_{k-1}}{y_k} \right) \\ &= \sum_{q=0}^{\ell} M_q \left(\frac{y_0}{y_k}, \dots, \frac{y_{k-1}}{y_k} \right). \end{aligned}$$

Therefore, the following limit converges by $|y_0| < \cdots < |y_k|$,

$$\begin{aligned}
\lim_{\ell \rightarrow \infty} \frac{M_\ell(y_0, \dots, y_k)}{y_k^\ell} &= \sum_{q=0}^{\infty} M_q \left(\frac{y_0}{y_k}, \dots, \frac{y_{k-1}}{y_k} \right) \\
&= \prod_{p=0}^{k-1} \left[1 + \left(\frac{y_p}{y_k} \right) + \left(\frac{y_p}{y_k} \right)^2 + \left(\frac{y_p}{y_k} \right)^3 + \cdots \right] \\
&= \frac{1}{\prod_{p=0}^{k-1} \left(1 - \frac{y_p}{y_k} \right)} \\
(2.7) \qquad &= \frac{y_k^k}{\prod_{p=0}^{k-1} (y_k - y_p)}.
\end{aligned}$$

From (2.5) we get

$$\begin{aligned}
\lim_{m \rightarrow \infty} \frac{t_{ij}^{(m)}}{t_{jj}^m} &= \sum_{s=(s_0, \dots, s_k) \in S_{ij}} \left[\frac{f_s(T)}{t_{jj}^{|s|}} \cdot \frac{t_{jj}^{|s|}}{\prod_{p=0}^{|s|-1} (t_{jj} - t_{s_p s_p})} \right] \\
&= \sum_{(s_0, \dots, s_k) \in S_{ij}} \frac{\prod_{p=0}^{k-1} t_{s_p s_{p+1}}}{\prod_{p=0}^{k-1} (t_{jj} - t_{s_p s_p})}.
\end{aligned}$$

This completes the proof. \square

The above results can be used to analyze the coefficients and the u -components in the Gelfand-Naimark decomposition of $X^m = C^{-1}T^mC$ where T is an upper triangular matrix with ascending moduli diagonal entries.

Suppose that X has distinct eigenvalue moduli: $|\lambda_1| < |\lambda_2| < \cdots < |\lambda_n|$. There are many possible ways to decompose $X = C^{-1}TC$ where the upper triangular matrix T has ascending moduli diagonal entries. For example, Jordan decomposition and Schur triangularization provide two different such decompositions. However, the permutation ω obtained from the Bruhat decomposition of C is uniquely determined by X . This could be seen from (1.2) and the fact that X has distinct eigenvalue moduli.

Because X has distinct eigenvalue moduli and thus is diagonalizable, it has a decomposition $X = C^{-1}TC$ where $T = \lambda(X)$ is an ascending moduli diagonal matrix. The equality $X^m = C^{-1}\lambda(X)^mC$ implies that

$$(2.8) \qquad x_{ij}^{(m)} = \sum_{p=1}^n c'_{ip} c_{pj} \lambda_p^m.$$

So $x_{11}^{(m)} = \sum_{p=1}^n c'_{1p} c_{p1} \lambda_p^m$. Suppose that C has the Bruhat decomposition $C = V' \omega U'$ for certain upper triangular matrices V' and U' , then $\omega(1)$ is the largest q such that $c_{q1} \neq 0$ in C . Thus

$$(2.9) \quad x_{11}^{(m)} = \sum_{p=1}^{\omega(1)} c'_{1p} c_{p1} \lambda_p^m.$$

Since $\sum_{p=1}^{\omega(1)} c'_{1p} c_{p1} = \sum_{p=1}^n c'_{1p} c_{p1} = 1$, there exists the largest integer $r \leq \omega(1)$ such that $c'_{1r} c_{r1} \neq 0$. Equality (2.9) implies that $x_{11}^{(m)} \neq 0$ when m is sufficiently large. Moreover, the following lemma holds:

Lemma 2.6. *Suppose that $X = C^{-1} \lambda(X) C$ has distinct eigenvalue moduli. Then*

$$(2.10) \quad \lim_{m \rightarrow \infty} |u_1(X^m)|^{1/m} = |\lambda_r|$$

where $r \leq \omega(1)$ is the largest integer such that $c'_{1r} c_{r1} \neq 0$.

As shown below, $r = \omega(1)$ in a Zariski dense open set. The Zariski topology on $\mathrm{GL}_n(\mathbb{C})$ is defined in the way that a Zariski closed set is the zeros of a set of polynomials on the matrix coefficients and \det^{-1} [2, page 7].

Lemma 2.7. *Suppose that Λ is a diagonal matrix with strictly ascending moduli, and ω is a permutation matrix, both in $\mathrm{GL}_n(\mathbb{C})$. Let $\mathcal{O}_{\Lambda, \omega}$ denote the set of all matrices $X = C^{-1} \Lambda C$ where C has a Bruhat decomposition $C = V' \omega U'$ for some upper triangular matrices V' and U' . Then on a Zariski dense open subset of the ω -orbit in the Bruhat decomposition of C ,*

$$\lim_{m \rightarrow \infty} |u_1(X^m)|^{1/m} = \lim_{m \rightarrow \infty} |x_{11}^{(m)}|^{1/m} = |\lambda_{\omega(1)}|.$$

Proof. The coefficient $c_{\omega(1)1}$ is always nonzero by the Bruhat decomposition $C = V' \omega U'$. Therefore, (2.9) implies that whenever $c'_{1\omega(1)} \neq 0$,

$$\lim_{m \rightarrow \infty} |u_1(X^m)|^{1/m} = \lim_{m \rightarrow \infty} |x_{11}^{(m)}|^{1/m} = |\lambda_{\omega(1)}|.$$

Clearly, $c'_{1\omega(1)} \neq 0$ if and only if the cofactor of the $(\omega(1), 1)$ entry of C is nonzero. This forms a Zariski dense open subset of the ω -orbit in the Bruhat decomposition of C . \square

Let $C_k(X)$ denote the k -compound matrix of $X \in \mathrm{GL}_n(\mathbb{C})$, $k = 1, \dots, n$. Let $Q_{k,n}$ denote the set of all k -subsequences of $\{1, 2, \dots, n\}$. The entries of $C_k(X)$ are of the form $x_{\alpha,\beta}$ ($\alpha, \beta \in Q_{k,n}$), where $x_{\alpha,\beta}$ is the determinant of the submatrix formed by the α -rows and the β -columns of X . So $C_k(X) \in \mathrm{GL}_{\binom{n}{k}}(\mathbb{C})$. The k -compound of a permutation (resp. diagonal, upper triangular, lower triangular) matrix is still a permutation (resp. diagonal, upper triangular, lower triangular) matrix. Therefore, the k -compound preserves the Gelfand-Naimark decomposition and the Bruhat decomposition.

Assume that the k -compound of X has distinct eigenvalue moduli for every k . In other words, $\prod_{i \in \alpha} |\lambda_i(X)|$ for all $\alpha \in Q_{k,n}$ are mutually distinct. Then Lemma 2.6 and Lemma 2.7 can be extended to a product of $u_i(X)$ by the compound matrix technique.

Theorem 2.8. *Suppose that the k -compound of $X \in \mathrm{GL}_n(\mathbb{C})$ has distinct eigenvalue moduli for every $k = 1, \dots, n$. Then X^m has LU decomposition when m is sufficiently large, and*

$$(2.11) \quad \lim_{m \rightarrow \infty} |u_1(X^m) \cdots u_k(X^m)|^{1/m} = \prod_{i \in \alpha_k} |\lambda_i| \quad \text{for some } \alpha_k \in Q_{k,n}.$$

In particular, $\lim_{m \rightarrow \infty} |u_k(X^m)|^{1/m}$ exists for $k = 1, \dots, n$.

Proof. Suppose that $X = C^{-1}\lambda(X)C$. Let $X(k|k)$ be the submatrix formed by the first k rows and the first k columns of X . By direct computation on $X^m = C^{-1}\lambda(X)^m C$, the $(1,1)$ entry in $C_k(X)^m = C_k(X^m)$ is

$$(2.12) \quad \det X^m(k|k) = \sum_{\alpha \in Q_{k,n}} \left(m_\alpha \prod_{i \in \alpha} \lambda_i^m \right) = \sum_{\alpha \in Q_{k,n}} \left[m_\alpha \left(\prod_{i \in \alpha} \lambda_i \right)^m \right]$$

where m_α are constants related to the first k -rows of C^{-1} and the first k -columns of C . Clearly $\sum_{\alpha \in Q_{k,n}} m_\alpha = 1$ by setting all $\lambda_i = 1$ in (2.12). By assumption, $\{\prod_{i \in \alpha} |\lambda_i| \mid \alpha \in Q_{k,n}\}$ are mutually distinct. There is one $\alpha_k \in Q_{k,n}$ such that $m_{\alpha_k} \neq 0$ and $\prod_{i \in \alpha_k} |\lambda_i|$ is maximal. Formula (2.12) implies that $\det X^m(k|k) \neq 0$ when m is sufficiently large. Moreover,

$$\begin{aligned} \lim_{m \rightarrow \infty} |u_1(X^m) \cdots u_k(X^m)|^{1/m} &= \lim_{m \rightarrow \infty} |u_1(C_k(X^m))|^{1/m} \\ &= \lim_{m \rightarrow \infty} |\det X^m(k|k)|^{1/m} = \prod_{i \in \alpha_k} |\lambda_i|. \end{aligned}$$

This completes the proof. \square

Remark 2.9. It is not necessarily true that $\alpha_{k-1} \subseteq \alpha_k$. So the limit $\lim_{m \rightarrow \infty} |u_k(X^m)|^{1/m}$ in Theorem 2.8 may not equal to an eigenvalue modulus of X . An example is presented in Example 3.1 (3).

Theorem 2.10. *Suppose that $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n) \in \text{GL}_n(\mathbb{C})$ is a diagonal matrix with strictly ascending moduli, and the elements in $\{\prod_{i \in \alpha} |\lambda_i| \mid \alpha \in Q_{k,n}\}$ are mutually distinct for every $k = 1, 2, \dots, n$. Suppose that $\omega \in \text{GL}_n(\mathbb{C})$ is a permutation matrix. Let $\mathcal{O}_{\Lambda, \omega}$ denote the set of all matrices $X = C^{-1}\Lambda C$ where C has a Bruhat decomposition $C = V'\omega U'$ for some upper triangular matrices V' and U' . Then on a Zariski dense open subset of the ω -orbit in the Bruhat decomposition of C ,*

$$\lim_{m \rightarrow \infty} |u_k(X^m)|^{1/m} = |\lambda_{\omega(k)}| \quad \text{for } k = 1, \dots, n.$$

Proof. The proof is done by applying Lemma 2.7 to the k -compound matrices of X for $k = 1, \dots, n$, and using the fact that the intersection of finitely many Zariski dense open subsets is still a Zariski dense open subset. Note that the diagonal entry moduli of $C_k(\Lambda)$ may not be in strictly ascending order. However, [7, Lemma 2.10] shows that: there exists a permutation matrix $P \in \text{GL}_{\binom{n}{k}}(\mathbb{C})$ such that the diagonal of $P^{-1}C_k(\Lambda)P$ is in ascending moduli order, and $P^{-1}C_k(V')P$ is still upper triangular for every upper triangular matrix $V' \in \text{GL}_n(\mathbb{C})$. Then

$$\begin{aligned} C_k(X) &= C_k(C)^{-1} C_k(\Lambda) C_k(C) \\ &= (P^{-1}C_k(C))^{-1} (P^{-1}C_k(\Lambda)P) (P^{-1}C_k(C)) \end{aligned}$$

where $P^{-1}C_k(C)$ has a Bruhat decomposition

$$P^{-1}C_k(C) = (P^{-1}C_k(V')P) (P^{-1}C_k(\omega)) C_k(U').$$

This leads to the proof of Theorem 2.10. \square

3. EXAMPLES

Let $X \in \text{GL}_n(\mathbb{C})$ such that the k -compound of X has distinct eigenvalue moduli for $k = 1, \dots, n$. By Theorem 2.8, $\lim_{m \rightarrow \infty} |u_k(X^m)|^{1/m}$ exists for $k = 1, \dots, n$. Example 3.1 indicates that $\lim_{m \rightarrow \infty} |u_k(X^m)|^{1/m}$ may or may not equal to an eigenvalue modulus of X .

Example 3.1. Consider the following two situations:

(1) Suppose $X = C^{-1}TC$ for $C := \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \in \mathrm{SU}_2(\mathbb{C})$

and $T := \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Then $\omega = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for a Bruhat decomposition $C = V'\omega U'$ of C . We have

$$X^m = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^m \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2^m & 2^m - 1 \\ 0 & 1 \end{bmatrix}.$$

It turns out that

$$\begin{aligned} \lim_{m \rightarrow \infty} |u_1(X^m)|^{1/m} &= 2 = |\lambda_{\omega(1)}|, \\ \lim_{m \rightarrow \infty} |u_2(X^m)|^{1/m} &= 1 = |\lambda_{\omega(2)}|. \end{aligned}$$

(2) Suppose $X = C^{-1}TC$ for $C := \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \in \mathrm{SU}_2(\mathbb{C})$

and $T := \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Then $\omega = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for a Bruhat decomposition $C = V'\omega U'$ of C . We have

$$X^m = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^m \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 - 2^m & 2^m \end{bmatrix}.$$

Clearly

$$\begin{aligned} \lim_{m \rightarrow \infty} |u_1(X^m)|^{1/m} &= 1 \neq |\lambda_{\omega(1)}| = 2, \\ \lim_{m \rightarrow \infty} |u_2(X^m)|^{1/m} &= 2 \neq |\lambda_{\omega(2)}| = 1. \end{aligned}$$

(3) Suppose $X = C^{-1}TC$ for

$$C := \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \quad T := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Then $C^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$, and

$$\begin{aligned} X^m &= \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1^m & 0 & 0 \\ 0 & 2^m & 0 \\ 0 & 0 & 3^m \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 + 2^m & -1 + 2^m & 1 - 2^m \\ -1 + 3^m & 1 + 3^m & -1 + 3^m \\ -2^m + 3^m & -2^m + 3^m & 2^m + 3^m \end{bmatrix}. \end{aligned}$$

It is clear that X^m has LU decomposition and

$$\begin{aligned}\lim_{m \rightarrow \infty} |u_1(X^m)|^{1/m} &= \lim_{m \rightarrow \infty} \left| \frac{1+2^m}{2} \right|^{1/m} = 2, \\ \lim_{m \rightarrow \infty} |u_1(X^m)u_2(X^m)|^{1/m} &= \lim_{m \rightarrow \infty} \left| \frac{\frac{1+2^m}{2}}{\frac{-1+3^m}{2}} \frac{-1+2^m}{\frac{1+3^m}{2}} \right|^{1/m} = 3.\end{aligned}$$

Therefore, $\lim_{m \rightarrow \infty} |u_2(X^m)|^{1/m} = \frac{3}{2}$ is not an eigenvalue of X .

The next example shows that: if the k -compound of X has no distinct eigenvalue moduli for certain k , then $\lim_{m \rightarrow \infty} |u_i(X^m)|^{1/m}$ may not exist. A lemma is needed to illustrate the example.

Lemma 3.2. *For every $t \in (0, 1)$, there exists an irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and two positive integer sequences $\{p_k\}_{k=1}^\infty \subseteq \mathbb{Z}^+$ and $\{m_k\}_{k=1}^\infty \subseteq \mathbb{Z}^+$, such that for $k = 1, 2, \dots$,*

- (1) m_k is divisible by k , and
- (2) $\alpha \in I_k := \left(\frac{p_k + 1/4}{m_k}, \frac{p_k + 1/4 + t^{m_k}}{m_k} \right)$.

Proof. First we use induction to construct $\alpha \in \mathbb{R}$, $\{p_k\}_{k=1}^\infty \subseteq \mathbb{Z}^+$ and $\{m_k\}_{k=1}^\infty \subseteq \mathbb{Z}^+$ that satisfy (1) and (2). Then we show that α must be irrational.

Denote $m_1 = p_1 = 1$ and $I_1 := (1 + 1/4, 1 + 1/4 + t)$. Suppose that $m_k, p_k \in \mathbb{Z}^+$, and

$$I_k := \left(\frac{p_k + 1/4}{m_k}, \frac{p_k + 1/4 + t^{m_k}}{m_k} \right) \subseteq \mathbb{R}^+$$

are well-defined. Then there exists a sufficiently large $m_{k+1} \in \mathbb{Z}^+$ and a suitable $p_{k+1} \in \mathbb{Z}^+$ such that $k+1$ divides m_{k+1} and the closure of

$$I_{k+1} := \left(\frac{p_{k+1} + 1/4}{m_{k+1}}, \frac{p_{k+1} + 1/4 + t^{m_{k+1}}}{m_{k+1}} \right)$$

is contained in I_k . By induction we obtain $\{p_k\}_{k=1}^\infty \subseteq \mathbb{Z}^+$, $\{m_k\}_{k=1}^\infty \subseteq \mathbb{Z}^+$, and the open interval sequence $\{I_k\}_{k=1}^\infty$ such that

$$\bar{I}_1 \supset I_1 \supset \bar{I}_2 \supset I_2 \supset \bar{I}_3 \supset I_3 \supset \bar{I}_4 \supset \dots$$

Because the lengths of closed intervals in $\{\bar{I}_k\}_{k=1}^\infty$ are decreasing to 0, by the Nested Interval Theorem [1], $\bigcap_{i=1}^\infty \bar{I}_i = \bigcap_{i=1}^\infty I_i$ contains exactly one number $\alpha \in \mathbb{R}^+$.

We show that $\alpha \notin \mathbb{Q}$. Suppose on the contrary, $\alpha = \frac{a}{b}$ where $a, b \in \mathbb{Z}^+$. By simultaneously multiplying a positive integer c on a and b , we

may assume that b is large enough so that $4t^{m_b} < 4t^b < 1$. Then

$$\alpha \in I_b = \left(\frac{p_b + 1/4}{m_b}, \frac{p_b + 1/4 + t^{m_b}}{m_b} \right).$$

Therefore,

$$(3.1) \quad 4m_b\alpha \in (4p_b + 1, 4p_b + 1 + 4t^{m_b}) \subset (4p_b + 1, 4p_b + 2).$$

However, $4m_b\alpha \in \mathbb{Z}^+$ since $\alpha = \frac{a}{b}$ and m_b is an integer multiple of b . This contradicts to (3.1). Thus $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and we are done. \square

Example 3.3. Fix a number $t \in (0, 1)$. Let $\theta := 2\pi\alpha$ where α is given in Lemma 3.2. Denote $X := \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$. Then $X^m =$

$$\begin{bmatrix} \cos m\theta & \sin m\theta \\ -\sin m\theta & \cos m\theta \end{bmatrix}. \text{ Obviously } \cos m\theta \neq 0 \text{ for all } m \in \mathbb{Z}^+ \text{ and}$$

$\overline{\{\cos m\theta \mid m \in \mathbb{Z}^+\}} = [-1, 1]$ since $\alpha = \frac{\theta}{2\pi}$ is irrational. So every X^m

has LU decomposition and $|u_1(X^m)|^{1/m} = |\cos m\theta|^{1/m}$. We claim that

$\lim_{m \rightarrow \infty} |u_1(X^m)|^{1/m} = \lim_{m \rightarrow \infty} |\cos m\theta|^{1/m}$ does not exist. On one hand,

it is easy to find a subsequence $\{n_1, n_2, \dots\} \subset \{1, 2, \dots\}$ such that

$\lim_{i \rightarrow \infty} |\cos n_i\theta|^{1/n_i} = 1$. On the other hand, let $\{m_k\}_{k=1}^\infty$ and $\{p_k\}_{k=1}^\infty$ be given as in Lemma 3.2. Then for $k = 1, 2, \dots$,

$$(3.2) \quad m_k\theta = 2m_k\pi\alpha \in \left(2p_k\pi + \frac{\pi}{2}, 2p_k\pi + \frac{\pi}{2} + 2\pi t^{m_k} \right).$$

Hence

$$|\cos m_k\theta|^{1/m_k} = \left| -\sin\left(m_k\theta - 2p_k\pi - \frac{\pi}{2}\right) \right|^{1/m_k} \leq |2\pi t^{m_k}|^{1/m_k} \rightarrow t.$$

Therefore, $\lim_{m \rightarrow \infty} |u_1(X^m)|^{1/m}$ does not exist.

REFERENCES

- [1] R.G. Bartle and D.R. Sherbert, *Introduction to Real Analysis*, Wiley, 1999.
- [2] R. Goodman and N.R. Wallach, *Representations and Invariants of the Classical Groups*, Cambridge University Press, 2000.
- [3] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge Univ. Press, 1991.
- [4] H. Huang and T.Y. Tam, An asymptotic behavior of QR decomposition, *Linear Algebra and Its Applications*, **424** (2007) 96–107.
- [5] H. Huang and T.Y. Tam, On Gelfand-Naimark decomposition of a nonsingular matrix, to appear, *Linear and Multilinear Algebra*.
- [6] H. Huang and T.Y. Tam, On the QR iterations of real matrices, *Linear Algebra and Its Applications*, **408** (2005) 161–176.

- [7] H. Huang and T.Y. Tam, Some asymptotic behaviors associated with matrix decomposition, *International J. of Information & Systems Sciences*, (2008) 148–159.
- [8] E. Tyrtysnikov, Matrix Bruhat decompositions with a remark on the QR (GR) algorithm, *Linear Algebra and Its Applications*, **250** (1997) 61–68.
- [9] T. Yamamoto, Extreme values of the roots of matrices, *J. Math. Soc. Japan*, **19** (1967) 171–178.

DEPARTMENT OF MATHEMATICS AND STATISTICS, AUBURN UNIVERSITY, AL
36849–5310, USA

E-mail address: `huanghu@auburn.edu`