Unspecified Journal
Volume 00, Number 0, Pages 000–000
S ????-????(XX)0000-0

SOME ASYMPTOTIC BEHAVIORS ASSOCIATED WITH MATRIX DECOMPOSITION

HUAJUN HUANG AND TIN-YAU TAM

ABSTRACT. We obtain several asymptotic results on the powers of a square matrix associated with SVD, QR decomposition and Cholesky decomposition.

1. Yamamoto's theorem

Given $X \in \mathbb{C}_{n\times n}$, the eigenvalue moduli of X are always less than or equal to the spectral norm of X since

$$\|X\| := \max_{\|v\|_2=1} \|Xv\|_2$$

and if $Xv = \lambda v$ for some unit vector v, then $|\lambda| = \|Xv\|_2 \le \|X\|$. So for all $m \in \mathbb{N}$,

$$r(X) \le \|X^m\|^{1/m} \le \|X\|,$$

where $r(X)$ denotes the spectral radius of X. The following is the celebrated Beruling-Gelfand's theorem (the finite dimensional case) [9, p.235, p.379].

Theorem 1.1. Let $X \in \mathbb{C}_{n\times n}$. then

$$\lim_{m\to\infty} \|X^m\|^{1/m} = r(X). \tag{1.1}$$

Needless to say, (1.1) is true for all norms since they are all equivalent. We now provide an elementary proof which is different from those in [11, 7, 2, 8].

Proof. Since $\|\cdot\|$ is invariant under unitary similarity, by Schur triangularization theorem, we may assume that $X = T$ is upper triangular with ascending diagonal moduli $|t_{11}| \le \cdots \le |t_{nn}|$. When X is nilpotent, that is, $r(X) = 0$, (1.1) is obviously true. Hence we may assume that X is not nilpotent so that $r(X) = |t_{nn}| \ne 0$. Write $T^m = [t_{ij}^{(m)}] \in \mathbb{C}_{n\times n}$ which is also upper triangular. For $1 \le i \le j \le n$,

$$t_{ij}^{(m)} = \sum_{i=p_0\le p_1\le\cdots\le p_m=j} \prod_{\ell=1}^{m} t_{p_{\ell-1}p_\ell}. \tag{1.2}$$

Clearly $t_{ii}^{(m)} = t_{ii}^m$, $i = 1, \ldots, n$. Since both sides of (1.1) are homogenous, by appropriate scaling, we may assume that $|t_{nn}| \ge 1$. Let us estimate

2000 Mathematics Subject Classification. Primary 15A23, 15A42

$|t_{ij}^{(m)}|$ for fixed $i < j$. The number of $(m+1)$-tuples $(p_0, p_1, \cdots, p_m)$, where $i = p_0 \le p_1 \le \cdots \le p_m = j$ are integers, is equal to $\binom{j-i+m-1}{m-1}$. For each of such $(p_0, \cdots, p_m)$, there are at most $j - i$ numbers ℓ's in $\{1, \cdots, m\}$ such that $p_{\ell-1} \ne p_\ell$. Let

$$c := \max_{1 \le p,q \le n} |t_{pq}| \ge 1$$

denote the maximal entry modulus of T. By (1.2) and the fact that $|t_{11}| \le \cdots \le |t_{nn}|$, when $m \ge n$,

$$\begin{aligned} |t_{ij}^{(m)}| &\le \sum_{i=p_0 \le p_1 \le \cdots \le p_m = j} \prod_{\ell=1}^{m} |t_{p_{\ell-1} p_\ell}| \\ &\le \sum_{i=p_0 \le p_1 \le \cdots \le p_m = j} c^{j-i} |t_{jj}|^{m-j+i} \\ &= \binom{j-i+m-1}{m-1} c^{j-i} |t_{jj}|^{m-j+i} \\ &\le \binom{n+m-2}{m-1} c^{n-1} |t_{nn}|^m. \end{aligned} \tag{1.3}$$

By (1.3), for $m \ge n$,

$$|t_{nn}|^m \le \|T^m\| \le \sum_{j=1}^{n} \sum_{i=1}^{j} |t_{ij}^m| \le n^2 c^{n-1} \binom{n+m-2}{m-1} |t_{nn}|^m.$$

Taking m-th roots on all sides and taking limits for $m \to \infty$ lead to

$$\lim_{m \to \infty} \|T^m\|^{1/m} = |t_{nn}| = r(T).$$

$\square$

Suppose that the singular values $s_1(X), \ldots, s_n(X)$ of X and the eigenvalues $\lambda_1(X), \ldots, \lambda_n(X)$ of X are arranged in descending order

$$s_1(X) \ge s_2(X) \ge \cdots \ge s_n(X), \qquad |\lambda_1(X)| \ge |\lambda_2(X)| \ge \cdots \ge |\lambda_n(X)|.$$

Since $\|X\| = s_1(X)$ and $r(X) = |\lambda_1(X)|$, the following result of Yamamoto [11] is a direct generalization of (1.1):

$$\lim_{m \to \infty} [s_i(X^m)]^{1/m} = |\lambda_i(X)|, \quad i = 1, \ldots, n. \tag{1.4}$$

We now provide a slight extension of Yamamoto's result and the proof makes use of compound matrix like Yamamoto's original proof.

Theorem 1.2. Let $A, B, X \in \mathbb{C}_{n \times n}$ such that A, B are nonsingular. Then

$$\lim_{m \to \infty} [s_k(AX^m B)]^{1/m} = |\lambda_k(X)|, \quad k = 1, \ldots, n. \tag{1.5}$$

Proof. It suffices to prove (1.5) for $B = I_n$ since $AX^mB = AB(B^{-1}XB)^m$ and the spectrum of X and $B^{-1}XB$ are identical. We first establish the case $k = 1$. Since $\|\cdot\|$ is submultiplicative, for all $m \in \mathbb{N}$,

$$\frac{1}{\|A^{-1}\|^{1/m}}\|X^m\|^{1/m} \le \|AX^m\|^{1/m} \le \|A\|^{1/m}\|X^m\|^{1/m}.$$

Since $\|A^{-1}\|^{1/m}$ and $\|A\|^{1/m}$ converge to 1,

$$\lim_{m\to\infty}[s_1(AX^m)]^{1/m} = |\lambda_1(X)| \tag{1.6}$$

is established by (1.1).

Let $C_k(X)$ denote the k-th compound of X. It is well-known that

$$|\lambda_1(C_k(X))| = \prod_{i=1}^k |\lambda_i(X)|, \quad s_1(C_k(X)) = \prod_{i=1}^k s_i(X). \tag{1.7}$$

So by (1.6),

$$\lim_{m\to\infty}[\prod_{i=1}^k s_i(AX^m)]^{1/m} = \prod_{i=1}^k |\lambda_i(X)|. \tag{1.8}$$

When X is nilpotent, that is, $|\lambda_1(X)| = 0$, (1.5) is obviously true. We now assume that X is not nilpotent. Let X have k nonzero eigenvalues for some $1 \le k \le n$, that is,

$$|\lambda_1(X)| \ge \cdots \ge |\lambda_k(X)| > |\lambda_{k+1}(X)| = \cdots = |\lambda_n(X)| = 0.$$

By (1.8) for sufficiently large m, we have $\prod_{i=1}^t s_i(AX^m) > 0$ when $1 \le t \le k$. By (1.8), for $1 \le j \le k+1$

$$\begin{aligned}
\lim_{m\to\infty}[s_j(AX^m)]^{1/m} &= \lim_{m\to\infty}\left[\frac{\prod_{i=1}^j s_i(AX^m)}{\prod_{i=1}^{j-1} s_i(AX^m)}\right]^{1/m} \\
&= \frac{\lim_{m\to\infty}\left[\prod_{i=1}^j s_i(AX^m)\right]^{1/m}}{\lim_{m\to\infty}\left[\prod_{i=1}^{j-1} s_i(AX^m)\right]^{1/m}} \\
&= \frac{\prod_{i=1}^j |\lambda_i(X)|}{\prod_{i=1}^{j-1} |\lambda_i(X)|} = |\lambda_j(X)|.
\end{aligned}$$

In particular $\lim_{m\to\infty}[s_{k+1}(AX^m)]^{\frac{1}{m}} = 0$ and the proof is completed. □

Remark 1.3. The nonsingularity of A and B is clearly needed. For example if

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix},$$

then $r(X) = 3$ but $AX^mB = 0$ for all $m \in \mathbb{N}$.

Remark 1.4. One may use the inequality [3, p.178] $s_i(AB) \le s_1(A)s_i(B)$, $A, B \in \mathbb{C}_{n\times n}$ and continuity argument to have

$$s_n(A)s_i(X^m) \le s_i(AX^m) \le s_1(A)s_i(X^m), \quad i = 1, \ldots, n.$$

Then by (1.4) we arrive at (1.5).

Remark 1.5. Yamamoto's theorem (1.4) was extended in the context of semisimple Lie group [10]. It involves Cartan decomposition and complete multiplicative Jordan decomposition.

2. Asymptotic result for QR decomposition

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{C}^n$, that is, $\mathbf{e}_i$ has 1 as the only nonzero entry at the i-th position. We identify a permutation $\omega \in S_n$ with the unique permutation matrix (also written as ω) in the general linear group $\mathrm{GL}_n(\mathbb{C})$, where $\omega\mathbf{e}_i = \mathbf{e}_{\omega(i)}$. The matrix representation of ω under the standard basis is

$$\omega = \left[\, \mathbf{e}_{\omega(1)}, \cdots, \mathbf{e}_{\omega(n)} \,\right].$$

Given $X \in \mathrm{GL}_n(\mathbb{C})$, the well-known QR decomposition asserts that $X = QR$, where Q is unitary and R is upper triangular with positive diagonal entries. The decomposition is unique. Denote

$$a(X) := \mathrm{diag}\, R,$$

where $\mathrm{diag}\, A \in \mathbb{C}_{n\times n}$ denotes the diagonal matrix whose diagonal is that of $A \in \mathbb{C}_{n\times n}$. When $v \in \mathbb{C}_{n\times n}$, $\mathrm{diag}\, v$ means the diagonal matrix with diagonal v. Recently it was shown in [4] that given $A, B \in \mathrm{GL}_n(\mathbb{C})$, the limit

$$\lim_{t\to\infty} [a(AX^mB)]^{1/m}$$

exists and is related to the eigenvalue moduli of X. More precisely,

Theorem 2.1. [4] Let $A, B, X \in \mathrm{GL}_n(\mathbb{C})$. Let $X = Y^{-1}JY$ be the Jordan decomposition of X, where J is the Jordan form of X, $\mathrm{diag}\, J = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$ satisfying $|\lambda_1| \ge \cdots \ge |\lambda_n|$. Then

$$\lim_{m\to\infty} [a(AX^mB)]^{1/m} = \mathrm{diag}\,(|\lambda_{\omega(1)}|, \ldots, |\lambda_{\omega(n)}|),$$

where the permutation ω is uniquely determined by the $L\omega U$ decomposition of $YB = L\omega U$, where L is unit lower triangular and U is upper triangular.

The quantity

$$a(X) = \mathrm{diag}\, R$$

has a very nice geometric meaning: $a_i(X)$ is the distance (in 2-norm) between the i-th column of X and the span of the previous $i-1$ columns, $i = 1, \ldots, n$ (we adopt the convention that the span of a empty set is the zero space). Since all norms of $\mathbb{C}^n$ are equivalent, we have

Corollary 2.2. Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$ and $i = 1, \ldots, n$. The m-th root of the distance in $\|\cdot\|$ between the i-th column of AX^mB and the span of the previous $i-1$ columns of AX^mB converges to $|\lambda_{\omega(i)}|$.

The $L\omega U$ decomposition is known as Gelfand-Naimark decomposition [1, p.434].

Remark 2.3. There are some interesting features of the result.

(1) The limit is independent of A but depends on B.
(2) The permutation matrix ω is not unique since Y is not unique. However the limit remains the same.

See [4] for explanations.

Remark 2.4. One may want to consider singular X. However the QR decomposition of a singular $X \in \mathbb{C}_{n\times n}$ is not unique. Indeed the a-component of X is not unique either. For example:

$$X := \begin{bmatrix} 1 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{bmatrix}$$

Remark 2.5. Theorem 2.1 was extended in the context of semisimple Lie group [5] which involves Iwasawa decomposition, complete multiplicative Jordan decomposition and Bruhat decomposition.

Remark 2.6. One may not replace the Jordan decomposition in Theorem 2.1 by Schur (upper) triangularization decomposition (with descending diagonal moduli) to get ω. For example, consider

$$X := \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}.$$

Then

$$X = V^{-1}TV, \quad T = \begin{bmatrix} 3 & 1 & 1 \\ & 3 & 2 \\ & & 2 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

is the Schur (upper) triangularization decomposition of X, where T is a Schur triangular form of X, with descending diagonal moduli. Clearly $V = I_3 \cdot V \cdot I_3$ is the Gelfand-Naimark decomposition, that is, ω_V of V is the transposition (23) so that $\lambda_{\omega_V} := (\lambda_{\omega_V(1)}, \lambda_{\omega_V(2)}, \lambda_{\omega_V(3)}) = (3, 2, 3)$. The following is a Jordan decomposition of X:

$$X = Y^{-1}JY, \quad J = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 0.5 & 0 & 0.25 \\ 0 & 1 & 0.5 \\ 0 & 1 & 0 \end{bmatrix},$$

and the Gelfand-Naimark decomposition of $Y = L\omega_Y U$ is given as

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 0.5 & 0 & 0.25 \\ 0 & 1 & 0.5 \\ 0 & 0 & -0.5 \end{bmatrix}$$

and $\omega_Y = I_3$ so that $\lambda_{\omega_Y} = (3, 3, 2)$.

The above example fails because of a "bad" choice of Schur decomposition as we shall see in Theorem 2.7. There is a variant of Gelfand-Naimark decomposition. Each matrix $Y \in \mathrm{GL}_n(\mathbb{C})$ can be written as

$$Y = V\omega U$$

where ω is a permutation matrix, V is unit upper triangular and U is upper triangular. The permutation ω is unique in the decomposition. It can easily be deduced from the Gelfand-Naimark decomposition because if

$$K := \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix} \in \mathbb{C}_{n\times n},$$

the Gelfand-Naimark decomposition for KY yields $KY = L\omega' U$. Then $Y = KLK(K\omega')U$ since $K^{-1} = K$. Notice that $V := KLK$ is unit upper triangular. Set $\omega = K\omega'$.

We have the following theorem.

Theorem 2.7. Suppose $X = Y^{-1}TY \in \mathrm{GL}_n(\mathbb{C})$, where $T \in \mathbb{C}_{n\times n}$ is an upper triangular matrix, and the diagonal entries of T satisfy that $|t_{11}| \le |t_{22}| \le \cdots \le |t_{nn}|$, so that $\lambda_k := t_{kk}$ are the eigenvalues of X with $|\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_n|$. Then

$$\lim_{m\to\infty} [a(AX^m B)]^{1/m} = \mathrm{diag}\,(|\lambda_{\omega(1)}|, \ldots, |\lambda_{\omega(n)}|), \tag{2.1}$$

where ω is a permutation determined by the decomposition $YB = V\omega U$ of YB, such that V is unit upper triangular and U is upper triangular.

Corollary 2.8. Suppose $X = Y^{-1}TY \in \mathrm{GL}_n(\mathbb{C})$ where $T \in \mathbb{C}_{n\times n}$ is a lower triangular matrix and the diagonal entries of T satisfy that $|t_{11}| \ge |t_{22}| \ge \cdots \ge |t_{nn}|$, so that $\lambda_k := t_{kk}$ are the eigenvalues of X with $|\lambda_1| \ge |\lambda_2| \ge \cdots \ge |\lambda_n|$. Then

$$\lim_{m\to\infty} [a(AX^m B)]^{1/m} = \mathrm{diag}\,(|\lambda_{\omega(1)}|, \ldots, |\lambda_{\omega(n)}|),$$

where ω is a permutation determined by the Gelfand-Naimark decomposition $YB = L\omega U$ of YB, such that L is unit lower triangular and U is upper triangular.

Proof. Notice that $X = Y^{-1}TY = Y^{-1}K(KTK)KY$, where KTK is upper triangular with ascending diagonal moduli $|\mu_1| \le \cdots \le |\mu_n|$ and $\mu_k := \lambda_{n-k+1}$, $k = 1, \ldots, n$. Moreover $KYB = (KLK)K\omega U$ in which KLK is unit upper triangular. Apply Theorem 2.7 to have

$$\lim_{m\to\infty} [a(AX^m B)]^{1/m} = \mathrm{diag}\,(|\mu_{K\omega(1)}|, \ldots, |\mu_{K\omega(n)}|) = \mathrm{diag}\,(|\lambda_{\omega(1)}|, \ldots, |\lambda_{\omega(n)}|).$$

∎

To prove Theorem 2.7 we need two lemmas on compound matrices. Suppose $1 \le k \le n$ and let

$$Q_{k,n} := \{\omega = (\omega(1), \dots, \omega(k)) : 1 \le \omega(1) < \cdots < \omega(k) \le n\}$$

be the set of strictly increasing sequences of length k chosen from $1, \dots, n$. The number of elements in $Q_{k,n}$ is $\binom{n}{k}$. The elements of $Q_{k,n}$ are ordered by the lexicographic order $\preceq$ (a total order) and traditionally the entries of the compound matrix $C_k(X)$ are indexed by $Q_{k,n}$. Let $\sigma : \{1, 2, \dots, \binom{n}{k}\} \to Q_{k,n}$ be the natural bijection such that $i < j$ if and only if $\sigma(i) \prec \sigma(j)$. Let $\omega \trianglelefteq \gamma$ denote the partial order in $Q_{k,n}$: $\omega(t) \le \gamma(t)$ for all $t = 1, \dots, k$.

Lemma 2.9. Let $T \in \mathbb{C}_{n\times n}$ be an upper triangular matrix. Then the kth compound $C_k(T) =: T' = \left[\, t'_{ij} \,\right] \in \mathbb{C}_{\binom{n}{k}\times\binom{n}{k}}$ is an upper triangular matrix. Moreover, $t'_{ij} \ne 0$ only if $\sigma(i) \trianglelefteq \sigma(j)$.

Proof. One has $t'_{ij} = \det T[\sigma(i)|\sigma(j)]$, where $T[\sigma(i)|\sigma(j)]$ denotes the submatrix of T by choosing the rows of T corresponding to $\sigma(i) \in Q_{k,n}$ and the columns of T corresponding to $\sigma(j) \in Q_{k,n}$. It suffices to prove that if $\sigma(i) \ntrianglelefteq \sigma(j)$ then $\det T[\sigma(i)|\sigma(j)] = 0$ for $i, j \in \{1, \cdots, \binom{n}{k}\}$.

Denote $\sigma_i := \sigma(i)$ and $\sigma_j := \sigma(j)$. Then $\sigma_i \ntrianglelefteq \sigma_j$ means that there is $t \in \{1, \cdots, k\}$ such that $\sigma_i(t) > \sigma_j(t)$. So

$$\sigma_i(k) > \sigma_i(k-1) > \cdots > \sigma_i(t) > \sigma_j(t) > \cdots > \sigma_j(2) > \sigma_j(1).$$

The lower left $(k+1-t)\times t$ submatrix of $T[\sigma_i|\sigma_j]$ is also the submatrix of T corresponding to the $\sigma_i(t), \cdots, \sigma_i(k)$ rows and the $\sigma_j(1), \cdots, \sigma_j(t)$ columns of T, which is a zero matrix. So the first t columns of $T[\sigma_i|\sigma_j]$ span a space of dimension no more than $k-(k+1-t) = t-1$. Therefore, $\det T[\sigma_i|\sigma_j] = 0$. □

Lemma 2.10. Suppose that $T \in \mathbb{C}_{n\times n}$ is upper triangular with ascending diagonal moduli $|t_{11}| \le \cdots \le |t_{nn}|$. Fix $1 \le k \le n$. Then there is an $\binom{n}{k}\times\binom{n}{k}$ permutation matrix P (depending on $|t_{11}|, \cdots, |t_{nn}|$) such that $P^{-1}C_k(T)P$ is upper triangular with ascending diagonal moduli and $P^{-1}C_k(S)P$ is upper triangular for all upper triangular matrices $S \in \mathbb{C}_{n\times n}$.

Proof. By Lemma 2.9, $C_k(T) =: T' = \left[\, t'_{ij} \,\right] \in \mathbb{C}_{\binom{n}{k}\times\binom{n}{k}}$ is a upper triangular matrix, with $t'_{ij} = \det T[\sigma(i)|\sigma(j)]$. Let P be the permutation of shortest length (that is, smallest number of transposition in its transposition decomposition) such that the diagonal moduli of $P^{-1}C_k(T)P$, namely, $|t'_{P(i)P(i)}|$, $i = 1, \dots, \binom{n}{k}$, are in ascending order. Then P rearranges $t'_{11}, \cdots, t'_{\binom{n}{k}\binom{n}{k}}$ in the way that it moves the smallest modulus term(s) to the beginning position(s), then moves the second smallest modulus term(s) to the consecutive position(s), and then moves the third smallest modulus term(s), and so on. Moreover, the fact that P has the shortest possible length among all such rearrangement(s) implies that if the i-th and the j-th diagonal terms of $P^{-1}C_k(T)P$ (namely $t'_{P(i)P(i)}$ and $t'_{P(j)P(j)}$) have the same modulus for certain $i > j$, then the positions of these two terms in the diagonal of $C_k(T)$ (namely the $P(i)$-th and $P(j)$-th positions) satisfy $P(i) > P(j)$.

With these facts in mind, we claim that if $S \in \mathbb{C}_{n\times n}$ is upper triangular, then $P^{-1}C_k(S)P \in \mathbb{C}_{\binom{n}{k}\times\binom{n}{k}}$ is also upper triangular. Now $C_k(S) = [s'_{ij}] \in \mathbb{C}_{\binom{n}{k}\times\binom{n}{k}}$ where $s'_{ij} = \det S[\sigma(i)|\sigma(j)]$ for $i, j \in \{1, \cdots, \binom{n}{k}\}$, and

$$P^{-1}C_k(S)P = [s'_{P(i)P(j)}] \in \mathbb{C}_{\binom{n}{k}\times\binom{n}{k}}.$$

If $P^{-1}C_k(S)P$ were not upper triangular, then we would have $i > j$ in $\{1, \cdots, \binom{n}{k}\}$ such that $s'_{P(i)P(j)} \neq 0$. So $\sigma(P(i)) \trianglelefteq \sigma(P(j))$ by Lemma 2.9. On one hand, we have $\sigma(P(i)) \preceq \sigma(P(j))$ and thus $P(i) \leq P(j)$. On the other hand,

$$|t'_{P(i)P(i)}| = |\det T[\sigma(P(i))|\sigma(P(i))]| = \prod_{r\in\sigma(P(i))} |t_{rr}| \leq \prod_{r\in\sigma(P(j))} |t_{rr}| = |t'_{P(j)P(j)}|$$

which would not occur because

(1) it is impossible that $|t'_{P(i)P(i)}| < |t'_{P(j)P(j)}|$ since $i > j$ and the diagonal moduli of $P^{-1}C_k(T)P$, namely, $|t'_{P(i)P(i)}|$, $i = 1, \ldots, \binom{n}{k}$, are in ascending order;
(2) it is also impossible that $|t'_{P(i)P(i)}| = |t'_{P(j)P(j)}|$ since this together with $i > j$ and $P(i) \leq P(j)$ contradicts the fact that P has the shortest possible length and the discussion at the end of the preceding paragraph.

This means that $P^{-1}C_k(S)P \in \mathbb{C}_{\binom{n}{k}\times\binom{n}{k}}$ is upper triangular. In particular, $P^{-1}C_k(T)P \in \mathbb{C}_{\binom{n}{k}\times\binom{n}{k}}$ is upper triangular. □

We now prove Theorem 2.7.

Proof. It suffices to prove (2.1) for $B = I_n$ since $AX^mB = AB(B^{-1}XB)^m$ and the spectrum of X and $B^{-1}XB$ are identical. Denote $a_k^{(m)} := a_k(AX^m)$, $k = 1, \ldots, n$. We will first prove that

$$\lim_{m\to\infty} [a_1^{(m)}]^{1/m} = |\lambda_{\omega(1)}| \tag{2.2}$$

and then use compound matrix. Notice that $a_1^{(m)} = \|AY^{-1}T^mY\mathbf{e}_1\|_2$ and

$$\frac{1}{\|(AY^{-1})^{-1}\|}\|T^mY\mathbf{e}_1\|_2 \leq \|AY^{-1}T^mY\mathbf{e}_1\|_2 \leq \|AY^{-1}\|\|T^mY\mathbf{e}_1\|_2$$

so (2.2) amounts to

$$\lim_{m\to\infty} \|T^mY\mathbf{e}_1\|_2^{1/m} = |\lambda_{\omega(1)}|. \tag{2.3}$$

Since $Y = V\omega U$,

$$\|T^mY\mathbf{e}_1\|_2 = \|T^mV\omega U\mathbf{e}_1\|_2 = |u_{11}|\|T^m\mathbf{v}_{\omega(1)}\|_2$$

where $\mathbf{v}_i$ denotes the i-th column of V. Since $|u_{11}| > 0$ is a constant, it suffices to show

$$\lim_{m\to\infty} \|T^m \mathbf{v}_{\omega(1)}\|_2^{1/m} = |\lambda_{\omega(1)}|. \tag{2.4}$$

Since both sides of (2.4) are homogenous, by an appropriate scaling, we may assume $|\lambda_i| \geq 1$ for all $i = 1, \dots, n$. To this end, write $T^m = [t_{ij}^{(m)}] \in \mathbb{C}_{n\times n}$ which is upper triangular since T is upper triangular.

Let $\mathbf{t}_j^{(m)}$ denote the j-th column of T^m. Since V is unit upper triangular, the vector $T^m\mathbf{v}_{\omega(1)}$ is a linear combination (with constant coefficients) of the first $\omega(1)$ columns of T^m, and the coefficient for the $\omega(1)$-th column of T^m in the linear combination is 1. On one hand, we have

$$|\lambda_{\omega(1)}|^m \leq \|T^m \mathbf{v}_{\omega(1)}\|_2. \tag{2.5}$$

On the other hand

$$\|T^m \mathbf{v}_{\omega(1)}\|_2 = \|\sum_{j=1}^{\omega(1)} \mathbf{t}_j^{(m)} v_{j\omega(1)}\|_2 \leq \sum_{j=1}^{\omega(1)} |v_{j\omega(1)}|\, \|\mathbf{t}_j^{(m)}\|_2 \leq b_{\omega(1)} \sum_{j=1}^{\omega(1)} \|\mathbf{t}_j^{(m)}\|_2,$$

where $b_{\omega(1)} := \max_{j=1,\dots,\omega(1)} |v_{j\omega(1)}| \geq 1$ is a constant. Notice that

$$\|\mathbf{t}_j^{(m)}\|_2 \leq \|\mathbf{t}_j^{(m)}\|_1 = \sum_{i=1}^{j} |t_{ij}^{(m)}|. \tag{2.6}$$

When $i \leq j$ and $m \geq n$, by $|t_{11}| \leq \cdots \leq |t_{nn}|$ and (1.3),

$$|t_{ij}^{(m)}| \leq \binom{j-i+m-1}{m-1} c^{j-i} |\lambda_j|^{m-j+i} \leq \binom{n+m-2}{m-1} c^{n-1} |\lambda_j|^m, \tag{2.7}$$

where $c := \max_{1\leq p,q\leq n} |t_{pq}| \geq 1$. Hence from (2.6) and (2.7)

$$\begin{aligned} \|T^m \mathbf{v}_{\omega(1)}\|_2 &\leq b_{\omega(1)} \sum_{j=1}^{\omega(1)} \sum_{i=1}^{j} \binom{n+m-2}{m-1} c^{n-1} |\lambda_j|^m \\ &\leq \binom{n+m-2}{m-1} b_{\omega(1)} \omega(1)^2 c^{n-1} |\lambda_{\omega(1)}|^m. \end{aligned} \tag{2.8}$$

So from (2.5) and (2.8),

$$\lim_{m\to\infty} \|T^m \mathbf{v}_{\omega(1)}\|_2^{1/m} = |\lambda_{\omega(1)}|,$$

or equivalently, (2.2) is established. If $AX^m = Q_m R_m$ is the QR decomposition of AX^m, then $C_k(AX^m) = C_k(Q_m)C_k(R_m)$ is the QR decomposition of $C_k(AX^m)$ so that

$$a_1(C_k(A)C_k(X)^m) = \prod_{i=1}^{k} a_i(AX^m) = \prod_{i=1}^{k} a_i^{(m)}.$$

Though $C_k(X) = C_k(Y)^{-1}C_k(T)C_k(Y)$ and $C_k(T)$ is upper triangular, we are not quite ready to apply (2.2) on

$$C_k(A)C_k(X)^m = C_k(AX^m).$$

It is because the diagonal moduli of $C_k(T)$ are not necessarily in ascending order. However by Lemma 2.10 there is an $\binom{n}{k} \times \binom{n}{k}$ permutation matrix P such that $P^{-1}C_k(T)P$ is upper triangular and $\operatorname{diag}(P^{-1}C_k(T)P)$ has descending diagonal moduli and $P^{-1}C_k(V)P$ is still unit upper triangular. Now

$$C_k(X) = (P^{-1}C_k(Y))^{-1}(P^{-1}C_k(T)P)(P^{-1}C_k(Y))$$

and

$$P^{-1}C_k(Y) = (P^{-1}C_k(V)P)(P^{-1}C_k(\omega))C_k(U).$$

The $(1,1)$ entry of

$$\begin{aligned} & (P^{-1}C_k(\omega))^{-1}[\operatorname{diag}(P^{-1}C_k(T)P)](P^{-1}C_k(\omega)) \\ = \; & C_k(\omega)^{-1}P(P^{-1}[\operatorname{diag} C_k(T)]P)P^{-1}C_k(\omega) \\ = \; & C_k(\omega)^{-1}C_k(\operatorname{diag} T)C_k(\omega) \\ = \; & C_k(\omega^{-1}(\operatorname{diag} T)\omega) \end{aligned}$$

is $\prod_{i=1}^k \lambda_{\omega(i)}$. By (2.2), for $1 \le k \le n$,

$$\lim_{m\to\infty} \prod_{i=1}^k |a_i^{(m)}|^{1/m} = \lim_{m\to\infty} (a_1(C_k(A)C_k(X)^m)^{1/m} = \prod_{i=1}^k |\lambda_{\omega(i)}|.$$

So

$$\begin{aligned} \lim_{m\to\infty} |a_k^{(m)}|^{1/m} &= \lim_{m\to\infty} \frac{\prod_{i=1}^k |a_i^{(m)}|^{1/m}}{\prod_{i=1}^{k-1} |a_i^{(m)}|^{1/m}} = \frac{\lim_{m\to\infty} \prod_{i=1}^k |a_i^{(m)}|^{1/m}}{\lim_{m\to\infty} \prod_{i=1}^{k-1} |a_i^{(m)}|^{1/m}} \\ &= \frac{\prod_{i=1}^k |\lambda_{\omega(i)}|}{\prod_{i=1}^{k-1} |\lambda_{\omega(i)}|} = |\lambda_{\omega(k)}|. \end{aligned}$$

This completes the proof. □

3. Cholesky component

Given a positive definite matrix $P \in \mathbb{C}_{n\times n}$, Cholesky decomposition asserts that

$$P = R^*R,$$

where $R \in \mathbb{C}_{n\times n}$ is an upper triangular matrix with positive diagonal entries and the decomposition is unique. Let

$$d(P) = (\operatorname{diag} R)^2.$$

The spectral theorem asserts that there is a unitary matrix $Y \in \mathbb{C}_{n\times n}$ such that

$$P = Y^{-1}\Lambda Y$$

where $\Lambda := \mathrm{diag}\,(\lambda_1, \dots, \lambda_n)$ with descending positive diagonal entries. So the power P^m remains positive definite and thus $d(P^m)$ is well defined.

Theorem 3.1. Let $P \in \mathbb{C}_{n\times n}$ be positive definite. Let $P = Y^{-1}\Lambda Y$ where $Y \in \mathbb{C}_{n\times n}$ is unitary and $\Lambda = \mathrm{diag}\,(\lambda_1, \dots, \lambda_n)$, where $\lambda_1 \geq \cdots \geq \lambda_n > 0$. Then

$$\lim_{m\to\infty} d(P^m)^{1/m} = \mathrm{diag}\,(\lambda_{\omega(1)}, \dots, \lambda_{\omega(n)}) \tag{3.1}$$

where ω is the permutation uniquely determined by the Gelfand-Naimark decomposition $Y = L\omega U$.

Proof. Notice that $P^{1/2} = Y^{-1}\Lambda^{1/2}Y$. Let $P^{m/2} = Q_m R_m$ be the QR decomposition of $P^{m/2}$. Then

$$P^m = P^{m/2}P^{m/2} = (P^{m/2})^* P^{m/2} = R_m^* R_m.$$

So

$$d(P^m) = [a(P^{m/2})]^2 \tag{3.2}$$

and thus

$$\begin{aligned} \lim_{m\to\infty} d(P^m)^{1/m} &= \lim_{m\to\infty} [a(P^{m/2})]^{2/m} \\ &= (\lim_{m\to\infty} a([P^{1/2}]^m)^{1/m})^2 \\ &= [\mathrm{diag}\,(\lambda_{\omega(1)}^{1/2}, \dots, \lambda_{\omega(n)}^{1/2})]^2 \\ &= \mathrm{diag}\,(\lambda_{\omega(1)}, \dots, \lambda_{\omega(n)}) \end{aligned}$$

by Theorem 2.1 or Corollary 2.8. □

Remark 3.2. When P is positive semidefinite, neither the Cholesky decomposition of P nor $d(P)$ is unique. For example

$$P := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = P^*P = R^*R, \quad R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We conclude our discussion with a result on the d-component of the set of positive definite matrices with prescribed eigenvalues. Kostant's convexity theorem [6] asserts that if $\Lambda = \mathrm{diag}\,(\lambda_1, \dots, \lambda_n)$, where $\lambda_1, \dots, \lambda_n$ are positive, then

$$\{a(\Lambda Y) : Y \in \mathbb{C}_{n\times n} \text{ is unitary}\} = \exp(\mathrm{conv}\, S_n \log \Lambda),$$

where S_n is the full symmetric group on $\{1, \dots, n\}$, $S_n x := \{(x_{\omega(1)}, \dots, x_{\omega(n)}) : \omega \in S_n\}$, and conv denotes the convex hull of the underlying set.

Theorem 3.3. Let $\Lambda = \mathrm{diag}\,(\lambda_1, \dots, \lambda_n)$, where $\lambda_1, \dots, \lambda_n$ are positive, and $O(\Lambda) := \{Y^{-1}\Lambda Y : Y \in \mathbb{C}_{n\times n} \text{ is unitary}\}$. Then

$$d(O(\Lambda)) = \exp(\mathrm{conv}\, S_n \log \Lambda),$$

where $\log \lambda = \mathrm{diag}\,(\log \lambda_1, \dots, \log \lambda_n)$.

Proof.

$$\begin{aligned} d(O(\Lambda)) &= \{d(Y^{-1}\Lambda Y) : Y \text{ unitary}\} \\ &= \{[a(Y^{-1}\Lambda^{1/2}Y)]^2 : Y \text{ unitary}\} \\ &= \{[a(\Lambda^{1/2}Y)]^2 : Y \text{ unitary}\} \\ &= [\exp(\operatorname{conv} S_n \log \Lambda^{1/2})]^2 \\ &= \exp(\operatorname{conv} S_n \log \Lambda). \end{aligned}$$

by Kostant's convexity theorem. □

Remark 3.4. In terms of inequalities $d \in \exp(\operatorname{conv} S_n \log \Lambda)$ means multiplicative majorization

$$\max_{\sigma\in S_n} \prod_{i=1}^{k} d_\sigma(i) \leq \max_{\sigma\in S_n} \prod_{i=1}^{k} \lambda_{\sigma(i)}, \quad k = 1, \ldots, n-1,$$

and $\prod_{i=1}^n d_i = \prod_{i=1}^n \lambda_n$.

Remark 3.5. Theorem 3.1 and Theorem 3.3 can be extended to semisimple Lie group along the same reasoning [5]. For generalized Cholesky decomposition see [1, p.272-273].

References

[1] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.
[2] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge Univ. Press, 1985.
[3] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge Univ. Press, 1991.
[4] H. Huang and T.Y. Tam, An asymptotic behavior of QR decomposition, to appear in Linear Alg. Appl.
[5] H. Huang and T.Y. Tam, An asymptotic result on the a-component in Iwasawa decomposition, manuscript.
[6] B. Kostant, On convexity, the Weyl group and Iwasawa decomposition, Ann. Sci. Ecole Norm. Sup. (4), **6** (1973) 413–460.
[7] R. Mathias, Two theorems on singular values and eigenvalues, Amer. Math. Monthly **97** (1990), 47–50.
[8] D. Serre, *Matrices: Theory and Applications*, Springer, New York, 2002.
[9] W. Rudin, *Functional Analysis*, McGraw-Hill Inc, New York, 1991.
[10] T.Y. Tam and H. Huang, An extension of Yamamoto's theorem on the eigenvalues and singular values of a matrix, Journal of Math. Soc. Japan, **58** (2006) 1197–1202.
[11] T. Yamamoto, On the extreme values of the roots of matrices, J. Math. Soc. Japan, **19** (1967) 173–178.

Department of Mathematics and Statistics, Auburn University, AL 36849–5310, USA
E-mail address: `huanghu@auburn.edu, tamtiny@auburn.edu`