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ON KOSTANT'S PARTIAL ORDER ON HYPERBOLIC ELEMENTS

HUAJUN HUANG AND SANGJIB KIM

ABSTRACT. We study Kostant's partial order on the elements of a semisimple Lie group in relations with the finite dimensional representations. In particular, we prove the converse statement of [3, Theorem 6.1] on hyperbolic elements.

A matrix in $\mathrm{GL}_n(\mathbb{C})$ is called *elliptic* (resp. *hyperbolic*) if it is diagonalizable with norm 1 (resp. real positive) eigenvalues. It is called *unipotent* if all its eigenvalues are 1. The *complete multiplicative Jordan decomposition* of $g \in \mathrm{GL}_n(\mathbb{C})$ asserts that $g = ehu$ for $e, h, u \in \mathrm{GL}_n(\mathbb{C})$, where e is elliptic, h is hyperbolic, u is unipotent, and these three elements commute (cf. [2, p430-431]). The decomposition can be easily seen when g is in a Jordan canonical form: if the diagonal entries (i.e. eigenvalues) of the Jordan canonical form are $z_1, \cdots, z_n$, then

$$e = \mathrm{diag}\left(\frac{z_1}{|z_1|}, \cdots, \frac{z_n}{|z_n|}\right), \qquad h = \mathrm{diag}\left(|z_1|, \cdots, |z_n|\right), \tag{1}$$

and $u = h^{-1}e^{-1}g$ is an upper triangular matrix with diagonal entries 1.

The above decomposition can be extended to semisimple Lie groups. Let G be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}$. An element $e \in G$ is *elliptic* if $\mathrm{Ad}\, e \in \mathrm{Aut}\, \mathfrak{g}$ is diagonalizable over $\mathbb{C}$ with eigenvalues of modulus 1. An element $h \in G$ is called *hyperbolic* if $h = \exp X$ where $X \in \mathfrak{g}$ is real semisimple, that is, $\mathrm{ad}\, X \in \mathrm{End}\, \mathfrak{g}$ is diagonalizable over $\mathbb{R}$ with real eigenvalues. An element $u \in G$ is called *unipotent* if $u = \exp X$ where $X \in \mathfrak{g}$ is nilpotent, that is, $\mathrm{ad}\, X \in \mathrm{End}\, \mathfrak{g}$ is nilpotent. The *complete multiplicative Jordan decomposition* [3, Proposition 2.1] for G asserts that each $g \in G$ can be uniquely written as

$$g = ehu, \tag{2}$$

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where e is elliptic, h is hyperbolic and u is unipotent and the three elements e, h, u commute. We write $g = e(g)h(g)u(g)$.

When $G = \mathrm{SL}_n(\mathbb{C})$, the above decomposition in G coincides with the complete multiplicative Jordan decomposition in $\mathrm{GL}_n(\mathbb{C})$ defined in the first paragraph. Let us elaborate on it. Given $g \in \mathrm{SL}_n(\mathbb{C})$, there exists $y \in \mathrm{SL}_n(\mathbb{C})$ such that $ygy^{-1} \in \mathrm{SL}_n(\mathbb{C})$ is in a Jordan canonical form. Since the complete multiplicative Jordan decompositions of g in both $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{SL}_n(\mathbb{C})$ are preserved by conjugations in $\mathrm{SL}_n(\mathbb{C})$, without lost of generality, we may assume that $g \in \mathrm{SL}_n(\mathbb{C})$ is already in a Jordan canonical form with diagonal entries $z_1, z_2, \cdots, z_n$. Then the complete multiplicative Jordan decomposition of g in $\mathrm{GL}_n(\mathbb{C})$ is $g = ehu$, where e and h are given in (1) and $u = h^{-1}e^{-1}g$ is an upper triangular matrix with diagonal entries 1. Let $E_{ij} \in \mathfrak{gl}_n(\mathbb{C})$ denote the matrix with 1 in the (i, j)-entry and 0 elsewhere. Then with respect to the following basis of $\mathfrak{sl}_n(\mathbb{C})$:

$$\{E_{ij} \mid i \neq j,\ 1 \leq i, j \leq n\} \cup \{E_{ii} - E_{(i+1)(i+1)} \mid 1 \leq i \leq n-1\},$$

the matrix $\mathrm{Ad}\, e$ is diagonal with norm 1 eignevalues $z_i z_j^{-1}/|z_i z_j^{-1}|$ $(i \neq j)$ and 1. This shows that e is elliptic in $\mathrm{SL}_n(\mathbb{C})$. Clearly, $h = \exp X$ for a real semisimple element $X := \log h \in \mathfrak{sl}_n(\mathbb{C})$, and $u = \exp Y$ for a nilpotent element

$$Y := \log u = -\frac{I_n - u}{1} - \frac{(I_n - u)^2}{2} - \cdots - \frac{(I_n - u)^{n-1}}{n-1} \in \mathfrak{sl}_n(\mathbb{C}),$$

where I_n is the identity matrix. This shows that h (resp. u) is hyperbolic (resp. unipotent) in $\mathrm{SL}_n(\mathbb{C})$. Thus $g = ehu$ is also the complete multiplicative Jordan decomposition of g in $\mathrm{SL}_n(\mathbb{C})$.

Let us return to a connected real semisimple Lie group G with Lie algebra $\mathfrak{g}$. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (e.g., [2]), and let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subalgebra. Let $A := \exp \mathfrak{a}$. Then the hyperbolic component $h(g)$ of g is conjugate to some element $a(g) \in A$. See, for example, [3, Proposition 2.4]. We denote

$$A(g) := \exp(\mathrm{conv}(W \cdot \log a(g))), \tag{3}$$

where W is the Weyl group of $(\mathfrak{g}, \mathfrak{a})$ and $\mathrm{conv}(W \cdot \log a(g))$ is the convex hull of the Weyl group orbit of $\log a(g)$ in $\mathfrak{a}$.

Kostant defined a partial order on the elements of G as

$$g_1 \geq g_2 \iff A(g_1) \supseteq A(g_2), \tag{4}$$

and then established the following results:

Theorem 1. [3, Theorem 3.1] Let $g_1, g_2 \in G$. Then $g_1 \geq g_2$ if and only if $|\pi(g_1)| \geq |\pi(g_2)|$ for all finite dimensional representations π of G where $|\cdot|$ denotes the spectral radius.

Theorem 2. [3, Theorem 6.1] Let $h_1, h_2 \in G$ be hyperbolic. Write χ_π for the character of a representation π of G. If $h_1 \geq h_2$, then $\chi_\pi(h_1) \geq \chi_\pi(h_2)$ for all finite dimensional representations π of G.

In [3, Remark 6.1], Kostant asked if the converse of Theorem 2 is true or not. Our goal is to answer this question affirmatively and thus we have the following result:

Theorem 3. Let h_1 and h_2 be two hyperbolic elements in a connected real semisimple Lie group G. Then $h_1 \geq h_2$ if and only if $\chi_\pi(h_1) \geq \chi_\pi(h_2)$ for all finite dimensional representations π of G.

Before proving the above result, we need the following two lemmas.

Lemma 4. Let C and D be two diagonal matrices in $\mathrm{GL}_n(\mathbb{C})$ with positive diagonal entries. If the spectral radii $|C| > |D|$, then there exists a finite dimensional representation π of $\mathrm{GL}_n(\mathbb{C})$ such that $\chi_\pi(C) > \chi_\pi(D)$.

Proof. Let π_m be the representation of $\mathrm{GL}_n(\mathbb{C})$ on the m-th symmetric power of the natural representation $\mathrm{Sym}^m(\mathbb{C}^n)$. Then, the character χ_m of π_m takes the following value on $\mathrm{diag}\,(x) = \mathrm{diag}\,(x_1, x_2, \cdots, x_n) \in \mathrm{GL}_n(\mathbb{C})$:

$$\chi_m(\mathrm{diag}\,(x)) = \sum_{\substack{\ell_1,\ell_2,\cdots,\ell_n \geq 0 \\ \ell_1+\cdots+\ell_n=m}} \prod_{i=1}^{n} x_i^{\ell_i}.$$

So $\chi_m(\mathrm{diag}\,(x))$ is the sum of all $\binom{m+n-1}{n-1}$ monomials of $x_1, \cdots, x_n$ of degree m. Suppose $c := |C| > |D| =: d > 0$. Because $\lim_{m\to\infty} (m+n)^{n/m} = 1$, there exists a sufficiently large m such that

$$\frac{c}{d} > (m+n)^{n/m}.$$

Write $C := \operatorname{diag}(c_1, \cdots, c_n)$ and $D := \operatorname{diag}(d_1, \cdots, d_n)$. Then

$$\begin{aligned}\chi_m(C) &= \sum_{\substack{\ell_1,\ell_2,\cdots,\ell_n\geq 0\\ \ell_1+\cdots+\ell_n=m}} \prod_{i=1}^{n} c_i^{\ell_i} \\ &\geq c^m > (m+n)^n d^m > \binom{m+n-1}{n-1} d^m \\ &\geq \sum_{\substack{\ell_1,\ell_2,\cdots,\ell_n\geq 0\\ \ell_1+\cdots+\ell_n=m}} \prod_{i=1}^{n} d_i^{\ell_i} = \chi_m(D).\end{aligned}$$

This completes the proof. □

The following results are shown in [3, Proposition 3.4 and its proof]: for a representation $\pi : G \to \mathrm{GL}_n(\mathbb{C})$,

- If $u \in G$ is unipotent, then all the eigenvalues of $\pi(u)$ are equal to 1;
- If $h \in G$ is hyperbolic in G, then $\pi(h)$ is diagonalizable and all the eigenvalues of $\pi(h)$ are positive;
- If $e \in G$ is elliptic in G, then $\pi(e)$ is diagonalizable and all the eigenvalues of $\pi(e)$ are of norm 1;
- If $g = ehu$ is the complete multiplicative Jordan decomposition of g in G, then $\pi(g) = \pi(e)\pi(h)\pi(u)$ where $\pi(e)$, $\pi(h)$, and $\pi(u)$ mutually commute. Hence $\pi(g) = \pi(e)\pi(h)\pi(u)$ is the complete multiplicative Jordan decomposition of $\pi(g)$ in $\mathrm{GL}_n(\mathbb{C})$.

In addition, if G is connected and semisimple, then $\pi(G)$ is generated by

$$\exp(d\pi(\mathfrak{g})) = \exp([d\pi(\mathfrak{g}), d\pi(\mathfrak{g})]) \subseteq \exp \mathfrak{sl}_n(\mathbb{C}) = \mathrm{SL}_n(\mathbb{C}).$$

Therefore, $\pi(g) \in \mathrm{SL}_n(\mathbb{C})$ and $\pi(g) = \pi(e)\pi(h)\pi(u)$ is the complete multiplicative Jordan decomposition of $\pi(g)$ in $\mathrm{SL}_n(\mathbb{C})$. Consequently, we have

Lemma 5. Let g be an element of a connected real semisimple Lie group G, and $\pi : G \to \mathrm{GL}_n(\mathbb{C})$ a finite dimensional representation of G. If $g = ehu$ is the complete multiplicative Jordan decomposition of g in G, then $\pi(g) = \pi(e)\pi(h)\pi(u)$ is the complete multiplicative Jordan decomposition of $\pi(g)$ in $\mathrm{SL}_n(\mathbb{C})$.

Now we prove Theorem 3:

Proof of Theorem 3. The necessary part is Theorem 2. It remains to prove the sufficient part. From the assumption, we have $\chi_\pi(h_1) \geq \chi_\pi(h_2)$ for all finite dimensional representations π of G. Now suppose

$h_1 \not\geq h_2$ in G. Then Theorem 1 implies that there exists a finite dimensional representation $\eta : G \to \mathrm{GL}_n(\mathbb{C})$ of G such that $|\eta(h_1)| < |\eta(h_2)|$. By Lemma 5, both $\eta(h_1)$ and $\eta(h_2)$ are hyperbolic in $\mathrm{SL}_n(\mathbb{C})$. So they are conjugate to certain diagonal matrices with positive diagonal entries. Using Lemma 4 and the fact that character values are independent of conjugacy, there is a finite dimensional representation π_m of $\mathrm{GL}_n(\mathbb{C})$ with character χ_m such that

$$\chi_m(\eta(h_1)) < \chi_m(\eta(h_2)).$$

Now $\rho := \pi_m \circ \eta$ is a finite dimensional representation of G with character χ_ρ satisfying that

$$\chi_\rho(h_1) = \chi_m(\eta(h_1)) < \chi_m(\eta(h_2)) = \chi_\rho(h_2).$$

It contradicts our assumption $\chi_\pi(h_1) \geq \chi_\pi(h_2)$ for all π. □

Note that if $g \in \mathrm{SL}_n(\mathbb{C})$ has the complete multiplicative Jordan decomposition $g = e(g)h(g)u(g)$ in $\mathrm{SL}_n(\mathbb{C})$, then the hyperbolic component $h(g)$ of g is diagonalizable and its eigenvalues are equal to the eigenvalue moduli of g.

Given a finite dimensional representation π of G with character χ_π, let us denote by $|\chi_\pi|(g)$ the sum of eigenvalue moduli of $\pi(g)$ for $g \in G$. Then, from the above observation, we have

$$|\chi_\pi|(g) := \mathrm{tr}\,(h(\pi(g))). \tag{5}$$

Then Theorem 3 can be extended to an equivalent condition for the partial ordering on *all elements* of G.

Corollary 6. Let G be a connected real semisimple Lie group and $g_1, g_2 \in G$. Then $g_1 \geq g_2$ in G if and only if $|\chi_\pi|(g_1) \geq |\chi_\pi|(g_2)$ for all finite dimensional representations π of G.

Proof. By the definitions (3) and (4) of the partial order, $g_1 \geq g_2$ in G if and only if $h(g_1) \geq h(g_2)$ in G; if and only if $\chi_\pi(h(g_1)) \geq \chi_\pi(h(g_2))$ for all finite dimensional representations π of G (Theorem 3), namely

$$\mathrm{tr}\,(\pi(h(g_1))) \geq \mathrm{tr}\,(\pi(h(g_2))).$$

According to Lemma 5, $\pi(h(g)) = h(\pi(g))$ for all $g \in G$. Thus $\mathrm{tr}\,(\pi(h(g_1))) \geq \mathrm{tr}\,(\pi(h(g_2)))$ if and only if $\mathrm{tr}\,(h(\pi(g_1))) \geq \mathrm{tr}\,(h(\pi(g_2)))$, namely $|\chi_\pi|(g_1) \geq |\chi_\pi|(g_2)$. □

For a finite dimensional representation $\pi : G \to \mathrm{GL}_n(\mathbb{C})$, we denote by

$$\lambda_\pi^{(1)}(g) \geq \lambda_\pi^{(2)}(g) \geq \cdots \geq \lambda_\pi^{(n)}(g) > 0 \tag{6}$$

the eigenvalue moduli of $\pi(g)$ in non-increasing order for $g \in G$, that is, $\lambda_\pi^{(1)}(g) \geq \cdots \geq \lambda_\pi^{(n)}(g)$ are the eigenvalues of $h(\pi(g)) = \pi(h(g))$. Then Theorem 1 says that $g_1 \geq g_2$ in G if and only if $\lambda_\pi^{(1)}(g_1) \geq \lambda_\pi^{(1)}(g_2)$ for all finite dimensional representations π of G, and Corollary 6 says that $g_1 \geq g_2$ in G if and only if $\sum_{i=1}^n \lambda_\pi^{(i)}(g_1) \geq \sum_{i=1}^n \lambda_\pi^{(i)}(g_2)$ for all finite dimensional representations $\pi : G \to \mathrm{GL}_n(\mathbb{C})$. In general, we have the following result.

Theorem 7. *Let G be a connected real semisimple Lie group and $g_1, g_2 \in G$.*

(1) *If $g_1 \geq g_2$ in G, then for every finite dimensional representation $\pi : G \to \mathrm{GL}_n(\mathbb{C})$, the following inequalities hold for $k = 1, \cdots, n$:*

$$\prod_{i=1}^k \lambda_\pi^{(i)}(g_1) \geq \prod_{i=1}^k \lambda_\pi^{(i)}(g_2), \qquad \sum_{i=1}^k \lambda_\pi^{(i)}(g_1) \geq \sum_{i=1}^k \lambda_\pi^{(i)}(g_2).$$

(2) *Fix $k \in \mathbb{Z}^+$. For every finite dimensional representation $\pi : G \to \mathrm{GL}_n(\mathbb{C})$ with $n \geq k$, if we have*

$$\sum_{i=1}^k \lambda_\pi^{(i)}(g_1) \geq \sum_{i=1}^k \lambda_\pi^{(i)}(g_2),$$

then $g_1 \geq g_2$ in G.

(3) *Fix $k \in \mathbb{Z}^+$. For every finite dimensional representation $\pi : G \to \mathrm{GL}_n(\mathbb{C})$ with $n \geq k$, if we have*

$$\prod_{i=1}^k \lambda_\pi^{(i)}(g_1) \geq \prod_{i=1}^k \lambda_\pi^{(i)}(g_2),$$

then $g_1 \geq g_2$ in G.

Proof.

(1) Suppose that $g_1 \geq g_2$ in G. Given any finite dimensional representation $\pi : G \to \mathrm{GL}_n(\mathbb{C})$, let $\rho_k : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(\wedge^k \mathbb{C}^n)$ denote the fundamental representation of $\mathrm{GL}_n(\mathbb{C})$ on the k-th exterior power of $\mathbb{C}^n$, $k = 1, \cdots, n$. Then by Theorem 1,

$$|\rho_k(\pi(g_1))| \geq |\rho_k(\pi(g_2))|, \qquad k = 1, \cdots, n-1,$$

and by $\pi(G) \subseteq \mathrm{SL}_n(\mathbb{C})$,

$$|\rho_n(\pi(g_1))| = |\det(\pi(g_1))| = 1 = |\det(\pi(g_2))| = |\rho_n(\pi(g_2))|.$$

Equivalently,

$$\begin{aligned}\prod_{i=1}^{k} \lambda_\pi^{(i)}(g_1) &\geq \prod_{i=1}^{k} \lambda_\pi^{(i)}(g_2), \qquad k = 1, 2, \cdots, n-1,\\ \prod_{i=1}^{n} \lambda_\pi^{(i)}(g_1) &= \prod_{i=1}^{n} \lambda_\pi^{(i)}(g_2) = 1.\end{aligned}$$

So the vector of eigenvalue moduli of $\pi(g_1)$ multiplicatively majorizes that of $\pi(g_2)$. Moreover, multiplicative majorization of vectors with real positive numbers implies additive majorization [1, Example II.3.5(vii)]. Hence

$$\sum_{i=1}^{k} \lambda_\pi^{(i)}(g_1) \geq \sum_{i=1}^{k} \lambda_\pi^{(i)}(g_2), \qquad k = 1, 2, \cdots, n.$$

(2) Fix $k \in \mathbb{Z}^+$. Let $\pi : G \to \mathrm{GL}_n(\mathbb{C})$ be an arbitrary finite dimensional representation of G. Let $k\pi$ denote the representation formed by taking the direct sum of k copies of π. Then the representation space of $k\pi$ has dimension $kn \geq k$. By assumption,

$$k\lambda_\pi^{(1)}(g_1) = \sum_{i=1}^{k} \lambda_{k\pi}^{(i)}(g_1) \geq \sum_{i=1}^{k} \lambda_{k\pi}^{(i)}(g_2) = k\lambda_\pi^{(1)}(g_2).$$

So $|\pi(g_1)| \geq |\pi(g_2)|$. Thus $g_1 \geq g_2$ in G by Theorem 1.

(3) The proof is similar to that of (2).

□

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Department of Mathematics and Statistics, Auburn University, AL 36849–5310, USA

E-mail address: huanghu@auburn.edu

Department of Mathematics, The University of Arizona, Tucson, AZ 85721–0089, USA

E-mail address: sangjib@math.arizona.edu