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ON GELFAND-NAIMARK DECOMPOSITION OF A NONSINGULAR MATRIX

HUAJUN HUANG AND TIN-YAU TAM

ABSTRACT. Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ and $A \in \mathrm{GL}_n(\mathbb{F})$. Let $s(A) \in \mathbb{R}^n_+$ be the singular values of A, $\lambda(A) \in \mathbb{C}^n$ the unordered n-tuple of eigenvalues of A, $a(A) := \operatorname{diag} R \in \mathbb{R}^n_+$, where $A = QR$ is the QR decomposition of A, $u(A) := \operatorname{diag} U \in \mathbb{C}^n$, where $A = L\omega U$ is any Gelfand-Naimark decomposition. We obtain complete relations between (1) $u(A)$ and $a(A)$, (2) $u(A)$ and $s(A)$, (3) $u(A)$ and $\lambda(A)$, and (4) $a(A)$ and $\lambda(A)$. We also study the relations between any three elements among u, λ, a, s.

1. Introduction

Let $\mathrm{GL}_n(\mathbb{F})$ denote the group of $n \times n$ matrices over $\mathbb{F}$ where $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$. There are several sets of important scalars associated with $A \in \mathrm{GL}_n(\mathbb{F})$. The well-known QR decomposition asserts that

$$A = QR,$$

where $Q \in \mathrm{U}_n(\mathbb{F})$ and R is upper triangular with positive diagonal entries. Here $\mathrm{U}_n(\mathbb{F})$ denotes the group of unitary matrices if $\mathbb{F} = \mathbb{C}$ and orthogonal matrices if $\mathbb{F} = \mathbb{R}$. The decomposition is unique and is the matrix version of the Gram-Schmidt orthonormalization process. The first set of scalars is

$$a(A) := \operatorname{diag} R = (r_{11}, \cdots, r_{nn}) \in \mathbb{R}^n_+. \tag{1.1}$$

Here $\operatorname{diag} X$ denotes the diagonal of $X \in \mathbb{F}_{n\times n}$. Notice that $a_i(A) := r_{ii}$ is the distance in terms of 2-norm of the i-th column of A to the span of the previous $i-1$ columns of A, $i = 1, \dots, n$ (we adopt the convention that the span of the empty set is the zero space).

Eigenvalues are certainly important and are often associated with Schur triangularization which asserts that there exists $U \in \mathrm{U}_n(\mathbb{C})$ and complex upper triangular T such that

$$A = U^*TU,$$

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where $\operatorname{diag} T = (\lambda_1, \dots, \lambda_n)$ and λ's are the eigenvalues of A. Moreover the order of λ's can be prefixed. We denote by

$$\lambda(A) := (\lambda_1, \dots, \lambda_n) \tag{1.2}$$

the *unordered* n-tuple of eigenvalues of A. However the real analog for $A \in \mathrm{GL}_n(\mathbb{R})$: $A = O^T T O$, for some $O \in \mathrm{U}_n(\mathbb{R})$ and T real upper triangular, is not true, since eigenvalues and eigenvectors may not be real.

Needless to say, singular values are important scalars. From the well-known Singular Value Decomposition (SVD) there are $U, V \in \mathrm{U}_n(\mathbb{F})$ such that

$$A = U \operatorname{diag}(s_1, \dots, s_n) V,$$

where $s_1 \geq \cdots \geq s_n$ are the singular values of $A \in \mathrm{GL}_n(\mathbb{F})$. Here $\operatorname{diag} v$ means the diagonal matrix with diagonal $v \in \mathbb{F}^n$. We denote by

$$s(A) := \operatorname{diag}(s_1, \dots, s_n). \tag{1.3}$$

The Gelfand-Naimark decomposition [1] asserts that

$$A = L\omega U,$$

where $L \in \mathrm{GL}_n(\mathbb{F})$ is unit lower triangular, $U \in \mathrm{GL}_n(\mathbb{F})$ is upper triangular and ω is a permutation matrix. It is different from Gauss elimination $A = \omega' L' U'$. Though Gelfand-Naimark decomposition is less well-known, it has very nice properties. For example, the permutation matrix ω and

$$u(A) := \operatorname{diag} U \in \mathbb{F}^n$$

in $A = L\omega U$ are uniquely determined by A (see Section 2). However none of the components in $A = \omega' L' U'$ (Gauss elimination) is unique.

We already mentioned four sets of scalars associated with $A \in \mathrm{GL}_n(\mathbb{F})$, namely, $a(A)$, $\lambda(A)$, $s(A)$ and $u(A)$. When $\mathbb{F} = \mathbb{C}$, the relation between $s(A)$ and $\lambda(A)$ are completely determined by Weyl [7] and Horn [2] in terms of log majorization $|\lambda(A)| \prec_{\log} s(A)$:

$$\begin{aligned} \prod_{i=1}^{k} |\lambda_i'(A)| &\leq \prod_{i=1}^{k} s_i(A), \quad k = 1, \dots, n-1, \\ \prod_{i=1}^{n} |\lambda_i'(A)| &= \prod_{i=1}^{n} s_i(A), \end{aligned} \tag{1.4}$$

where $\lambda_1'(A), \dots, \lambda_n'(A)$ are rearrangements of $\lambda_1(A), \dots, \lambda_n(A)$ such that $|\lambda_1'(A)| \geq \cdots \geq |\lambda_n'(A)|$; and conversely if $|\lambda| \prec_{\log} s$ ($\lambda \in \mathbb{C}^n, s \in \mathbb{R}_+^n, s_1 \geq \cdots \geq s_n$), then there exists $A \in \mathrm{GL}_n(\mathbb{C})$ such that $\lambda(A) = \lambda$ and $s(A) = s$. Moreover A may be chosen to be a real matrix if the non-real numbers among $\lambda_1, \dots, \lambda_n$ occur in complex conjugate pairs [6]. See [5, Theorem 5.4] for a nice generalization. Likewise, given $A \in \mathbb{F}_{n\times n}$, the relation between $s(A)$ and $a(A)$ is completely determined by log majorization, that is,

$$a(A) \prec_{\log} s(A), \tag{1.5}$$

and conversely if $a \prec_{\log} s$ ($a, s \in \mathbb{R}_+^n$), then there exists $A \in \mathrm{GL}_n(\mathbb{F})$ such that $s(A) = s$ and $a(A) = a$. The result is a special case of Kostant's nonlinear convexity theorem [5, Theorem 4.1] on Iwasawa decomposition of a semisimple Lie group. We want to find complete relations

(1) between $u(A)$ and $a(A)$,
(2) between $u(A)$ and $s(A)$,
(3) between $u(A)$ and $\lambda(A)$, and
(4) between $a(A)$ and $\lambda(A)$.

Relations (1) and (2) will be given by the following partial order $\trianglelefteq$ on the set of positive n-tuples $\mathbb{R}_+^n$: Given $a, b \in \mathbb{R}_+^n$, $a \trianglelefteq b$ means

$$\begin{aligned} \prod_{i=1}^k a_i &\le \prod_{i=1}^k b_i, \quad k = 1, \ldots, n-1, \\ \prod_{i=1}^n a_i &= \prod_{i=1}^n b_i. \end{aligned}$$

It turns out the relation (1) is given by $|u(A)| \trianglelefteq a(A)$ and the relation (2) is given by $|u(A)| \trianglelefteq s(A)$. The partial order $\trianglelefteq$ looks very similar to log majorization $a \prec_{\log} b$. However they are different since $\trianglelefteq$ does not require the entries of a and b in the above inequalities having descending order.

We will show that relation (3) is given by

$$\pm \prod_{i=1}^n u_i(A) = \prod_{i=1}^n \lambda_i(A),$$

and relation (4) is given by

$$\prod_{i=1}^n a_i(A) = \prod_{i=1}^n |\lambda_i(A)|.$$

We organize the paper in the following way. We first review some basic facts of Gelfand-Naimark decomposition in Section 2. In Section 3 we obtain complete relations between $u(A)$ and $a(A)$, and between $u(A)$ and $s(A)$. In Section 4 a relation between $u(A)$ and $\lambda(A)$ is given. In Section 5 a relation between $a(A)$ and $\lambda(A)$ is also given. Together with the results of Weyl-Horn-Thompson and Kostant, namely (1.4) and (1.5), the relations between any two among u, s, λ, a are completely known. Finally in Section 6 we make some remarks on the relations between any three among u, s, λ, a.

2. Basics of Gelfand-Naimark decomposition

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{F}^n$, that is, $\mathbf{e}_i$ has 1 as the only nonzero entry at the i-th position. We identify a permutation $\omega \in S_n$ with the unique permutation matrix (also written as ω) in the general linear group $\mathrm{GL}_n(\mathbb{F})$, where $\omega \mathbf{e}_i = \mathbf{e}_{\omega(i)}$. The matrix representation of ω under the standard basis is

$$\omega = \left[\mathbf{e}_{\omega(1)}, \cdots, \mathbf{e}_{\omega(n)} \right].$$

So if $A = \begin{bmatrix} \mathbf{a}_1, \cdots, \mathbf{a}_n \end{bmatrix}$ in column form, then $A\omega = \begin{bmatrix} \mathbf{a}_{\omega(1)}, \cdots, \mathbf{a}_{\omega(n)} \end{bmatrix}$. Moreover, if $x_1, \ldots, x_n \in \mathbb{F}$, then

$$\omega^{-1}\mathrm{diag}\,(x_1, \ldots, x_n)\omega = \mathrm{diag}\,(x_{\omega(1)}, \ldots, x_{\omega(n)}). \tag{2.1}$$

Given a matrix $A \in \mathbb{F}_{n\times n}$, let $A(i|j)$ denote the submatrix formed by the first i rows and the first j columns of A, $1 \le i, j \le n$. The following proposition establishes the existence of Gelfand-Naimark decomposition.

Proposition 2.1. Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$. Each $A \in \mathrm{GL}_n(\mathbb{F})$ has $A = L\omega U$, for a permutation matrix ω, a unit lower triangular matrix $L \in \mathrm{GL}_n(\mathbb{F})$, and an upper triangular $U \in \mathrm{GL}_n(\mathbb{F})$. The permutation matrix ω is uniquely determined by A:

$$\mathrm{rank}\,\omega(i|j) = \mathrm{rank}\,A(i|j) \qquad \text{for} \qquad 1 \le i, j \le n.$$

Moreover $\mathrm{diag}\,U$ is uniquely determined by A.

Proof. We first prove the existence of the decomposition $A = L\omega U$ which is indeed a matrix version of some sequence of elementary row and column operations applying on A.

Let a_{k1} be the first nonzero entry of the first column of A. By multiplying the first column of A by $1/a_{k1}$, we turn the $(k, 1)$ entry to 1. Using the 1 as a pivot to consecutively eliminate other nonzero entries on the first column (using row operations) and the kth row (using column operations).

The above operations are equivalent to the following post- and pre-matrix multiplications: Let $D_1 = \mathrm{diag}\,(1/a_{k1}, 1, \ldots, 1) \in \mathrm{GL}_n(\mathbb{F})$. Denote by $A' = AD_1$. Let $E_{ij} \in \mathbb{R}_{n\times n}$ with (i, j) entry 1 as the only nonzero entry. Let

$$L_1 = (I - a'_{k+1,1}E_{k+1,k})(I - a'_{k+2,1}E_{k+2,k}) \cdots (I - a'_{n1}E_{nk}) \in \mathrm{GL}_n(\mathbb{F}),$$

a unit lower triangular matrix, and

$$U_1 = (I - a'_{k2}E_{12})(I - a'_{k3}E_{13}) \cdots (I - a'_{kn}E_{1n}) \in \mathrm{GL}_n(\mathbb{F}),$$

a unit upper triangular matrix. Then

$$A \qquad \rightarrow \qquad L_1AD_1 \qquad \rightarrow \qquad L_1AD_1U_1$$

$$\begin{bmatrix} 0 & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * & * & * & \cdots & * \\ a_{k1} & * & \cdots & * & * & * & \cdots & * \\ * & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & * & \cdots & * & * & * & \cdots & * \end{bmatrix} \rightarrow \begin{bmatrix} 0 & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * & * & * & \cdots & * \\ 1 & * & \cdots & * & * & * & \cdots & * \\ 0 & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * & * & * & \cdots & * \end{bmatrix} \rightarrow \begin{bmatrix} 0 & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * & * & * & \cdots & * \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * & * & * & \cdots & * \end{bmatrix}$$

Repeat the procedure on the second column of $L_1AD_1U_1$ and so on. Eventually we obtain a permutation matrix ω, unit lower triangular matrices $L_1, \ldots, L_n \in \mathrm{GL}_n(\mathbb{F})$, diagonal matrices $D_1, \ldots, D_n \in \mathrm{GL}_n(\mathbb{F})$, and unit upper triangular matrices $U_1, \ldots, U_n \in \mathrm{GL}_n(\mathbb{F})$ such that

$$L_n \cdots L_1AD_1U_1 \cdots D_nU_n = \omega.$$

Denote

$$\begin{aligned} L^{-1} &= L_n \cdots L_1, \qquad \text{and} \\ U^{-1} &= D_1 U_1 \cdots D_n U_n. \end{aligned}$$

Then $A = L\omega U$ as desired. Since the group of nonsingular diagonal matrices normalizes the group of unit upper triangular matrices, $U^{-1} = U'D$ for some unit upper triangular matrix U', where $D := D_1 \cdots D_n$. So $U = D^{-1}U'^{-1}$. In other words, the i-th diagonal entry u_{ii} of U is indeed the first nonzero entry of the i-th column in the i-th elimination step.

By block multiplication we notice that

$$\begin{aligned} A(i|j) &= \begin{bmatrix} L(i|i) & 0 \end{bmatrix} \begin{bmatrix} \omega(i|j) & * \\ * & * \end{bmatrix} \begin{bmatrix} U(j|j) \\ 0 \end{bmatrix} \\ &= L(i|i)\omega(i|j)U(j|j). \end{aligned}$$

So $\operatorname{rank}\omega(i|j) = \operatorname{rank}A(i|j)$, $1 \le i, j \le n$. Obviously $\operatorname{rank}\omega(i|j)$ is the number of nonzero entries in $\omega(i|j)$. Thus it is easy to verify that ω_{ij} is nonzero if and only if

$$\operatorname{rank}\omega(i|j) - \operatorname{rank}\omega(i|j-1) - \operatorname{rank}\omega(i-1|j) + \operatorname{rank}\omega(i-1|j-1) = 1.$$

So the permutation matrix ω is uniquely determined by $\operatorname{rank}\omega(i|j)$, $1 \le i, j \le n$. Hence ω is uniquely determined by A.

If $L\omega U = L'\omega U'$ for another unit lower triangular L' and upper triangular U', then $\omega^{-1}L'^{-1}L\omega = U'U^{-1}$. Clearly the diagonal entries of $\omega^{-1}L'^{-1}L\omega$ are ones and thus $\operatorname{diag}U = \operatorname{diag}U'$. □

Remark 2.2. Although ω and $u(A)$ are unique in Gelfand-Naimark decomposition $A = L\omega U$ of A, the L and U components may be not unique. For example,

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

So the elimination given in the proof of Proposition 2.1 corresponds to one but not all Gelfand-Naimark decompositons. Nevertheless $\operatorname{diag}U$ is unique and its entries can be thought of the "pivots" in the elimination.

In contrast, the permutation ω' in a Gauss elimination $A = \omega'L'U'$ may be not unique, but L' and U' are uniquely determined by ω'. For example,

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}.$$

Moreover, the ω in a Gelfand-Naimark decomposition $A = L\omega U$ of A can also be a permutation in a Gauss elimination $A = \omega L'U'$ of A. To see this, we notice that $\omega^{-1}A = (\omega^{-1}L\omega)U$ and $\det[(\omega^{-1}L\omega)(k|k)] = 1$ since $(\omega^{-1}L\omega)(k|k)$ is the submatrix formed by choosing the $\omega(1), \cdots, \omega(k)$ rows and columns of L. Therefore, by LU algorithm [3], $\omega^{-1}L\omega = L_1U_1$ for some unit lower triangular L_1 and unit upper triangular U_1, and

$$A = L\omega U = \omega(\omega^{-1}L\omega)U = \omega L_1(U_1U) = \omega L'U', \tag{2.2}$$

where $L' := L_1$ and $U' := U_1U$.

From (2.2) we also have $\omega^{-1}A = L_1U_1U$. Then $u(A)$ can be computed by

$$\det[(\omega^{-1}A)(k|k)] = \det[(L_1U_1U)(k|k)] = \det[U(k|k)] = \prod_{i=1}^{k} u_{ii}. \tag{2.3}$$

Remark 2.3. The above proof constructs a Gelfand-Naimark decomposition of A via an elimination process. Indeed each Gelfand-Naimark decomposition of $A = L\omega U$ is a matrix version of some elimination process. It is because that $L^{-1}AU^{-1} = \omega$ where L^{-1} corresponds to a sequences of elementary row operations and U^{-1} corresponds to a sequences of elementary column operations.

Remark 2.4. When ω is the identity, it is well-known that [3] the decomposition $A = LU$ is unique.

Now the following is well-defined

$$u(A) := \operatorname{diag} U = \operatorname{diag}(u_{11}, \dots, u_{nn}), \tag{2.4}$$

where $A = L\omega U$ is *any* Gelfand-Naimark decomposition of A.

3. U-COMPONENTS, A-COMPONENTS AND SINGULAR VALUES

Let $\mathbb{R}^n_+$ be the set of positive n-tuples. If $a, b \in \mathbb{R}^n_+$, denote by $a \trianglelefteq b$ if

$$\prod_{i=1}^{k} a_i \leq \prod_{i=1}^{k} b_i, \quad k = 1, \dots, n-1, \tag{3.1}$$

$$\prod_{i=1}^{n} a_i = \prod_{i=1}^{n} b_i \tag{3.2}$$

The partial order $\trianglelefteq$ is different from log majorization. For example, if $a = (3, 2)$ and $b = (1, 6)$, then $a \prec_{\log} b$ but $a \ntrianglelefteq b$. Indeed $b \trianglelefteq a$.

The following theorem gives a complete relation between the a-component and the u-component of $A \in \mathrm{GL}_n(\mathbb{F})$, in terms of the partial order $\trianglelefteq$. We denote by $C_k(A)$ the k-compound of $A \in \mathrm{GL}_n(\mathbb{F})$ where $1 \leq k \leq n$.

Theorem 3.1. Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. If $A \in \mathrm{GL}_n(\mathbb{F})$, then $|u(A)| \trianglelefteq a(A)$. Conversely if $a := (a_1, \dots, a_n)$ where $a_1, \dots, a_n > 0$ and $u := (u_1, \dots, u_n)$ where $u_1, \dots, u_n \in \mathbb{F}$ are nonzero numbers such that $|u| \trianglelefteq a$, then there exists $A \in \mathrm{GL}_n(\mathbb{F})$ such that $a(A) = a$ and $u(A) = u$ and A has LU decomposition. Indeed, if $u \trianglelefteq a$, where $u_1, \dots, u_n > 0$, then there exists $Q \in \mathrm{SO}(n)$ such that $u((Q \operatorname{diag} a)) = u$ and $Q \operatorname{diag} a$ has LU decomposition.

Proof. Suppose $A \in \mathbb{C}_{n\times n}$ and $A = QR = L\omega U$. Since $u_1(A)$ is the first nonzero entry of the first column of A and $a_1(A)$ is the 2-norm of the first column of A,

$$|u_1(A)| \leq a_1(A). \tag{3.3}$$

Since

$$C_k(A) = C_k(Q)C_k(R), \qquad C_k(A) = C_k(L)C_k(\omega)C_k(U)$$

are the QR decomposition and Gelfand-Naimark decomposition of $C_k(A)$, respectively, we apply (3.3) on $C_k(A)$. Since $u_1(C_k(A)) = \prod_{i=1}^k u_i(A)$ and $a_1(C_k(A)) = \prod_{i=1}^k a_i(A)$, $k = 1, \ldots, n-1$ and $\prod_{i=1}^n |u_i(A)| = \prod_{i=1}^n a_i(A)$ (follows from determinant consideration), we have $|u(A)| \trianglelefteq a(A)$.

Given $u \in \mathbb{C}^n, a \in \mathbb{R}_+^n$, we say that the pair (u, a) is $\mathbb{F}$-*realizable* if there exists $A \in \mathrm{GL}_n(\mathbb{F})$ such that $u(A) = u$ and $a(A) = a$. It is not hard to see that (u, a) is $\mathbb{F}$-realizable if and only if there exists $Q \in \mathrm{U}_n(\mathbb{F})$ such that $u = u(Q \operatorname{diag} a)$. We remark that

(1) if (u, a) is $\mathbb{C}$-realizable, so is $(D_\theta u, a)$, where $D_\theta := \operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$, $\theta_1, \ldots, \theta_n \in \mathbb{R}$.

(2) if (u, a) is $\mathbb{R}$-realizable, so is (Du, a), where $D := \operatorname{diag}(\pm 1, \ldots, \pm 1)$.

The reason is that (1) if $Q \operatorname{diag} a = L\omega U$, then from (2.1)

$$\begin{aligned}
\operatorname{diag}(e^{i\theta_{\omega^{-1}(1)}}, \ldots, e^{i\theta_{\omega^{-1}(n)}})Q \operatorname{diag} a &= \operatorname{diag}(e^{i\theta_{\omega^{-1}(1)}}, \ldots, e^{i\theta_{\omega^{-1}(n)}})L\omega U \\
&= L'\operatorname{diag}(e^{i\theta_{\omega^{-1}(1)}}, \ldots, e^{i\theta_{\omega^{-1}(n)}})\omega U \\
&= L'(\omega\operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})\omega^{-1})\omega U \\
&= L'\omega\operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})U
\end{aligned}$$

where $L' := \operatorname{diag}(e^{i\theta_{\omega^{-1}(1)}}, \ldots, e^{i\theta_{\omega^{-1}(n)}})L \operatorname{diag}(e^{-i\theta_{\omega^{-1}(1)}}, \ldots, e^{-i\theta_{\omega^{-1}(n)}})$ is still unit lower triangular. The real case (2) is similar.

We now prove the converse by induction on n. Because of the above remark it is sufficient to consider $u \in \mathbb{R}_+^n$ and prove the last statement in Theorem 3.1. Suppose $(u_1, u_2) \trianglelefteq (a_1, a_2)$, that is, $u_1 \le a_1$ and $u_1u_2 = a_1a_2$. Let

$$Q := \begin{bmatrix} p & -(1-p^2)^{1/2} \\ (1-p^2)^{1/2} & p \end{bmatrix} \in \mathrm{SO}(2),$$

where $p := u_1/a_1 \in (0, 1]$. The first column of

$$A := Q \operatorname{diag}(a_1, a_2) = \begin{bmatrix} a_1p & -a_2(1-p^2)^{1/2} \\ a_1(1-p^2)^{1/2} & a_2p \end{bmatrix}$$

has the first nonzero entry $a_1p = u_1$ so that A has LU decomposition. Clearly $a(A) = a$. Since $u_1u_2 = a_1a_2$, we have $u(A) = u$. Suppose the statement is true for $n \le k$. Let $a = (a_1, \ldots, a_k, a_{k+1}), u = (u_1, \ldots, u_k, u_{k+1}) \in \mathbb{R}_+^{k+1}$ and $u \trianglelefteq a$. Then

$$u' := (u_1, \ldots, u_{k-1}, a_1 \cdots a_k/u_1 \cdots u_{k-1}) \trianglelefteq a' := (a_1, \ldots, a_k).$$

By the induction hypothesis there is $Q' \in \mathrm{SO}(k)$ such that

$$A' := Q' \operatorname{diag}(a_1, \ldots, a_k)$$

has LU decomposition satisfying $u(A') = u'$ and $a(A') = a'$. Set

$$Q_2 := \begin{bmatrix} t & -(1-t^2)^{1/2} \\ (1-t^2)^{1/2} & t \end{bmatrix} \in \mathrm{SO}(2),$$

where $t := u_1 \cdots u_k / a_1 \cdots a_k \le 1$. The first $(k-1)$ rows of

$$A := (I_{k-1} \oplus Q_2)(Q' \oplus 1) \operatorname{diag} a$$

are those of $(Q' \oplus 1)\operatorname{diag} a$ so that the first $k-1$ rows of the matrices $A(k|k)$ and A' are identical and the last row of $A(k|k)$ is t times the last row of A'. So

$$\det A(k|k) = t a_1 \cdots a_k = u_1 \cdots u_k.$$

Moreover $A(k|k)$ has LU decomposition and $u_i(A) = u_i$ for all $i = 1, \dots, k$. Hence $A = Q \operatorname{diag} a$ has LU decomposition and is the required matrix where $Q := (I_{k-1} \oplus Q_2)(Q' \oplus 1) \in \mathrm{SO}(k+1)$. □

Theorem 3.2. Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. Let $A \in \mathrm{GL}_n(\mathbb{F})$. Then $|u(A)| \trianglelefteq a(A) \prec_{\log} s(A)$. Conversely, if $a := (a_1, \dots, a_n)$, $s := (s_1, \dots, s_n)$, $u := (u_1, \dots, u_n)$, where $a_1, \dots, a_n > 0$, $s_1 \ge \cdots \ge s_n > 0$, $u_1, \dots, u_n \in \mathbb{F}$ are nonzero numbers such that $|u| \trianglelefteq a \prec_{\log} s$, then there exists $A \in \mathrm{GL}_n(\mathbb{F})$ such that $u(A) = u$, $a(A) = a$ and $s(A) = s$.

Proof. The relations $|u(A)| \trianglelefteq a(A) \prec_{\log} s(A)$ follow from Theorem 3.1 and (1.5).

Suppose $s_1 \ge \cdots \ge s_n$. By Kostant's nonlinear convexity theorem [5, Theorem 4.1] and SVD, there exist $Q_1, Q_2 \in \mathrm{U}_n(\mathbb{F})$ such that $Q_1 (\operatorname{diag} s) Q_2$ has QR-decomposition $Q_1 (\operatorname{diag} s) Q_2 = Q_3 R$, where $Q_3 \in \mathrm{U}_n(\mathbb{F})$, R is upper triangular and $\operatorname{diag} R = a$. Write $R = (\operatorname{diag} a) V$ where V is unit upper triangular. By Theorem 3.1, there is $Q \in \mathrm{U}_n(\mathbb{F})$ such that $Q \operatorname{diag} a = LU$ where $u(Q \operatorname{diag} a) = u$. Set

$$A := L(UV) = (LU)V = Q\, (\operatorname{diag} a)\, V = QR = QQ_3^{-1} Q_1 s Q_2.$$

Then $s(A) = s$, $u(A) = \operatorname{diag}(UV) = u$ and $a(A) = a$. □

The following theorem gives a complete relation between u-component and the singular values of $A \in \mathrm{GL}_n(\mathbb{F})$, also in terms of the partial order $\trianglelefteq$.

Corollary 3.3. Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. If $A \in \mathrm{GL}_n(\mathbb{F})$, then $|u(A)| \trianglelefteq s(A)$. Conversely, if $s := (s_1, \dots, s_n)$ and $u := (u_1, \dots, u_n)$, where $s_1 \ge \cdots \ge s_n > 0$ and $u_1, \dots, u_n \in \mathbb{F}$ are nonzero numbers such that $|u| \trianglelefteq s$, then there exists $A \in \mathrm{GL}_n(\mathbb{F})$ such that A has LU decomposition and $u(A) = u$ and $s(A) = s$.

Proof. By Theorem 3.2, $|u(A)| \trianglelefteq s(A)$. For the converse, choose $a = s$ and apply Theorem 3.2. □

Remark 3.4. There is a slight difference between Theorem 3.1 and Theorem 3.3, that is, the entries of a may not be in descending order but the entries of s are already in descending order.

4. U-COMPONENTS AND EIGENVALUES

The following theorem gives a complete relation between u-component and the eigenvalues of $A \in \mathrm{GL}_n(\mathbb{C})$.

Theorem 4.1. If $A \in \mathrm{GL}_n(\mathbb{C})$, then $\pm u_1(A) \cdots u_n(A) = \lambda_1(A) \cdots \lambda_n(A)$. Conversely, when $n \geq 2$, if $\pm u_1 \cdots u_n = \lambda_1 \cdots \lambda_n \neq 0$, where $u_1, \ldots, u_n \in \mathbb{C}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are nonzero numbers, then there exists $A \in \mathrm{GL}_n(\mathbb{C})$ such that $u(A) = u$ and $\lambda(A) = \lambda$. Moreover A may be chosen so that

(a) if $u_1 \cdots u_n = \lambda_1 \cdots \lambda_n$, then A has LU decomposition,
(b) if $-u_1 \cdots u_n = \lambda_1 \cdots \lambda_n$, then A has $L\omega U$ decomposition where ω is the transposition $(n-1, n)$.

When $n = 1$, only (a) is true.

Proof. Suppose $A = L\omega U$ and $A = Q^* T Q$ are the Gelfand-Naimark decomposition and Schur Triangularization of $A \in \mathrm{GL}_n(\mathbb{C})$, where $Q \in \mathrm{U}_n(\mathbb{C})$ and T is upper triangular with $\operatorname{diag} T = \lambda(A)$. Taking determinant yields $\pm u_1(A) \cdots u_n(A) = \lambda_1(A) \cdots \lambda_n(A)$.

We will prove the converse by induction on n for both (a) and (b).

When $n = 1$, (a) is obviously true and (b) will not happen. Suppose the statements are true for $n \leq k$. Let

$$\lambda = (\lambda_1, \ldots, \lambda_k, \lambda_{k+1}), \quad u = (u_1, \ldots, u_k, u_{k+1}) \in \mathbb{C}^{k+1}$$

such that $\lambda_1 \cdots \lambda_k \lambda_{k+1} = \pm u_1 \cdots u_k u_{k+1} \neq 0$. Then

$$u' := (u_1, \ldots, u_{k-1}, \frac{\lambda_1 \cdots \lambda_k}{u_1 \cdots u_{k-1}}), \quad \lambda' := (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$$

satisfy $\lambda'_1 \cdots \lambda'_k = u'_1 \cdots u'_k \neq 0$. So by the induction hypothesis there exists $A' \in \mathrm{GL}_n(\mathbb{C})$ such that $u(A') = u'$, $\lambda(A') = \lambda'$, and A' has LU decomposition. Let

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \in \mathrm{SO}(2).$$

Set

$$A := \begin{bmatrix} I_{k-1} & 0 \\ 0 & Q^* \end{bmatrix} \begin{bmatrix} A' & y \\ 0 & \lambda_{k+1} \end{bmatrix} \begin{bmatrix} I_{k-1} & 0 \\ 0 & Q \end{bmatrix} \tag{4.1}$$

where $y \in \mathbb{C}^k$ is an indeterminate vector. Clearly $\lambda(A) = \lambda$. Moreover, $A(k-1|k-1) = A'(k-1|k-1)$. Since A' has LU decomposition, we have $u_i(A) = u_i(A') = u_i$ for $i = 1, \cdots, k-1$.

Case (a): $u_1 \cdots u_{k+1} = \lambda_1 \cdots \lambda_{k+1}$. Let $y := t\mathbf{e}_k \in \mathbb{C}^k$ where $t \in \mathbb{C}$ is to be determined. From (4.1) and $\det A(k-1|k-1) = u_1 \cdots u_{k-1} \neq 0$, $\det A(k|k)$ is a polynomial of t of degree one. By choosing an appropriate $t \in \mathbb{C}$, we can make $\det A(k|k) = u_1 \cdots u_k \neq 0$, which implies that

$$u_k(A) = \det A(k|k) / \det A(k-1|k-1) = u_k.$$

From (4.1), $\det A = \lambda_{k+1} \det A' = \lambda_1 \cdots \lambda_{k+1} = u_1 \cdots u_{k+1}$. So

$$u_{k+1}(A) = \det A / \det A(k|k) = u_{k+1}$$

and A has the LU decomposition.

Case (b): $-u_1\cdots u_{k+1}=\lambda_1\cdots\lambda_{k+1}$. Denote $A'':=A'(k-1|k-1)$. Write

$$A'=\begin{bmatrix} A'' & \beta \\ \alpha^T & \mu \end{bmatrix}, \qquad y=\begin{bmatrix} y' \\ y_k \end{bmatrix}\in\mathbb{C}^k$$

where $\alpha,\beta,y'\in\mathbb{C}^{k-1}$ and $\mu,y_k\in\mathbb{C}$. Direct computation on (4.1) yields

$$\begin{aligned} A &= \begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} A'' & \beta & y' \\ \alpha^T & \mu & y_k \\ 0 & 0 & \lambda_{k+1} \end{bmatrix} \begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} A'' & \frac{1}{\sqrt{2}}(\beta-y') & \frac{1}{\sqrt{2}}(\beta+y') \\ \frac{1}{\sqrt{2}}\alpha^T & \frac{1}{2}(\mu-y_k+\lambda_{k+1}) & \frac{1}{2}(\mu+y_k-\lambda_{k+1}) \\ \frac{1}{\sqrt{2}}\alpha^T & \frac{1}{2}(\mu-y_k-\lambda_{k+1}) & \frac{1}{2}(\mu+y_k+\lambda_{k+1}) \end{bmatrix}. \end{aligned}$$

We want to have a Gelfand-Naimark decomposition $A=L\omega U$ where ω is the transposition $(k,k+1)$. Set $y'=\beta$ and $y_k=\mu+\lambda_{k+1}$ so that

$$\beta-y'=0\in\mathbb{C}^k, \qquad \mu-y_k+\lambda_{k+1}=0\in\mathbb{C},$$

and thus

$$A=\begin{bmatrix} A'' & 0 & \sqrt{2}\beta \\ \frac{1}{\sqrt{2}}\alpha^T & 0 & \mu \\ \frac{1}{\sqrt{2}}\alpha^T & -\lambda_{k+1} & \mu+\lambda_{k+1} \end{bmatrix}.$$

By Proposition 2.1, the permutation ω in the Gelfand-Naimark decomposition of A is the transposition $(k,k+1)$. However,

$$u(A)=(u_1,\cdots,u_{k-1},-\lambda_{k+1},\frac{\lambda_1\cdots\lambda_k}{u_1\cdots u_{k-1}}).$$

Let $D:=I_k\oplus(-\lambda_{k+1}/u_k)$ and $A_0:=D^{-1}AD$. Then A_0 has the same ω as A in its Gelfand-Naimark decomposition, and clearly $\lambda(A_0)=\lambda$ and $u(A_0)=u$. Reset $A=A_0$ and we are done. □

If $A\in\mathrm{GL}_n(\mathbb{R})$, the non-real eigenvalues of A appear in complex conjugate pairs. It turns out that this is the only additional requirement for the real case but the proof is more involved.

Theorem 4.2. If $A\in\mathrm{GL}_n(\mathbb{R})$, then $\pm u_1(A)\cdots u_n(A)=\lambda_1(A)\cdots\lambda_n(A)$. Conversely, when $n\geq 2$, if $\pm u_1\cdots u_n=\lambda_1\cdots\lambda_n\neq 0$, where $u_1,\dots,u_n\in\mathbb{R}$, and the non-real numbers of $\lambda_1,\dots,\lambda_n$ appear in complex conjugate pairs, then there exists $A\in\mathrm{GL}_n(\mathbb{R})$ such that $u(A)=u$ and $\lambda(A)=\lambda$, where $u:=(u_1,\dots,u_n)$ and $\lambda:=(\lambda_1,\dots,\lambda_n)$. Moreover A may be chosen so that

(a) if $u_1\cdots u_n=\lambda_1\cdots\lambda_n$, then A has LU decomposition,
(b) if $-u_1\cdots u_n=\lambda_1\cdots\lambda_n$, then A has $L\omega U$ decomposition where ω is the transposition $(1,2)$, provided that $n\geq 2$.

When $n=1$, only (a) is true.

Proof. The relation $\pm u_1(A)\cdots u_n(A) = \lambda_1(A)\cdots\lambda_n(A)$ is contained in Theorem 4.1. We now prove the converse by induction.

When $n = 1$, (a) is obviously true and (b) will not happen.

When $n = 2$, suppose $\pm u_1 u_2 = \lambda_1\lambda_2 (\neq 0)$. Let

$$A := \begin{cases} \begin{bmatrix} u_1 & \lambda_1+\lambda_2-u_1-u_2 \\ u_1 & \lambda_1+\lambda_2-u_1 \end{bmatrix} & \text{if } u_1u_2 = \lambda_1\lambda_2, \\ \\ \begin{bmatrix} 0 & u_2 \\ u_1 & \lambda_1+\lambda_2 \end{bmatrix} & \text{if } -u_1u_2 = \lambda_1\lambda_2. \end{cases} \tag{4.2}$$

Then $A \in \mathrm{GL}_2(\mathbb{R})$ since $\lambda_1 + \lambda_2$ is real. Clearly $\lambda(A) = (\lambda_1, \lambda_2)$, $u(A) = (u_1, u_2)$, and (a) and (b) are true.

When $n = 3$, suppose $\pm u_1u_2u_3 = \lambda_1\lambda_2\lambda_3 (\neq 0)$ and the non-real numbers of $\lambda_1, \lambda_2, \lambda_3$ appear in complex conjugate pair. Then at least one of $\lambda_1, \lambda_2, \lambda_3$ is real. Without lost of generality, suppose $\lambda_3 \in \mathbb{R}$. Let

$$\rho_1 := \lambda_1+\lambda_2+\lambda_3, \quad \rho_2 := \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1, \quad \rho_3 := \lambda_1\lambda_2\lambda_3. \tag{4.3}$$

Denote

$$A = \begin{cases} \begin{bmatrix} u_1 & -u_2 & -\frac{\rho_2}{u_1}+\rho_1-u_1+u_2 \\ u_1 & 0 & \rho_1-u_1-u_3 \\ u_1 & 0 & \rho_1-u_1 \end{bmatrix} & \text{if } u_1u_2u_3 = \lambda_1\lambda_2\lambda_3, \\ \\ \begin{bmatrix} 0 & u_2 & -\frac{\lambda_1\lambda_2}{u_1}-u_2 \\ u_1 & 0 & \lambda_1+\lambda_2 \\ u_1 & -\lambda_3 & \rho_1 \end{bmatrix} & \text{if } -u_1u_2u_3 = \lambda_1\lambda_2\lambda_3. \end{cases} \tag{4.4}$$

Then $A \in \mathrm{GL}_3(\mathbb{R})$ and $\lambda(A) = (\lambda_1, \lambda_2, \lambda_3)$, $u(A) = (u_1, u_2, u_3)$, and (a) and (b) are true.

Suppose the statements are true for $n \le k$ where $k \ge 3$. Let

$$\lambda = (\lambda_1, \dots, \lambda_k, \lambda_{k+1}) \in \mathbb{C}^{k+1}, \quad u = (u_1, \dots, u_k, u_{k+1}) \in \mathbb{R}^{k+1}$$

such that $\pm u_1\cdots u_k u_{k+1} = \lambda_1\cdots\lambda_k\lambda_{k+1} (\neq 0)$ and the non-real λ's occur in complex conjugate pairs. We now consider two situations.

Situation A: $\lambda_i \in \mathbb{R}$ for some $i = 1, \cdots, k+1$. Without lost of generality we assume that $\lambda_{k+1} \in \mathbb{R}$.

Case (a): If $u_1\cdots u_{k+1} = \lambda_1\cdots\lambda_{k+1}$, let

$$u' := (u_1, \dots, u_{k-1}, \frac{\lambda_1\cdots\lambda_k}{u_1\cdots u_{k-1}}), \quad \lambda' := (\lambda_1, \dots, \lambda_k), \tag{4.5}$$

then $u'_1\cdots u'_k = \lambda'_1\cdots\lambda'_k \neq 0$. By the induction hypothesis there exists $A' \in \mathrm{GL}_k(\mathbb{R})$ such that $u(A') = u'$, $\lambda(A') = \lambda'$, and A' has LU decomposition.

Case (b): If $-u_1 \cdots u_{k+1} = \lambda_1 \cdots \lambda_{k+1}$, let

$$u' := (u_1, \dots, u_{k-1}, -\frac{\lambda_1 \cdots \lambda_k}{u_1 \cdots u_{k-1}}), \quad \lambda' := (\lambda_1, \dots, \lambda_k), \tag{4.6}$$

then $-u'_1 \cdots u'_k = \lambda'_1 \cdots \lambda'_k \neq 0$. By the induction hypothesis there exists $A' \in \mathrm{GL}_k(\mathbb{R})$ such that $u(A') = u'$, $\lambda(A') = \lambda'$, and A' has $L\omega U$ decomposition where ω is the transposition $(1, 2)$.

In both cases, write $A' = \begin{bmatrix} A'' & \beta \\ \alpha^T & \mu \end{bmatrix}$ where $A'' := A'(k-1|k-1)$, $\alpha, \beta \in \mathbb{R}^{k-1}$ and $\mu \in \mathbb{R}$. Let $t \in \mathbb{R}$ be an indeterminate and set

$$A := \begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}^{-1} \begin{bmatrix} A'' & \beta & 0 \\ \alpha^T & \mu & t \\ 0 & 0 & \lambda_{k+1} \end{bmatrix} \begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \tag{4.7}$$

$$= \begin{bmatrix} A'' & \frac{1}{2}\beta & \beta \\ \alpha^T & \frac{1}{2}(\mu - t + \lambda_{k+1}) & \mu + t - \lambda_{k+1} \\ \frac{1}{2}\alpha^T & \frac{1}{4}(\mu - t - \lambda_{k+1}) & \frac{1}{2}(\mu + t + \lambda_{k+1}) \end{bmatrix}. \tag{4.8}$$

Then $\lambda(A) = \lambda$ and $A(k-1|k-1) = A'' = A'(k-1|k-1)$ where $k-1 \geq 2$. By Proposition 2.1 (about ω) and (2.3), in both cases $u_i(A) = u_i(A') = u_i$ for $i = 1, \cdots, k-1$. From (4.8) and the fact that $\det A'' = \pm u_1 \cdots u_{k-1} \neq 0$, $\det A(k|k)$ is a polynomial of t of degree 1. Choose $t \in \mathbb{R}$ such that

$$\det A(k|k) = \begin{cases} u_1 \cdots u_k & \text{in case (a)} \\ -u_1 \cdots u_k & \text{in case (b).} \end{cases} \tag{4.9}$$

Then $u_k(A) = u_k$ and $A(k|k)$ has the same ω as A'. From (4.7), $\det A = \lambda_1 \cdots \lambda_{k+1}$. So $u_{k+1}(A) = u_{k+1}$ by (4.9). Moreover, A has the LU decomposition in case (a), and A has the $L\omega U$ decomposition in case (b), where ω is the transposition $(1, 2)$.

In the above two cases $\lambda(A) = \lambda$, $u(A) = u$.

Situation B: All $\lambda_1, \cdots, \lambda_{k+1}$ appear in complex conjugate pairs. So $k \geq 3$ is odd. Without lost of generality, we assume $\lambda_{k+1} = \overline{\lambda_k}$.

Case (a): If $u_1 \cdots u_{k+1} = \lambda_1 \cdots \lambda_{k+1}$, let

$$u' = (u_1, \cdots, u_{k-2}, \frac{\lambda_1 \cdots \lambda_{k-1}}{u_1 \cdots u_{k-2}}), \qquad \lambda' = (\lambda_1, \cdots, \lambda_{k-1}), \tag{4.10}$$

then $u'_1 \cdots u'_{k-1} = \lambda'_1 \cdots \lambda'_{k-1}$. By the induction hypothesis, there exists $A' \in \mathrm{GL}_{k-1}(\mathbb{R})$ where $u(A') = u'$, $\lambda(A') = \lambda'$, and A' has LU decomposition.

Case (b): If $-u_1 \cdots u_{k+1} = \lambda_1 \cdots \lambda_{k+1}$, let

$$u' := (u_1, \dots, u_{k-2}, -\frac{\lambda_1 \cdots \lambda_{k-1}}{u_1 \cdots u_{k-2}}), \quad \lambda' := (\lambda_1, \dots, \lambda_{k-1}), \tag{4.11}$$

then $-u'_1 \cdots u'_{k-1} = \lambda'_1 \cdots \lambda'_{k-1} \neq 0$. By the induction hypothesis there exists $A' \in \mathrm{GL}_{k-1}(\mathbb{R})$ such that $u(A') = u'$, $\lambda(A') = \lambda'$, and A' has $L\omega U$ decomposition, where ω is the transposition $(1, 2)$.

In both cases, denote $A'' := A'(k-2|k-2)$. Write $A' := \begin{bmatrix} A'' & \beta \\ \alpha^T & \mu \end{bmatrix}$ where $\alpha, \beta \in \mathbb{R}^{k-2}$ and $\mu \in \mathbb{R}$. Let $t \in \mathbb{R}$ and $0 \neq s \in \mathbb{R}$ be indeterminates. Set

$$p := \lambda_k + \lambda_{k+1} - s - \frac{\lambda_k \lambda_{k+1}}{s}, \tag{4.12}$$

$$q := \lambda_k + \lambda_{k+1} - s. \tag{4.13}$$

Then $\begin{bmatrix} s & p \\ s & q \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R})$ has eigenvalues λ_k and λ_{k+1}.

Possibility (i): Suppose $k > 3$, or $k = 3$ and $u_1u_2u_3u_4 = \lambda_1\lambda_2\lambda_3\lambda_4$ (case (a)). We have $\det A'' = \pm u_1 \cdots u_{k-2} \neq 0$. Set

$$A = \begin{bmatrix} I_{k-2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A'' & \beta & 0 & 0 \\ \alpha^T & \mu & t & 0 \\ 0 & 0 & s & p \\ 0 & 0 & s & q \end{bmatrix} \begin{bmatrix} I_{k-2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.14}$$

$$= \begin{bmatrix} A'' & \frac{1}{2}\beta & \beta & 0 \\ \alpha^T & \frac{1}{2}(\mu - t + s) & \mu + t - s & -p \\ \frac{1}{2}\alpha^T & \frac{1}{4}(\mu - t - s) & \frac{1}{2}(\mu + t + s) & \frac{1}{2}p \\ 0 & -\frac{1}{2}s & s & q \end{bmatrix}. \tag{4.15}$$

Then $A \in \mathrm{GL}_{k+1}(\mathbb{R})$ and $\lambda(A) = \lambda$ by (4.14). On one hand,

$$\det A(k|k) = \det \begin{bmatrix} A'' & \beta & 0 \\ \alpha^T & \mu & t \\ 0 & 0 & s \end{bmatrix} = s\lambda_1 \cdots \lambda_{k-1} \tag{4.16}$$

so that we can choose $0 \neq s \in \mathbb{R}$ such that

$$\det A(k|k) = \begin{cases} u_1 \cdots u_k & \text{in case (a),} \\ -u_1 \cdots u_k & \text{in case (b).} \end{cases}$$

On the other hand, by (4.15), $A(k-2|k-2) = A'' = A'(k-2|k-2)$. Then

$$\det A(k-2|k-2) = \det A'' = \pm u_1 \cdots u_{k-2} \neq 0$$

and $u_i(A) = u_i$ for $i = 1, \cdots, k-2$. By (4.15), $\det A(k-1|k-1)$ is a real polynomial of t of degree 1. We can choose $t \in \mathbb{R}$ such that

$$\det A(k-1|k-1) = \begin{cases} u_1 \cdots u_{k-1} & \text{in case (a)} \\ -u_1 \cdots u_{k-1} & \text{in case (b).} \end{cases}$$

Then $u_i(A) = u_i$ for $i = k-1, k, k+1$. Clearly, A has LU decomposition in case (a), and A has $L\omega U$ decomposition, where $\omega = (1, 2)$ in case (b).

Possibility (ii): Suppose $k = 3$ and $-u_1u_2u_3u_4 = \lambda_1\lambda_2\lambda_3\lambda_4$ (case (b)). Let $s := -\frac{u_1u_2u_3}{\lambda_1\lambda_2}$, and let p and q be defined as in (4.12) and (4.13). Let A

be the matrix

$$A := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -\frac{\lambda_1\lambda_2}{u_1} & -\frac{\lambda_1\lambda_2+2u_1u_2}{u_1} & 0 \\ u_1 & \lambda_1+\lambda_2 & 0 & 0 \\ 0 & 0 & s & p \\ 0 & 0 & s & q \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & u_2 & \frac{-2(\lambda_1\lambda_2+u_1u_2)}{u_1} & 0 \\ u_1 & \frac{1}{2}(\lambda_1+\lambda_2+s) & \lambda_1+\lambda_2-s & -p \\ \frac{1}{2}u_1 & \frac{1}{4}(\lambda_1+\lambda_2-s) & \frac{1}{2}(\lambda_1+\lambda_2+s) & \frac{1}{2}p \\ 0 & -\frac{1}{2}s & s & q \end{bmatrix}.$$

Then $A \in \mathrm{GL}_4(\mathbb{R})$, $\lambda(A) = \lambda$, $u_1(A) = u_1$ and $u_2(A) = u_2$. Notice that

$$\det A(3|3) = \det \begin{bmatrix} 0 & -\frac{\lambda_1\lambda_2}{u_1} & -\frac{\lambda_1\lambda_2+2u_1u_2}{u_1} \\ u_1 & \lambda_1+\lambda_2 & 0 \\ 0 & 0 & s \end{bmatrix} = -u_1u_2u_3.$$

So $u_3(A) = u_3$ and thus $u_4(A) = u_4$. Moreover, the ω in a Gelfand-Naimark decomposition of A is the transposition $(1, 2)$.

Therefore, we complete the proof by induction. □

5. A-COMPONENT, EIGENVALUES AND SINGULAR VALUES

Let $A \in \mathrm{GL}_n(\mathbb{F})$ where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. We first determine the relation among $s(A)$, $\lambda(A)$, and $a(A)$. Then we use the result to determine the relation between a-component and eigenvalues of A.

Theorem 5.1. If $A \in \mathrm{GL}_n(\mathbb{C})$, then $a(A) \prec_{\log} s(A)$ and $|\lambda(A)| \prec_{\log} s(A)$. Conversely, if $a := (a_1, \cdots, a_n)$, $s := (s_1, \cdots, s_n)$, $\lambda := (\lambda_1, \cdots, \lambda_n)$ where $a_1, \cdots, a_n > 0$, $s_1 \geq \cdots \geq s_n > 0$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are nonzero numbers such that $a \prec_{\log} s$ and $|\lambda| \prec_{\log} s$, then there exists $A \in \mathrm{GL}_n(\mathbb{C})$ such that $a(A) = a$, $\lambda(A) = \lambda$ and $s(A) = s$. Moreover, A may be chosen to be real if the non-real numbers among $\lambda_1, \ldots, \lambda_n$ appear in complex conjugate pairs.

Proof. The relation $a(A) \prec_{\log} s(A)$ and $|\lambda(A)| \prec_{\log} s(A)$ are known [5, 7].

We now establish the converse. Since $|\lambda| \prec_{\log} s$, by [2], there is $A_0 \in \mathrm{GL}_n(\mathbb{C})$ such that $\lambda(A_0) = \lambda$ and $s(A_0) = s$. By SVD write $A_0 = K_1 (\operatorname{diag} s) K_2$ where $K_1, K_2 \in \mathrm{U}_n(\mathbb{C})$. Since $a \prec_{\log} s$ by [5, Theorem 4.1] there is $V \in \mathrm{U}_n(\mathbb{C})$ such that $a((\operatorname{diag} s) V) = a$. Let

$$\begin{aligned} A &:= V^{-1}K_2K_1 (\operatorname{diag} s) V \\ &= (K_2^{-1}V)^{-1}(K_1 (\operatorname{diag} s) K_2)(K_2^{-1}V) \\ &= (K_2^{-1}V)^{-1}A_0(K_2^{-1}V). \end{aligned} \tag{5.1}$$

Then $a(A) = a((\operatorname{diag} s) V) = a$, $\lambda(A) = \lambda(A_0) = \lambda$, and $s(A) = s(A_0) = s$, as desired.

Suppose that the non-real numbers among $\lambda_1, \ldots, \lambda_n$ appear in complex conjugate pairs. Since $|\lambda| \prec_{\log} s$, by Thompson's result [6] there is $A_0 \in \mathrm{GL}_n(\mathbb{R})$ such that $\lambda(A_0) = \lambda$ and $s(A_0) = s$. Write $A_0 = K_1 (\operatorname{diag} s) K_2$

where $K_1, K_2 \in \mathrm{U}_n(\mathbb{R})$. Since $a \prec_{\log} s$ by [5, Theorem 4.1] there is $V \in \mathrm{U}_n(\mathbb{R})$ such that $a((\operatorname{diag} s)\, V) = a$. Then follow the argument in (5.1) to have the desired result. □

Corollary 5.2. If $A \in \mathrm{GL}_n(\mathbb{C})$, then $a_1(A) \cdots a_n(A) = |\lambda_1(A) \cdots \lambda_n(A)|$. Conversely, if $a := (a_1, \cdots, a_n)$ and $\lambda := (\lambda_1, \cdots, \lambda_n)$, where $a_1, \cdots, a_n > 0$ and $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ are nonzero numbers such that $a_1 \cdots a_n = |\lambda_1 \cdots \lambda_n|$, then there exists $A \in \mathrm{GL}_n(\mathbb{C})$ such that $a(A) = a$ and $\lambda(A) = \lambda$. Moreover, A may be chosen to be real if the non-real numbers among $\lambda_1, \ldots, \lambda_n$ appear in complex conjugate pairs.

Proof. We only need to prove the converse. Choose $s_1 \geq \cdots \geq s_n > 0$ such that $s_1, s_2, \ldots, s_{n-1}$ are sufficiently large and $s_1 \cdots s_n = a_1 \cdots a_n$ so that $a \prec_{\log} s$ and $|\lambda| \prec_{\log} s$. By Theorem 5.1 there is $A \in \mathrm{GL}_n(\mathbb{C})$ such that $a(A) = a$, $\lambda(A) = \lambda$, and $s(A) = s$. Moreover A may be chosen to be real if the non-real numbers among $\lambda_1, \ldots, \lambda_n$ appear in complex conjugate pairs. □

6. u, λ, s and u, λ, a

In Theorem 3.2 we see that $|u| \trianglelefteq a \prec_{\log} s$ are necessary and sufficient for the existence of $A \in \mathrm{GL}_n(\mathbb{F})$ such that $u(A) = u$, $a(A) = a$ and $s(A) = s$. Notice that the conditions $|u| \trianglelefteq a \prec_{\log} s$ are equivalent to the three conditions $|u| \trianglelefteq a$, $a \prec_{\log} s$ and $|u| \trianglelefteq s$, since the last is implied by the first two conditions ($s_1 \geq \cdots \geq s_n$). In other words, the totality of the pairwise conditions (see Theorem 3.1, (1.5), Theorem 3.3) among u, a, s are the necessary and sufficient conditions for the existence of $A \in \mathrm{GL}_n(\mathbb{F})$ such that $u(A) = u$, $a(A) = a$ and $s(A) = s$.

Similarly in Theorem 5.1 the conditions $a \prec_{\log} s$ and $|\lambda| \prec_{\log} s$ are necessary and sufficient for the existence of $A \in \mathrm{GL}_n(\mathbb{C})$ such that $a(A) = a$, $\lambda(A) = \lambda$ and $s(A) = s$. Notice that $a \prec_{\log} s$ and $|\lambda| \prec_{\log} s$ imply $a_1 \cdots a_n = |\lambda_1 \cdots \lambda_n|$. In other words, the totality of the pairwise conditions (see (1.5), (1.4), Corollary 5.2) among u, λ, a are the necessary and sufficient conditions for the existence of $A \in \mathrm{GL}_n(\mathbb{C})$ such that $u(A) = u$, $a(A) = a$ and $\lambda(A) = \lambda$. The real case is similar and the only difference is that the non-real numbers among $\lambda_1, \ldots, \lambda_n$ appear in complex conjugate pairs.

However the three pairwise conditions among u, λ, s: $|u| \trianglelefteq s$ (in Corollary 3.3), $|\lambda| \prec_{\log} s$ in (1.4) and $\pm u_1 \cdots u_n = \lambda_1 \cdots \lambda_n$ (in Theorem 4.1) do not suffice to ensure the existence of $A \in \mathrm{GL}_n(\mathbb{F})$ such that $u(A) = u$, $\lambda(A) = \lambda$ and $s(A) = s$. For example if we set $\lambda = s$, $s_1 \geq \cdots \geq s_n$, then A with $\lambda(A) = s(A) = s$ must be a positive definite matrix. By Cholesky decomposition, $A = T^*T$ for some upper triangular matrix $T \in \mathrm{GL}_n(\mathbb{F})$ with positive diagonal entries. Notice that

$$u_i(A) = |\lambda_i(T)|^2, \quad s_i(A) = s_i^2(T), \qquad i = 1, \ldots, n$$

so that $u(A) \prec_{\log} s(A)$ by Weyl's result (1.4) on T. Clearly $u(A) \prec_{\log} s(A)$ is not necessarily implied by $|u| \trianglelefteq s$. For example $s = \lambda = (3, 2)$, $u = (1, 6)$.

If we consider u, λ, a, the pairwise conditions $\pm u_1 \cdots u_n = \lambda_1 \cdots \lambda_n$ (in Theorem 4.1), $|u| \trianglelefteq a$ (in Theorem 3.1) and $|\lambda_1 \cdots \lambda_n| = a_1 \cdots a_n$ (in Theorem 5.2) are not sufficient to ensure the existence of an $A \in \mathrm{GL}_n(\mathbb{C})$ such that $u(A) = u$, $\lambda(A) = \lambda$ and $a(A) = a$ (indeed the last condition $|\lambda_1 \cdots \lambda_n| = a_1 \cdots a_n$ is implied by the first two). For example, consider $n = 2$. Suppose $u_1 u_2 = \lambda_1 \lambda_2$. Choose u_1 such that $|u_1| = a_1$ (then $|u_2| = a_2$). If there is $A \in \mathrm{GL}_2(\mathbb{C})$ such that $u(A) = u$, $\lambda(A) = \lambda$ and $a(A) = a$, then A would be of the form

$$A = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}$$

and thus λ_1, λ_2 would be u_1, u_2. It does not necessarily follow from $u_1 u_2 = \lambda_1 \lambda_2$, $(u_1, u_2) \trianglelefteq (a_1, a_2)$, say, $u = a = (3, 2)$, $\lambda = (1, 6)$.

The following proposition is straightforward computation.

Proposition 6.1. Suppose that $u_1, u_2, \lambda_1, \lambda_2 \in \mathbb{C}$ are nonzero numbers and $a_1, a_2 > 0$.

(1) If $u_1 u_2 = \lambda_1 \lambda_2$ and $(|u_1|, |u_2|) \trianglelefteq (a_1, a_2)$ such that $|u_1| \neq a_1$, then

$$A = \begin{bmatrix} u_1 & \frac{u_1(\lambda_1 + \lambda_2 - u_1) - \lambda_1 \lambda_2}{\sqrt{a_1^2 - |u_1|^2}} \\ \sqrt{a_1^2 - |u_1|^2} & \lambda_1 + \lambda_2 - u_1 \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

satisfies $\lambda(A) = \lambda$, $u(A) = u$, $a(A) = a$. In addition, if $u_1, u_2 \in \mathbb{R}$ and if λ_1, λ_2 are real or are complex conjugate pair, then $A \in \mathrm{GL}_2(\mathbb{R})$.

(2) If $u_1 u_2 = \lambda_1 \lambda_2$ and $(|u_1|, |u_2|) = (a_1, a_2)$, then $A \in \mathrm{GL}_2(\mathbb{C})$ satisfying $u(A) = u$, $\lambda(A) = \lambda$ and $a(A) = a$ must be of the form:

$$A = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}$$

so that λ_1, λ_2 are u_1, u_2.

(3) If $-u_1 u_2 = \lambda_1 \lambda_2$ and $(|u_1|, |u_2|) \trianglelefteq (a_1, a_2)$, then $A \in \mathrm{GL}_2(\mathbb{C})$ satisfying $u(A) = u$, $\lambda(A) = \lambda$ and $a(A) = a$ must be of the form:

$$A = \begin{bmatrix} 0 & u_2 \\ u_1 & \lambda_1 + \lambda_2 \end{bmatrix}$$

so that $(|u_1|, |u_2|) = (a_1, a_2)$. In addition, if $u_1, u_2 \in \mathbb{R}$ and if λ_1, λ_2 are real or are complex conjugate pair, then $A \in \mathrm{GL}_2(\mathbb{R})$.

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Department of Mathematics and Statistics, Auburn University, AL 36849–5310, USA
E-mail address: huanghu@auburn.edu, tamtiny@auburn.edu