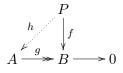
## 4.3 **Projective and Injective Modules**

Given *R*-modules *A* and *B*, the set  $[\operatorname{Hom}_R(A, B)]$  of all *R*-module homomorphisms  $A \to B$  is naturally an *R*-module.

## 4.3.1 **Projective Modules**

**Def.** An *R*-module *P* is **projective** if for any exact sequence  $A \rightarrow B \rightarrow 0$  (i.e. *g* is an epimorphism) and any homomorphism  $P \xrightarrow{f} B$ , there exists an *R*-module homomorphism  $h: P \rightarrow A$  such that  $g \circ h = f$ :



In other words, P is projective iff

$$A \twoheadrightarrow B \to 0$$
 is exact  $\implies$  the induced sequence  $Hom_R(P, A) \twoheadrightarrow Hom_R(P, B) \to 0$  is exact

Thm 4.15. Every free R-module is projective.

Cor 4.16. Every R-module is the homomorphic image of a projective R-module.

**Thm 4.17.** Let P be an R-module. The following are equivalent:

- 1. P is projective;
- 2. if  $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$  is exact, then

$$0 \to Hom_R(P,A) \xrightarrow{f^*} Hom_R(P,B) \xrightarrow{g^*} Hom_R(P,C) \to 0 \qquad is \ exact;$$

3. if  $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$  is exact, then it is split exact (so  $B \simeq A \oplus P$ );

4. there is a free R-module F and an R-module K such that  $F \simeq K \oplus P$ ;

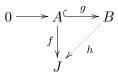
**Ex.**  $\mathbb{Z}_6 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_3$  as  $\mathbb{Z}_6$ -modules. So  $\mathbb{Z}_2$  and  $\mathbb{Z}_3$  are projective (but not free)  $\mathbb{Z}_6$ -modules. **Ex.**  $\mathbb{Z}_2$  is NOT a projective  $\mathbb{Z}_4$ -module.

**Thm 4.18.** A direct sum of R-modules  $\bigoplus_{i \in I} P_i$  is projective iff each  $P_i$  is projective.

## 4.3.2 Injective Modules

Injectivity is dual to projectivity.

**Def.** An *R*-module *J* is **injective** if for any exact sequence  $0 \to A \stackrel{g}{\hookrightarrow} B$ , (i.e. *g* is a monomorphism) and any homomorphism  $A \stackrel{f}{\to} J$ , there exists a homomorphism  $h : B \to J$  such that  $h \circ g = f$ :



In other words, J is injective iff

 $0 \to A \hookrightarrow B$  is exact  $\implies$  the induced sequence  $\operatorname{Hom}_R(B,J) \twoheadrightarrow \operatorname{Hom}_R(A,J) \to 0$  is exact.

Prop 4.19. Every R-module A can be embedded in an injective R-module.

Thm 4.20. Let J be an R-module. The following conditions are equivalent:

- 1. J is injective;
- 2. if  $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$  is exact, then

 $0 \to Hom_R(C,J) \xrightarrow{g_*} Hom_R(B,J) \xrightarrow{f_*} Hom_R(A,J) \to 0 \qquad is \ exact;$ 

3. every short exact sequence  $0 \to J \xrightarrow{f} B \xrightarrow{g} C \to 0$  is split exact (hence  $B \simeq J \oplus C$ ).

4. J is a direct summand of any module B of which J is a submodule.

**Thm 4.21.** A direct product of R-modules  $\prod_{i \in I} J_i$  is injective iff each  $J_i$  is injective.