4.3 Projective and Injective Modules

Given R-modules A and B, the set $\boxed{\operatorname{Hom}_R(A,B)}$ of all R-module homomorphisms $A \to B$ is naturally an R-module.

4.3.1 Projective Modules

Def. *An R-module P is* **projective** *if for any exact sequence $A \twoheadrightarrow B \to 0$ (i.e. g is an epimorphism) and any homomorphism $P \xrightarrow{f} B$, there exists an R-module homomorphism $h : P \to A$ such that $g \circ h = f$:*

$$\begin{array}{ccccc} & & P & & \\ & {\scriptstyle h}\swarrow & \downarrow{\scriptstyle f} & & \\ A & \overset{g}{\twoheadrightarrow} & B & \longrightarrow & 0 \end{array}$$

In other words, P is projective iff

$$A \twoheadrightarrow B \to 0 \text{ is exact} \implies \text{the induced sequence } Hom_R(P,A) \twoheadrightarrow Hom_R(P,B) \to 0 \text{ is exact.}$$

Thm 4.15. *Every free R-module is projective.*

Cor 4.16. *Every R-module is the homomorphic image of a projective R-module.*

Thm 4.17. *Let P be an R-module. The following are equivalent:*

1. *P is projective;*
2. *if $0 \to A \overset{f}{\hookrightarrow} B \overset{g}{\twoheadrightarrow} C \to 0$ is exact, then*
$$0 \to Hom_R(P,A) \overset{f^*}{\hookrightarrow} Hom_R(P,B) \overset{g^*}{\twoheadrightarrow} Hom_R(P,C) \to 0 \qquad \text{is exact;}$$
3. *if $0 \to A \overset{f}{\hookrightarrow} B \overset{g}{\twoheadrightarrow} P \to 0$ is exact, then it is split exact (so $B \simeq A \oplus P$);*
4. *there is a free R-module F and an R-module K such that $F \simeq K \oplus P$;*

Ex. $\mathbb{Z}_6 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_3$ *as $\mathbb{Z}_6$-modules. So $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are projective (but not free) $\mathbb{Z}_6$-modules.*

Ex. $\mathbb{Z}_2$ *is NOT a projective $\mathbb{Z}_4$-module.*

Thm 4.18. *A direct sum of R-modules $\bigoplus_{i \in I} P_i$ is projective iff each P_i is projective.*

4.3.2 Injective Modules

Injectivity is dual to projectivity.

Def. *An R-module J is* **injective** *if for any exact sequence* $0 \to A \overset{g}{\hookrightarrow} B$, *(i.e. g is a monomorphism) and any homomorphism* $A \overset{f}{\to} J$, *there exists a homomorphism* $h : B \to J$ *such that* $h \circ g = f$*:*

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \overset{g}{\hookrightarrow} & B \\ & & {\scriptstyle f}\downarrow & \swarrow {\scriptstyle h} & \\ & & J & & \end{array}$$

In other words, J is injective iff

$$0 \to A \hookrightarrow B \text{ is exact} \implies \text{the induced sequence } Hom_R(B, J) \twoheadrightarrow Hom_R(A, J) \to 0 \text{ is exact.}$$

Prop 4.19. *Every R-module A can be embedded in an injective R-module.*

Thm 4.20. *Let J be an R-module. The following conditions are equivalent:*

1. *J is injective;*
2. *if* $0 \to A \overset{f}{\hookrightarrow} B \overset{g}{\twoheadrightarrow} C \to 0$ *is exact, then*
$$0 \to Hom_R(C, J) \overset{g_*}{\hookrightarrow} Hom_R(B, J) \overset{f_*}{\twoheadrightarrow} Hom_R(A, J) \to 0 \qquad \text{is exact;}$$
3. *every short exact sequence* $0 \to J \overset{f}{\hookrightarrow} B \overset{g}{\twoheadrightarrow} C \to 0$ *is split exact (hence* $B \simeq J \oplus C$*).*
4. *J is a direct summand of any module B of which J is a submodule.*

Thm 4.21. *A direct product of R-modules* $\prod_{i \in I} J_i$ *is injective iff each* J_i *is injective.*