Chapter 4

Modules

We always assume that R is a ring with unity 1_R.

4.1 Modules, Homomorphisms, and Exact Sequences

A fundamental example of groups is the symmetric group S_Ω on a set Ω. By Cayley's Theorem, every group G is isomorphic to a subgroup of the transformation group S_G.

Similarly, a fundamental example of rings is $\text{End}(A)$, the ring of endomorphisms of an abelian group A. Every ring R with unity is isomorphic to a subring of $\text{End}(R)$, determined by $f : R \to \text{End}(R)$, $r \mapsto g_r$, where $g_r(x) := rx$ for $x \in R$.

In general, an R-module is an abelian group A together with a ring homomorphism $f : R \to \text{End}(A)$ such that $1_R \mapsto id_A$.

Def. *Let R be a ring. A (left) R-***module** *is an abelian group A together with a function $R \times A \to A$, $(r, a) \mapsto ra$, such that for all $r, s \in R$ and $a, b \in A$:*

1. $r(a + b) = ra + rb$,
2. $(r + s)a = ra + sa$,
3. $r(sa) = (rs)a$,
4. $1_R a = a$ *for all* $a \in A$.

Right R-modules are similarly defined.

Ex. *Every abelian group A is a $\mathbb{Z}$-module by $(n, a) \mapsto na$ for $n \in \mathbb{Z}$ and $a \in A$.*

Ex. *A vector space V over a division ring $\mathbb{F}$ is an $\mathbb{F}$-module.*

Ex. *Let I be a left ideal of R.*

- *I is a (left) R-module by $(r,x) \mapsto rx$ for $r \in R$ and $x \in I$;*
- *the quotient ring R/I is a (left) R-module by $(r, s+I) \mapsto rs+I$ for $r \in R$ and $s+I \in R/I$.*

Ex. *If S is a subring of R, then R is a S-module by $(s,x) \mapsto sx$ for $s \in S$ and $x \in R$.*

Ex. *Let $\varphi : R \to S$ be a ring homomorphism. Then every S-module A can be made into an R-module by $(r,x) \mapsto \varphi(r)x$. The R-module structure of A is given by* **pullback along** φ.

We use A, B, $\cdots$, to denote R-modules for a ring R with unity.

Def. *A subgroup B of A is called a* **submodule** *of A (notation: $B \leq A$) provided that $rb \in B$ for all $r \in R$ and $b \in B$.*

Ex. *A subspace of a vector space is a submodule.*

Ex. *A subgroup H of an abelian group G is a $\mathbb{Z}$-submodule of G.*

Thm 4.1.

1. *If B_i $(i \in I)$ are submodules of A, then $\bigcap_{i\in I} B_i$ is a submodule of A.*
2. *If $B_1, \cdots, B_n$ are submodules of A, then $B_1 + \cdots + B_n$ is a submodule of A.*

Def. *Let X be a subset of an R-module A. Then*

$$\bigcap_{\substack{B \text{ is a submodule of } A\\ \text{that contains } X}} B$$

is called the **submodule generated by** X.

Thm 4.2. *Let A be an R-module.*

1. *The* **cyclic submodule** *generated by $a \in A$ is $Ra = \{ra \mid r \in R\}$.*
2. *The submodule generated by $X \subseteq A$ is*

$$\left\{ \sum_{i=1}^{s} r_i a_i \,\middle|\, s \in \mathbb{N} \cup \{0\};\ a_i \in X;\ r_i \in R \right\} = \sum_{x \in X} Rx.$$

Def. *Let A and B be R-modules. A function $f : A \to B$ is an R*-**module homomorphism** *if*

$$f(a+c) = f(a) + f(c) \qquad \text{and} \qquad f(ra) = rf(a)$$

for $a, c \in A$ and $r \in R$.

- *The* **kernel** *of f is:*

$$\ker f = \{a \in A \mid f(a) = 0\} \ \leq \ A.$$

- *The* **image** *of f is*

$$Im\, f = \{f(a) \mid a \in A\} \ \leq \ B.$$

The R-module **monomorphism, epimorphism, isomorphism** are similarly defined.

Ex. *An R-module homomorphism over a division ring R is called a* **linear transformation** *of vector spaces.*

Ex. *An abelian group homomorphism $f : A \to B$ is a $\mathbb{Z}$-module homomorphism.*

Ex. *Let $f : A \to B$ be an R-module homomorphism.*

- *If $C \leq A$, then $f(C)$ is a submodule of B.*
- *If $D \leq B$, then $f^{-1}(D) = \{a \in A \mid f(a) \in D\}$ is a submodule of A.*

Ex. *Let A be an R-module. Given $a \in A$, the function $\phi_a : R \to Ra$, $r \mapsto ra$, is an epimorphism. The kernel*

$$\ker \phi_a = \{r \in R \mid ra = 0_A\} := \boxed{Ann(a)}$$

is a left ideal of R.

Thm 4.3. *Let $B \leq A$ as R-modules. Then A/B is an R-module such that*

$$r(a + B) = ra + B \qquad \text{for} \qquad r \in R,\ a \in A.$$

The map $\pi : A \to A/B$ given by $a \mapsto a + B$ is an R-module epimorphism with kernel B (called **canonical epimorphism** *or* **projection***).*

The three isomorphism theorems and the (external/internal) products and coproducts of abelian groups can be extended to their counterparts in modules.

Def. *A pair of module homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is said to be* **exact** *at B if $Im\, f = \ker g$. A sequence of module homomorphisms*

$$\cdots \xrightarrow{f_{i-1}} A_{i-1} \xrightarrow{f_i} A_i \xrightarrow{f_{i+1}} A_{i+1} \xrightarrow{f_{i+2}} \cdots$$

is **exact** *provided that $Im\, f_i = \ker f_{i+1}$ for all indices i.*

For any module A, there are unique module homomorphisms $0 \to A$ and $A \to 0$.

1. $0 \to A \xrightarrow{f} B$ is exact if and only if f is a monomorphism.
2. $B \xrightarrow{g} C \to 0$ is exact if and only if g is a epimorphism.
3. If $A \xrightarrow{f} B \xrightarrow{g} C$ is exact, then $g \circ f = 0$.
4. An exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is called a **short exact sequence**, in which

$$A \simeq \operatorname{Im} f = \ker g, \qquad B/\ker g \simeq \operatorname{Im} g = C.$$

   Whenever $A \leq B$, there is a short exact sequence

$$0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} B/A \to 0$$

Ex. *Let $f : A \to B$ be an R-module homomorphism.*

- *$A/\ker f$ is the* **coimage** *of f (denoted $Coim\, f$), and*
- *$B/Im\, f$ is the* **cokernel** *of f (denoted $Coker\, f$).*

The following sequences are exact:

$$0 \to \ker f \to A \to Coim\, f \to 0$$
$$0 \to Im\, f \to B \to Coker\, f \to 0$$
$$0 \to \ker f \to A \xrightarrow{f} B \to Coker\, f \to 0$$

Thm 4.4. *(The Short Five Lemma) Let R be a ring and*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

a commutative diagram of R-module homomorphisms such that each row is a short exact sequence. Then

1. *α and γ are monomorphisms $\Longrightarrow$ β is a monomorphism;*
2. *α and γ are epimorphisms $\Longrightarrow$ β is a epimorphism;*
3. *α and γ are isomorphisms $\Longrightarrow$ β is a isomorphism. In such case, the row short exact sequences are said to be* **isomorphic**.

Proof.

1. Suppose α and γ are monomorphisms. Let $b \in B$ such that $\beta(b) = 0$. Then

$$\gamma \circ g(b) = g' \circ \beta(b) = g'(0) = 0.$$

So $g(b) = 0$ since γ is injective. Then $b \in \operatorname{Ker} g = \operatorname{Im} f$. There is $a \in A$ such that $b = f(a)$. Then

$$f' \circ \alpha(a) = \beta \circ f(a) = \beta(b) = 0.$$

So $a = 0$ since both f' and α are injective. Therefore $b = f(a) = 0$. This shows that f is a monomorphism.

2. Suppose α and β are epimorphisms. Let $b' \in B'$. Then

$$g'(b') = \gamma(c) = \gamma \circ g(b) = g' \circ \beta(b)$$

for some $c \in C$ and $b \in B$, since γ and g are surjective. Therefore $g'(b' - \beta(b)) = 0$. Thus

$$b' - \beta(b) \in \operatorname{Ker} g' = \operatorname{Im} f' \overset{(*)}{=} \operatorname{Im} f' \circ \alpha = \operatorname{Im} \beta \circ f,$$

where $(*)$ is implied by the surjectivity of α. There is $a \in A$ such that $b' - \beta(b) = \beta \circ f(a)$. Then $b' = \beta(b + f(a)) \in \operatorname{Im} \beta$. Therefore β is an epimorphism.

3. Obviously by 1. and 2.

□

Thm 4.5. *Let R be a ring and $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ a short exact sequence of R-module homoms. Then the following conditions are equivalent:*

1. *There is an R-module homomorphism $h : A_2 \to B$ with $gh = 1_{A_2}$;*

2. *There is an R-module homomorphism $k : B \to A_1$ with $kf = 1_{A_1}$;*

3. *The given sequence is isomorphic to the direct sum short exact sequence*

$$0 \to A_1 \xrightarrow{\iota_1} A_1 \oplus A_2 \xrightarrow{\pi_2} A_2 \to 0;$$

in particular $B \simeq A_1 \oplus A_2$; such a sequence is called a **split exact sequence***.*

Proof.

1. (i)⇒(iii): Define $\varphi : A_1 \oplus A_2 \to B$ by $(a_1, a_2) \mapsto f(a_1) + h(a_2)$. The following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{\iota_1} & A_1 \oplus A_2 & \xrightarrow{\pi_2} & A_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_{A_1}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle 1_{A_2}} & & \\
0 & \longrightarrow & A_1 & \xrightarrow{f} & B & \xrightarrow{g} & A_2 & \longrightarrow & 0
\end{array}$$

   The Short Five Lemma implies that φ is an isomorphism.

2. (ii)⇒(iii): Define $\psi : B \to A_1 \oplus A_2$ by $b \mapsto (k(b), g(b))$. The following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{f} & B & \xrightarrow{g} & A_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_{A_1}} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle 1_{A_2}} & & \\
0 & \longrightarrow & A_1 & \xrightarrow{\iota_1} & A_1 \oplus A_2 & \xrightarrow{\pi_2} & A_2 & \longrightarrow & 0
\end{array}$$

   The Short Five Lemma implies that ψ is an isomorphism.

3. Obvious.

□