4.4 Review

Group

$$\textbf{Def:}\quad \text{Group } (G,*) = \begin{cases} \text{Closed under } * \\ \text{Associativity} \\ \text{Identity} \\ \text{Inverse} \end{cases}$$

If $a * b = b * a$, then $(G, *)$ is an *abelian group.*

Typical examples of groups:

1. Abelian:

   (a) Additive: $\mathbf{Z}$, $r\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{Z}_m$, $\mathbf{R}^n$, $\mathbf{R}^{m\times n}$, the set of all functions (continuous functions, differentiable functions, etc),

   (b) Multiplicative: $\mathbf{Q}^*$, $\mathbf{R}^*$, $\mathbf{C}^*$, $\mathbf{R}^+$, the set of all functions $f : \mathbf{R} \to \mathbf{R}$ with $f(x) \neq 0$ for all x.

   (c) Direct product of abelian groups.

   (*Important!*) The structure theorem of finitely generated abelian groups.

2. Non-abelian: $GL(\mathbf{R}, n)$, $SL(\mathbf{R}, n)$, S_n, A_n, D_n, etc.

Defs: The order of a group, the order of an element, cyclic groups.

Homomorphism

Def: A map $\phi : G \to G'$ between groups $(G, *)$ and $(G', *')$ that satisfies the homomorphism property:

$$\phi(a * b) = \phi(a) *' \phi(b) \qquad \text{for all} \qquad a, b \in G.$$

Homomorphism property implies many nice results:

1. ϕ sends identity to identity, inverse to inverse.

2. ϕ sends subgroups (of G) to subgroups (of G'). Conversely, the inverse image of a subgroup (of G') is a subgroup (of G). In particular, the inverse of the trivial subgroup of G' is a normal subgroup $\ker(\phi)$ of G.

3. The compositions of homomorphisms are homomorphisms

4. If a homomorphism is one-to-one and onto, then it is an isomorphism. An isomorphism of G to G is called an automorphism of G.

Subgroup, Coset, Normal Subgroup

Def: (subgroup of G) A subset H of a group G that is closed under multiplication and inverse (so it must contain the identity of G).

Then we define the cosets gH for $g \in H$.

A subgroup H of G is a *normal subgroup* of G if and only if one of the following holds:

1. $(aH)(bH) = abH$ for all $a, b \in G$;
2. Given $h \in H$, there is an element $h' \in H$ such that $aha^{-1} = h'$;
3. $aH = Ha$ for all $a \in G$;
4. $aHa^{-1} = H$ for all $a \in G$.

We can define the factor group G/H for a normal subgroup H.

Advanced Group Theory

Requirements: Know the meanings of the major theorems and know how to apply the theorems.

Three isomorphism Theorems:

1. If $\phi : G \to G'$ is a homomorphism, $K = \ker(\phi)$, and $\gamma_K : G \to G/K$ is the canonical homomorphism, then there is a unique $\mu : G/K \to \phi(G)$ such that $\phi = \mu \circ \gamma_K$
2. If H is a subgroup, and K is a normal subgroup, of G, then
$$(HN)/N \simeq H/(H \cap N).$$
3. Let H and K be normal subgroups of G with $K \leq H$, then $G/H \simeq (G/K)/(H/K)$.

Defs: Normal/Subnormal series, composition series.
Schreier thm: Every two series of G have isomorphic refinements.
Jordan-Hölder thm: Every two composition series of G are isomorphic.
Solvable groups:
G is solvable if it has a series $\{e\} = H_0 \leq H_1 \leq \cdots \leq H_n = G$, where H_{i+1}/H_i is abelian for $i = 0, \cdots, n-1$.

3 Sylow Theorems: Let G be a finite group with $|G| = p^n m$, where p^n is the maximal p-power factor of $|G|$.

1. (First Sylow thm) G contains a subgroup of order p^i for $0 \leq i \leq n$. Moreover, each subgroup of order p^i is a normal subgroup of order p^{i+1} for $0 \leq i < n$.

2. (Second Sylow thm) All *Sylow p-subgroups* of G (i.e. all subgroups of order p^n) are conjugate to each other.

3. (Third Sylow thm) The number N_p of Sylow p-subgroups of G is congruent to 1 modulo p and divides $|G|$.

Sylow Theorems are useful in studying the structures of finite groups.