Chapter 2

Permutations, Cosets, and Direct Products

2.1 Groups of Permutations

If we rearrange $A := \{1, 2, 3, 4, 5\}$ to another order, say $\{3, 2, 5, 1, 4\}$, then we get a permutation of A. This "rearrangement" is essentially the bijection from $\{1, 2, 3, 4, 5\}$ to $\{1, 2, 3, 4, 5\}$ that

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 2 & 5 & 1 & 4 \end{array}$$

We write this bijection/permutation as $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 1 & 4 \end{smallmatrix}\right)$. [1]

Def 2.1. A permutation of a set A is a bijection $\phi : A \to A$, that is, a one-to-one and onto function from A to A.

Ex 2.2. The inverse of the permutation $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 1 & 4 \end{smallmatrix}\right)$ is $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 1 & 5 & 3 \end{smallmatrix}\right)$.

[2]

2.1.1 Permutations Groups

If σ and τ are both permutations of A, then $\sigma \circ \tau$ is defined by

$$A \xrightarrow{\tau} A \xrightarrow{\sigma} A.$$

[1] In the literature, the permutation $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 1 & 4 \end{smallmatrix}\right)$ may be further simplified as (3 2 5 1 4).

[2] 1st HW: 2, 4, 16, 30, 33, **35**

For simplicity, we use $\sigma\tau$ to represent $\sigma \circ \tau$.

CAUTION: The action of $\sigma\tau$ on A is in right-to-left order: first apply τ and then σ. So $(\sigma\tau)(a) := \sigma(\tau(a))$ for $a \in A$.

Ex 2.3 (Ex 8.4, p.76).

Ex 2.4 (HW 5, p.83).

Thm 2.5. *Let A be a nonempty set. The set S_A of all bijections from A to A (i.e. permutations of A) with functional composition forms a group.*

(Proof)

Def 2.6. The group S_A is called the *permutation group* of A. The functional composition is also called *permutation multiplication.* When A is the finite set $\{1, 2, \cdots, n\}$, the group of all permutations of A is the *symmetric group on n letters*, and is denoted by S_n.

Notes:

1. If $|A| = |B|$, then $S_A \simeq S_B$ (draw a diagram to demonstrate it). So the permutation group of a finite set is isomorphic to some S_n.

2. The group S_n has $n!$ elements.

Ex 2.7 (Ex 8.7, p.78). $S_3 \simeq D_3$.

Def 2.8. The *n-th dihedral group* D_n is the group of symmetries of the regular n-gon. (Some may write D_{2n} instead of D_n since it has $2n$ elements.)

Ex 2.9 (Ex 8.10, p.79). View D_4 as a subgroup of S_4.

2.1.2 Cayley's Theorem

Every group G can be viewed as a subgroup of a permutation group.

In the group table of G, each row gives us a permutation of the group elements. So we can build up a map between G and S_G. Indeed this map is an isomorphism between G and a subgroup of S_G. In other words, it "realizes" G as a subgroup of S_G.

Lem 2.10. Let G and G' be groups and let $\phi : G \to G'$ be a one-to-one function such that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in G$, then $\phi[G]$ is a subgroup of G' and ϕ provides an isomorphism of G with $\phi[G]$.

Thm 2.11 (Cayley's Theorem). *Every group is isomorphic to a group of permutations.*

(proof by left regular representation)

Def 2.12. Let G be a group.

1. Let $\phi : G \to S_G$ be defined by $\phi(x) = \lambda_x$, where the permutation $\lambda_x : G \to G$ is defined by $\lambda_x(g) = xg$ for all $g \in G$. Then ϕ is the *left regular representation* of G.

2. Let $\mu : G \to S_G$ be defined by $\mu(x) := \rho_{x^{-1}}$, where the permutation $\rho_{x^{-1}}$ is defined by $\rho_{x^{-1}}(g) = gx^{-1}$ for all $g \in G$. Then μ is the *right regular representation* of G.

2.1.3 Homework, Section 8, p.83-p.87

1st: 2, 4, 16, 30, 33, **35**

2nd: 18, 27, **46**,

(opt) 10, 20, 36, 49,