2.4 Direct Products and Finitely Generated Abelian Groups

2.4.1 Direct Products

**Def 2.43.** The Cartesian product of sets $S_1, S_2, \cdots, S_n$ is the set of all ordered $n$-tuples $(a_1, \cdots, a_n)$ where $a_i \in S_i$ for $i = 1, 2, \cdots, n$.

\[
\prod_{i=1}^{n} S_i = S_1 \times S_2 \times \cdots \times S_n
\]

\[= \{(a_1, a_2, \cdots, a_n) \mid a_i \in S_i \text{ for } i = 1, 2, \cdots, n\}.
\]

**Caution:** The notation $(a_1, a_2, \cdots, a_n)$ here has different meaning from the cycle notation in permutation groups.

**Thm 2.44.** If $G_1, G_2, \cdots, G_n$ are groups with the corresponding multiplications, then $\prod_{i=1}^{n} G_i$ is a group under the following multiplication:

\[(a_1, a_2, \cdots, a_n)(b_1, b_2, \cdots, b_n) := (a_1b_1, a_2b_2, \cdots, a_nb_n)\]

for $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ in $\prod_{i=1}^{n} G_i$.

**Proof.**

1. **Closed:** $a_i \in G_i, b_i \in G_i \Rightarrow a_ib_i \in G_i \Rightarrow (a_1b_1, \cdots, a_nb_n) \in \prod_{i=1}^{n} G_i$.

2. **Associativity:** By the associativity in each $G_i$.

3. **Identity:** Let $e_i$ be the identity in $G_i$. Then $e := (e_1, e_2, \cdots, e_n)$ is the identity of $G$.

4. **Inverse:** The inverse of $(a_1, \cdots, a_n)$ is $(a_1^{-1}, \cdots, a_n^{-1})$.

\[\square\]

**Ex 2.45.** $\mathbb{R}^2, \mathbb{R}^3$ are abelian groups.
(Every finite dimensional real vector space is an abelian group that is isomorphic to certain $\mathbb{R}^n$.)

**Ex 2.46.** $\mathbb{Z}^2 = \{(a, b) \mid a, b \in \mathbb{Z}\}$ is a subgroup of $\mathbb{R}^2$.
(If $H_i \leq G_i$ for $i = 1, \cdots, n$, then $\prod_{i=1}^{n} H_i \leq \prod_{i=1}^{n} G_i$.)

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Ex 2.47. If \( e_i \) is the identity of \( G_i \) for \( i = 1, \cdots, n \). Then
\[
G_j \simeq \{e_1\} \times \{e_2\} \times \cdots \times \{e_{j-1}\} \times G_j \times \{e_{j+1}\} \times \cdots \times \{e_n\} \leq \prod_{i=1}^{n} G_i.
\]
So every \( G_j \) is isomorphic to a subgroup of \( \prod_{i=1}^{n} G_i \) for \( j = 1, \cdots, n \).

Ex 2.48. The Klein 4-group \( V \) is isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), which is not cyclic.

Ex 2.49. \( \mathbb{Z}_2 \times \mathbb{Z}_3 \simeq \mathbb{Z}_6 \) is cyclic, and \((1,1)\) is a generator.

Thm 2.50. Let \( m \) and \( n \) be positive integers.

1. If \( \gcd(m, n) = 1 \) (i.e. \( m \) and \( n \) are relative prime), then \( \mathbb{Z}_m \times \mathbb{Z}_n \) is cyclic and is isomorphic to \( \mathbb{Z}_{mn} \), and \((1,1)\) is a generator of \( \mathbb{Z}_m \times \mathbb{Z}_n \).

2. If \( \gcd(m, n) \neq 1 \), then \( \mathbb{Z}_m \times \mathbb{Z}_n \) is not cyclic.

Proof. 

1. Suppose \( \gcd(m, n) = 1 \). The sum of \( k \) copies of \((1,1)\) is equal to \((k \mod m, k \mod n)\). The smallest positive integer \( k \) that makes \((k \mod m, k \mod n) = (0,0)\) is \( k = mn \) because \( m \) and \( n \) are relative prime. So the cyclic subgroup of \((1,1)\) in \( \mathbb{Z}_m \times \mathbb{Z}_n \) has the order \( mn = |\mathbb{Z}_m \times \mathbb{Z}_n|\). So \((1,1)\) generates \( \mathbb{Z}_m \times \mathbb{Z}_n \). It implies that \( \mathbb{Z}_m \times \mathbb{Z}_n \) is a cyclic group isomorphic to \( \mathbb{Z}_{mn} \).

2. Suppose \( \gcd(m, n) = d \neq 1 \). Given \((a, b) \in \mathbb{Z}_m \times \mathbb{Z}_n\), the sum of \( k \) copies of \((a, b)\) is \((ka \mod m, kb \mod n)\). Notice that both \( m \) and \( n \) divide \( \frac{mn}{d} = m \cdot \frac{n}{d} = \frac{m}{d} \cdot n \). Then
\[
\left( \frac{mn}{d}a \mod m, \frac{mn}{d}b \mod n \right) = (0,0).
\]
So the cyclic subgroup generated by \((a,b)\) has an order no more than \( \frac{mn}{d} < mn = |\mathbb{Z}_m \times \mathbb{Z}_n|\). Thus \( \mathbb{Z}_m \times \mathbb{Z}_n \) can not be generated by any one of its elements, which implies that \( \mathbb{Z}_m \times \mathbb{Z}_n \) is not cyclic.

Cor 2.51. The group \( \prod_{i=1}^{n} \mathbb{Z}_{m_i} \) is cyclic and is isomorphic to \( \mathbb{Z}_{m_1m_2\cdots m_n} \) if and only if the numbers \( m_i \) for \( i = 1, \cdots, n \) are pairwise relative prime, that is, the gcd of any two of them is 1.

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Proof. (For reference only) If \( m_i \) for \( i = 1, \cdots, n \) are pairwise relative prime, then
\[
\prod_{i=1}^{n} Z_{m_i} \cong Z_{m_1 m_2} \left( \prod_{i=3}^{n} Z_{m_i} \right) \cong Z_{m_1 m_2 m_3} \left( \prod_{i=4}^{n} Z_{m_i} \right) \cong \cdots \cong Z_{m_1 m_2 \cdots m_n}.
\]
Conversely, if \( \prod_{i=1}^{n} Z_{m_i} \cong Z_{m_1 m_2 \cdots m_n} \), then \( \prod_{i=1}^{n} Z_{m_i} \) contains a subgroup isomorphic to \( Z_{m_i} \times Z_{m_j} \) for \( 1 \leq i < j \leq n \). Every subgroup of the cyclic group \( \prod_{i=1}^{n} Z_{m_i} \cong Z_{m_1 m_2 \cdots m_n} \) is cyclic. Hence \( Z_{m_i} \times Z_{m_j} \) is cyclic. So \( \gcd(m_i, m_j) = 1 \) for \( 1 \leq i < j \leq n \).

Cor 2.52. If a positive integer \( n \) is factorized as a product of powers of distinct prime numbers:
\[
n = (p_1)^{n_1} (p_2)^{n_2} \cdots (p_r)^{n_r},
\]
then
\[
Z_n \cong Z_{(p_1)^{n_1}} \times Z_{(p_2)^{n_2}} \times \cdots \times Z_{(p_r)^{n_r}}.
\]

Note: Each \( Z_{(p_i)^{n_i}} \) can not be further decomposed into a product of proper nontrivial subgroups.

Ex 2.53. \( (Z_5)^2 \not\cong Z_{5^2} \). The group \( (Z_5)^2 \) is not cyclic, but \( Z_{5^2} = Z_{25} \) is cyclic.

Def 2.54. The least common multiple (abbreviated lcm) of positive integers \( r_1, r_2, \cdots, r_n \) is the smallest positive integer that is a multiple of each \( r_i \) for \( i = 1, 2, \cdots, n \).

If \( M \) is a multiple of \( r_i \) for \( i = 1, \cdots, n \), then \( M \) is a multiple of \( \text{lcm}(r_1, \cdots, r_n) \).

Thm 2.55. Let \( (a_1, a_2, \cdots, a_n) \in \prod_{i=1}^{n} G_i \). If \( a_i \) is of finite order \( r_i \) in \( G_i \), then the order of \((a_1, a_2, \cdots, a_n)\) in \( \prod_{i=1}^{n} G_i \) is \( \text{lcm}(r_1, r_2, \cdots, r_n) \).

Proof. \( (a_1, a_2, \cdots, a_n)^k = (a_1^k, a_2^k, \cdots, a_n^k) \).

So \( (a_1^m, a_2^m, \cdots, a_n^m) = (e_1, e_2, \cdots, e_n) \) if and only if \( m \) is a multiple of \( r_i \) for \( i = 1, 2, \cdots, n \), if and only if \( m \) is a multiple of \( \text{lcm}(r_1, r_2, \cdots, r_n) \). The smallest such positive integer is \( \text{lcm}(r_1, r_2, \cdots, r_n) \). So the order of \((a_1, a_2, \cdots, a_n)\) is \( \text{lcm}(r_1, r_2, \cdots, r_n) \).

Ex 2.56. The order of \((8, 4, 10)\) in \( Z_{12} \times Z_{60} \times Z_{24} \).

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2.4.2 The Structure of Finitely Generated Abelian Groups

Thm 2.57 (Fundamental Theorem of Finitely Generated Abelian Groups). Every finitely generated abelian group $G$ is isomorphic to a direct product of cyclic groups in the form

$$Z_{(p_1)^{r_1}} \times Z_{(p_2)^{r_2}} \times \cdots \times Z_{(p_n)^{r_n}} \times Z \times Z \times \cdots \times Z$$

where the $p_i$ are primes, not necessarily distinct, and the $r_i$ are positive integers. The direct product is unique except for possible rearrangement of the factors; that is, the number (Betti number of $G$) of factor $Z$ is unique and the prime powers $(p_i)^{r_i}$ are unique.

The proof (omitted) is challenging.

In particular, every finite abelian group is isomorphic to a product of finite cyclic groups. (Caution: finite group $\neq$ finitely generated group)

Ex 2.58. Decompose the following three groups:

- $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15} \simeq \mathbb{Z}_2^2 \times (\mathbb{Z}_2 \times \mathbb{Z}_3) \times (\mathbb{Z}_3 \times \mathbb{Z}_5) \simeq (\mathbb{Z}_2^2)^2 \times (\mathbb{Z}_3)^2 \times \mathbb{Z}_5$,
- $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_{60} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{2^2} \times (\mathbb{Z}_{2^2} \times \mathbb{Z}_4 \times \mathbb{Z}_5) \simeq (\mathbb{Z}_{2^2})^2 \times (\mathbb{Z}_3)^2 \times \mathbb{Z}_5$,
- $\mathbb{Z}_{20} \times \mathbb{Z}_{36} \simeq (\mathbb{Z}_{2^2} \times \mathbb{Z}_5) \times (\mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2}) \simeq (\mathbb{Z}_{2^2})^2 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5$.

Therefore, $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15} \cong \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_{60} \not\cong \mathbb{Z}_{20} \times \mathbb{Z}_{36}$.

Ex 2.59 (Ex 11.13, p.109). Find all abelian groups, up to isomorphism, of order 360.

2.4.3 Applications

Def 2.60. A group $G$ is decomposable if it is isomorphic to a direct product of two proper nontrivial subgroups. Otherwise $G$ is indecomposable.

Thm 2.61. The finite indecomposable abelian groups are exactly the cyclic groups with order a power of a prime.

Thm 2.62. If $G$ is a finite abelian group, then $G$ has a subgroup of order $m$ for every $m$ that divides $|G|$.

Ex 2.63. If $m$ is a square-free integer, then every abelian group of order $m$ is isomorphic to $\mathbb{Z}_m$ and is cyclic.
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2.4.4 Homework, II-11, p.110-p.113

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2nd 13, 15, 32, 47, 50

3rd 18, 26, 36, 53

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