# WEAKLY CONTINUOUSLY URYSOHN SPACES

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Abstract. We study weakly continuously Urysohn spaces, which were introduced in [Z]. We show that every weakly continuously Urysohn $w\Delta$-space has a base of countable order, that separable weakly continuously Urysohn spaces are submetrizable, hence continuously Urysohn, that monontonically normal weakly continuously Urysohn spaces are hereditarily paracompact, and that no linear extension of any uncountable subspace of the Sorgenfrey line is weakly continuously Urysohn. These results generalize various results in the literature concerning continuously Urysohn spaces.

## 1. Introduction

A space $X$ is said to be *continuously Urysohn (cU)* if there is a continuous function $\varphi : X^2 \setminus \Delta \to C(X)$ such that $\varphi(x, y)(x) \neq \varphi(x, y)(y)$. Here $C(X)$ is the space of bounded real-valued functions with the norm topology. This notion was first explored in [S1] and it was named in [HH]. In [Z], we say that a space is *weakly continuously Urysohn (wcU)* if there is a continuous function $\varphi : (X^2 \setminus \Delta) \times X \to \mathbb{R}$ such that $\varphi(x, y, x) \neq \varphi(x, y, y)$. The significance of the wcU property is demonstrated in [Z], where it is shown that a space $X$ is wcU if and only if there is a continuous function $e : \{f : f \in C(H), H \text{ is a compact subset of } X\} \to C(X)$, where $e(f)$ is a continuous extension of $f$ and $\{f : f \in C(H), H \text{ is a compact subset of } X\}$ is endowed with the Vietoris topology.

A theme of this paper is that the wcU property is simpler than the cU property, yet many results in the literature concerning cU spaces carry over to wcU spaces. A good example of this is Stepanova's theorem [S1] that a paracompact $p$-space which is cU must be metrizable; the proof she gives is rather lengthy and involved. In Section 2, we provide a short proof that a submetacompact (=$\theta$-refinable) w$\Delta$-space that is wcU is a Moore space; Stepanova's result is an immediate corollary.

A submetrizable space is continuously Urysohn [S1] and a space with a zero-set diagonal is wcU [Z]. Unfortunately, we don't know of an example of a space which is wcU but not cU.

**Question.** Is every wcU space cU?

In Section 3, we show that separable wcU-spaces are submetrizable (and hence cU).

In Section 4, we extend Bennett and Lutzer's result that monotonically normal cU-spaces are hereditarily paracompact to the class of wcU spaces.

Bennett and Lutzer showed that the linearly ordered extension $S^*$ of the Sorgenfrey line $S$ is, consistently, not cU, and asked if this were true in ZFC. Shi and

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Gao[SG] obtained an affirmative answer, and further showed that no linear extension of $S$ is cU. In Section 5 we generalize Shi and Gao's result in two ways, by showing that no linear extension of any uncountable subspace of the Sorgenfrey line is wcU.

In the last section, we show that any nonarchimedean space is cU; it easily follows that there is a cU Souslin space, which answers a question in [BL].

All spaces are assumed to be regular and $T_1$. We use the following fact througout: If $a < b$ are real numbers and $X$ is wcU, we we may assume that the witnessing function $\varphi$ satisfies $\varphi(x, y, x) = a$ and $\varphi(x, y, y) = b$ [S1][Z].

## 2. wcU-spaces that are Moore Spaces

A space $X$ is a w$\Delta$-space provided that there is a sequence $\{\mathcal{W}_i\}$ of open covers of $X$ such that if, for each $i$, $x \in u_i \in \mathcal{W}_i$ and $y_i \in u_i$, then $\{y_i\}$ has a cluster point.

The notion of a base of countable order is due to Arhangel'skii[A]. We write $X^{<\omega} = \{\langle x_0, x_1, \ldots, x_n\rangle : n < \omega, x_i \in X\}$. Chaber, Choban, and Nagami[CCN] show that a space $(X, \tau)$ has a base of countable order if there is a function $\sigma : X^{<\omega} \to \tau$ satisfying:

1. $\{\sigma(\langle x\rangle) : x \in X\}$ covers $X$;
2. If $x \in \sigma(\langle x_0, x_1, \ldots, x_n\rangle)$, then $\sigma(\langle x_0, x_1, \ldots, x_n, x\rangle) \subset \sigma(\langle x_0, x_1, \ldots, x_n\rangle)$;
3. $\{\sigma(\langle x_0, x_1, \ldots, x_n, x\rangle) : x \in \sigma(\langle x_0, x_1, \ldots, x_n\rangle)\}$ covers $\sigma(\langle x_0, x_1, \ldots, x_n\rangle)$;
4. If $x$ and the sequence $x_0, x_1, \ldots$ satisfy $x, x_{n+1} \in \sigma(\langle x_0, x_1, \ldots, x_n\rangle)$ for all $n$, then
$\{\sigma(\langle x_0, x_1, \ldots, x_i\rangle)\}_{i\in\omega}$ is a base at $x$.

**Theorem 1.** *If $X$ is a weakly continuously Uryshon w$\Delta$-space, then $X$ has a base of countable order.*

**Proof:** Let $\varphi$ be a continuous separating function such that $\varphi(x, y, x) = -1$ and $\varphi(x, y, y) = 1$. Let $\mathcal{W}_i$ be a sequence of open covers witnessing that $X$ is a w$\Delta$-space. Let $U$ be an open set. If $U = \{x\}$ is a degenerate open set, then define $\beta_i(x, U) = U$. Otherwise, let $p(U) \neq q(U)$ be points of $U$. For each $x \in U$ let $\beta_i(x, U)$ be an open set containing $x$ such that $\overline{\beta_i(x, U)} \subset U$ such that

1. $\beta_i(x, U)$ is a subset of some member of $\mathcal{W}_i$,
2. if $x \neq p(U)$ and if $\{y, y'\} \subset \beta_i(x, U)$, then $\varphi(y, p(U), y') < 0$.
3. if $x \neq q(U)$ and if $\{y, y'\} \subset \beta_i(x, U)$, then $\varphi(y, q(U), y') < 0$.

Now, let $\sigma(\langle x\rangle) = \beta_0(x, X)$, and if $\sigma(\langle x_0, x_1, \ldots, x_n\rangle)$ has been defined and $x \in \sigma(\langle x_0, x_1, \ldots, x_n\rangle)$, let

$$\sigma(\langle x_0, x_1, \ldots, x_n, x\rangle) = \beta_n(x, \sigma(\langle x_0, x_1, \ldots, x_n\rangle)).$$

Suppose $x$ and the sequence $x_0, x_1, \ldots$ satisfy $x, x_{n+1} \in \sigma(\langle x_0, x_1, \ldots, x_n\rangle)$ for all $n$. Let $U_n = \sigma(\langle x_0, x_1, \ldots, x_n\rangle)$. Suppose that $\{U_i\}$ is not a base at $x$. Then there is an open set $O$ containing $x$ such that if $i \in \omega$, then there is a point $y_i \in (X \setminus O) \cap U_i$. Since each $U_i$ is a subset of some member of $\mathcal{W}_i$ and $\overline{U_{i+1}} \subset U_i$, $\{y_i\}$ has a cluster point $y \in (X \setminus O) \cap (\cap\{U_i\})$. For each $i$, $U_i = \beta_i(x_i, U_{i-1})$. Note that if $j > i$, then $U_j \subset U_i$ and if $\{x', x''\} \subset U_j$, then $\varphi(x', p(U_j), x'') < 0$ or $\varphi(x', q(U_j), x'') < 0$. We may assume that there is a subsequence $n_i$ of $\omega$ such that $x_{n_i} \neq p(U_{n_i})$ for each $i$. Now, $\{p(U_{n_i})\}$ has a cluster point $z$ distinct from either $x$ or from $y$. We assume that $z$ is distinct from $x$. There is an open set $V$ containing $z$ such that if $\{z', z''\} \subset V$, then $\varphi(x, z', z'') > 1/2$. Choose integers $j > i$ such that $p(U_{n_j})$ and

$p(U_{n_i})$ are both in $V$. But, by construction, $\varphi(x, p(U_{n_i}), p(U_{n_j})) < 0$ which is a contradiction from which the theorem follows. □

In [WW] it is shown that a regular submetacompact space with a base of countable order is a Moore space. Thus, we have the following corollary.

**Corollary 2.** *If $X$ is submetacompact, w$\Delta$, and wcU, then $X$ is a Moore space.*

Since a paracompact p-space is submetacompact and w$\Delta$, we have that Corollary 2 generalizes Stepanova's result. Corollary 2 should also be compared to second result of Stepanova in [S2] where it is argued that a paracompact p-space $X$ is metrizable provided that there is a continuous function $\varphi : X^3 \setminus \Delta \to \mathbb{R}$ such that $\varphi(x, y, x) \neq \varphi(x, y, y)$.

## 3. Separable wcU-spaces

**Theorem 3.** *If $X$ is separable, then the following are equivalent.*

1. *$X$ is wcU.*
2. *$X$ is submetrizable.*
3. *$X$ is continuously Uryshon.*
4. *$X$ has a zero-set diagonal.*

The equivalence of 2 and 4 is a theorem of H. Martin[M]. That 2 implies 3 was done in [S1].

We need only show that 1 implies 2: Let $\varphi : (X^2 \setminus \Delta) \times X \to [0, 1]$ be the continuous function given by the fact that $X$ is a wcU-space. Let $A$ be a countable dense subset of $X$. Then $B = A^2 \setminus \Delta$ is dense in $X^2$. For each $(x, y) \in B$, let $f_{xy} : X \to [0, 1]$ be defined by $f_{xy}(t) = \varphi(x, y, t)$.

Then $\mathcal{F} = \{f_{xy} \,|\, (x, y) \in B\}$ is a countable set of continuous functions from $X$ into $[0, 1]$. We need to show that $\mathcal{F}$ separates points in $X$. To this end, suppose that $(x, y) \in (X^2 \setminus \Delta)$. Let $U \times V$ be a rectangular open set in $X^2 \setminus \Delta$ containing $(x, y)$ such that if $\{(x', y'), (x'', y'')\} \subset U \times V$, then $\varphi(x', y', x'') < 1/4$ and $\varphi(x', y', y'') > 3/4$. Let $(a, b) \in B \cap (U \times V)$. Then $f_{ab}(x) < 1/4$ and $f_{ab}(y) > 3/4$. □

As mentioned earlier, we do not have an example of a wcU-space that is not continuously Urysohn.

In [BL] it is shown that a continuously Urysohn GO-space with countable cellularity is submetrizable. We have the following.

**Theorem 4.** *If $X^2 \setminus \Delta$ satisfies the countable chain condition, then $X$ is continuously Urysohn if and only if $X$ is submetrizable.*

Proof: For each point $(x, y)$ of $X^2 \setminus \Delta$, let $W_{x,y} = (U_{x,y} \times V_{x,y})$ be a rectangular open set in $X^2 \setminus \Delta$ containing $(x, y)$ such that if $(x', y')$ and $(x'', y'')$ are in $W_{x,y}$, then $\varphi(x', y', x'') < 1/8$ and $\varphi(x', y', y'') > 7/8$. Since $X^2 \setminus \Delta$ satisfies the ccc condition, there is a countable subset $A$ of $X^2 \setminus \Delta$ such that $\cup\{W_{x,y} \,|\, (x, y) \in A\}$ is dense in $X^2 \setminus \Delta$.

For $(x, y) \in X^2 \setminus \Delta$ let $f_{x,y} : X \to [0, 1]$ be defined by $f_{x,y}(t) = \varphi(x, y, t)$. We want to show that $\{f_{x,y} \,|\, (x, y) \in A\}$ separates points of $X$. It will follow that there is a one-to-one continuous function from $X$ into $[0, 1]^\omega$ and that $X$ is submetrizable. To this end suppose otherwise. That is, suppose that there is a point $(w, z) \in X^2 \setminus \Delta$ such that if $(x, y) \in A$, then $\varphi(x, y, w) = \varphi(x, y, z)$. Let $B = \{(x, y) \in A \,|\, \varphi(x, y, w) > 1/2\}$ and $C = \{(x, y) \in A \,|\, \varphi(x, y, w) \leq 1/2\}$. Then

$A = B \cup C$ and $(w, z)$ is a limit point of $\cup\{W_{x,y} \,|\, (x, y) \in B\}$ or of $\cup\{W_{x,y} \,|\, (x, y) \in C\}$. We assume $(w, z)$ is a limit point of $\cup\{W_{x,y} \,|\, (x, y) \in B\}$. Then there are $(x, y) \in B$ and $(x', y) \in (W_{x,y}) \cap (W_{w,z})$. Then

$$\begin{aligned} 1/2 &< |\varphi(x, y, w) - \varphi(x, y, x)| \\ &= |\varphi(x, y, w) - \varphi(x, y, x') + \varphi(x, y, x') - \varphi(x, y, x)| \\ &\leq |\varphi(x, y, w) - \varphi(x, y, x')| + |\varphi(x, y, x') - \varphi(x, y, x)| < 1/8 + 1/8 = 1/4 \end{aligned}$$

which is a contradiction, from which the theorem follows. □

## 4. Monotonically Normal wcU-Spaces

**Theorem 5.** *If $S$ is a stationary subset of a regular uncountable cardinal, then $S$ is not wcU.*

**Proof:** Let $S$ be a stationary subset of the regular uncountable cardinal $\lambda$ and suppose that $\varphi : (X^2 \setminus \Delta) \times X \to \mathbb{R}$ witnesses that $S$ is wcU. We may assume $\varphi(x, y, x) = 0$ and $\varphi(x, y, y) = 1$.

Let $\alpha \in S$. For each $\gamma \in S$, there is a $\beta_\alpha(\gamma) \in \gamma$ such that if $\gamma_1, \gamma_2 \in (\beta_\alpha(\gamma), \gamma] \cap S$, then $\varphi(\alpha, \gamma_1, \gamma_2) > 3/4$. By the Pressing Down Lemma, there are an ordinal $\nu(\alpha)$ and a stationary subset $T_\alpha$ of $S$ such that if $\gamma \in T_\alpha$, then $\beta_\alpha(\gamma) = \nu(\alpha)$. It follows that if $\gamma_1, \gamma_2 \in \lambda \setminus \nu(\alpha) + 1$, then $\varphi(\alpha, \gamma_1, \gamma_2) > 3/4$.

Let

$$C = \{\delta \in \lambda \,|\, \delta \text{ is a limit point of } S \text{ and } \nu(\alpha) < \delta \text{ for every } \alpha \in S \cap \delta\}.$$

$C$ is a closed and unbounded subset of $\lambda$. Let $\delta \in C \cap S$ be a limit point of $C \cap S$. Choose $\delta' \in C \cap S \cap \delta$. Since $\varphi(\delta', \delta, \delta') = 0$, there is $\delta'' < \delta'$ such that, if $\gamma \in (\delta'', \delta'] \cap S$, then $\varphi(\gamma, \delta, \delta') < 1/4$. But since $\delta' \in C$, we have $\nu(\alpha) < \delta'$ and so $\varphi(\gamma, \delta, \delta') > 3/4$, a contradiction. □

In [BR], Balogh and Rudin show that a monotonically normal space is paracompact if and only if it contains no subspace that is homeomorphic to a stationary subset of a regular uncountable cardinal. Since monotone normality and the weak continuous Urysohn property are hereditary, we have the following corollary which generalizes Proposition 2.3 in [BL].

**Corollary 6.** *wcU monotonically normal spaces are paracompact.*

## 5. Subsets of the Sorgenfrey line

Shi and Gao[SG], answering a question in [BL], showed that the linearly ordered extension $S^*$ of the Sorgenfrey line $S$ is not cU, and further showed that no linear extension of $S$ is cU. We show:

**Theorem 7.** *No linear extension of any uncountable subspace of the Sorgenfrey line $S$ is wcU.*

**Proof:** Let $T \subset S$ be uncountable. Since the wcU property is hereditary, by removing countably many points if necessary, we may assume every $x \in T$ is a limit point of $(x, \to) \cap T$.

Let $L$ be any linearly ordered space containing $T$ whose order extends the usual order on $T$ and such that the subspace topology $T$ inherits from $L$ is the Sorgenfrey topology. For each $t \in T$, there must be a point $t' \in L \setminus T$ such that $t' < t$, and

$s < t'$ for any $s \in T$ with $s < t$. For $U \subset T$, let $U' = \{t' : t \in U\}$. We will show that $L$ is not wcU by showing that its subspace $T \cup T'$ is not wcU.

Suppose $\varphi$ witnesses that $T \cup T'$ is wcU. Let $C \subset T$ be countable and dense in $T$. Note that each $t \in T$ has a local base of sets of the form $[t, c)$, where $c \in C$ and the interval $[t, c)$ is computed in $L$. We may assume $\varphi(x, y, y) = 1$ and $\varphi(x, y, x) = 0$. Then for each $t \in T$, there is $c_t \in C$ such that

$$\varphi(t', [t, c_t)^2) > 3/4.$$

Let

$$A_c = \{t \in T : c_t = c\}.$$

Choose $c \in C$ such that $|A_c| > \omega$. There exist $x, t_n, t \in A_c$ such that $t_n \to x$ from the right and $t > t_n$ for all $n$. Note that $t'_n \to x$ also, and that $(t, t_n) \in [t_n, c)^2$. Thus $\varphi(t'_n, t, t_n) > 3/4$, but $(t'_n, t, t_n) \to (x, t, x)$ and $\varphi(x, t, x) = 0$, a contradiction. □

## 6. Nonarchimedian spaces

Recall that a space $X$ is *nonarchimedian* if $X$ has a base $\mathcal{B}$ which is a tree under reverse inclusion. Several examples that were proven to be cU in [BL] are nonarchimedian (e.g., Examples 3.1-3.3). Here we show:

**Theorem 8.** *Any nonarchimedian space is cU.*

**Proof:** Let $\mathcal{B}$ be a tree base for $X$. Note that members of $\mathcal{B}$ are clopen sets (see,.e.g., [N]). For $x \in X$, and ordinal $\alpha$, let $B(x, \alpha)$ be that element of $\mathcal{B}$ at level $\alpha$ in the tree which contains $x$, if such an element exists.

We define a function $\varphi : X^2 \setminus \Delta \to C(X)$ as follows. Given $x \neq y \in X$, let $\alpha$ be least such that $B(x, \alpha) \neq B(y, \alpha)$, and let $\varphi(x, y)$ be the characteristic function of $B(y, \alpha)$. This clearly satisfies $\varphi(x, y)(x) \neq \varphi(x, y)(y)$, and $\varphi$ is continuous because, for any $(x', y') \in B(x, \alpha) \times B(y, \alpha)$, we have $B(x', \alpha) = B(x, \alpha)$ and $B(y', \alpha) = B(y, \alpha)$; so $\varphi$ is locally constant. □

It is mentioned in remarks after Question 2.5 in [BL] that the first author has answered 2.5 in the affirmative by showing that if there is a Souslin space (i.e., a linearly ordered non-separable space satisfying the countable chain condition), then there is one that is cU. We use the above result to give more details. Let $T$ be a Souslin tree. Let $B(T)$ be the space whose points are the branches of $T$, and for each $t \in T$, let the set $[t] = \{b \in B(T) : t \in b\}$ be a basic open set. This space is clearly nonarchimedian, so cU. If each node of $T$ has infinitely many successors, then $B(T)$ is the same as the branch space of $T$ as defined in [T] if we order each set of successors so there is no first or last point. It is well-known that the branch space of a Souslin tree is a Souslin space.

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