ON THIN, VERY THIN, AND SLIM DENSE SETS

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Abstract

The notions of thin and very thin dense subsets of a product space were introduced by the third author, and in this article we also introduce the notion of a slim dense set in a product. We obtain a number of results concerning the existence and non-existence of these types of small dense sets, and we study the relations among them.

1 Introduction

In the paper we will consider three kinds of dense subsets in products of topological spaces. Assume that $\kappa > 1$ is a cardinal number, X_α, $\alpha < \kappa$, are topological spaces, and $X = \Pi_{\alpha<\kappa} X_\alpha$. We say that the set $D \subset X$ is

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- *thin* if $|\{\alpha < \kappa \colon x_\alpha \neq y_\alpha\}| > 1$ whenever $x, y \in D$, $x \neq y$, $x = (x_\alpha)_{\alpha<\kappa}$, $y = (y_\alpha)_{\alpha<\kappa}$;
- *very thin* if for every $x, y \in D$, $x \neq y$, and $\alpha < \kappa$ we have $x_\alpha \neq y_\alpha$.
- *slim* if for every non-empty proper subset $K \subset \kappa$ and $v \in \Pi_{\alpha\in K} X_\alpha$ the set $D \cap C(v)$ is nowhere dense in $C(v)$.

Here $C(v)$ denotes the cross-section of X at v, i.e., $C(v) = \{x \in X \colon x \restriction K = v\}$.

Notice that every very thin set in X is also thin. In the product of two spaces those notions coincide. If all X_α are dense-in-themselves T_1 spaces, then every very thin set in X is slim, but we will see that the notions of thinness and slimness are not comparable in general.

The notions of thin and very thin were introduced in [P], where it is shown that the product of $\mathfrak{c}$ many separable spaces always has a countable dense thin set. Answering a question in [P], P. J. Szeptycki [Sz] and, independently, J. Schröder [S], showed that one cannot necessarily produce a very thin dense subset in such a product, for if τ is the topology on ω generated by a maximal independent family, then there is no very thin dense set in $\tau \times \tau$, and hence not in $\tau^{\mathfrak{c}}$. On the other hand, in [S] the author shows that there is a very thin dense subset in a product of $\mathfrak{c}$ spaces if each factor is dense-in-itself and has a countable (weak) π-base.

In this note, we give a number of results which assert the existence or non-existence of these kinds of dense sets. In Section 2, we show that there is a metrizable space X such that X^n does not have a thin dense set for any $n < \omega$, and X^ω has no very thin dense set. We show that any infinite power of a space X has a thin dense set; on the other hand, this does not hold for products with different factors. We also construct, under CH, a space X such that X^2 has a (very) thin dense set, but X^3 does not have a thin dense set.

In Section 3, we show that any product of dense-in-themselves metrizable spaces has a slim dense set, but there are spaces X and Y, one of which is metrizable, such that $X \times Y$ does not have a slim dense set. Assuming CH, we construct a regular separable space whose square does not have a slim dense set, but we show that any infinite product of separable Hausdorff spaces has a dense set which is both slim and thin. In Section 4, we introduce properties (NC) and (GC) of the ideal of nowhere dense sets. Every metrizable space satisfies (GC) and every product of spaces satisfying (GC) has a slim dense set, as does every power of a space satisfying (NC).

Recall that a space X is *irresolvable* if it does not have disjoint dense sets. Noting that the examples of Szeptycki and Schröder mentioned earlier are irresolvable, it is not surprising that some of our results involve this notion. For example, we show that in any model in which there are no irresolvable Baire spaces (e.g., V=L), any infinite power of any dense-in-itself space has a slim dense subset (Theorem 3.12). Also, while we do not know of any example of a dense-in-itself space X such that X^ω has no slim dense set, we show that if there is such an X, then it must contain an irresolvable Baire subspace.

All spaces are assumed to be at least Hausdorff.

2 Very thin and thin dense sets

We begin with a result relating the existence of a very thin dense subset of a product space to certain cardinal functions on the factors. Recall that $\Delta(X)$ is the least cardinal of a non-empty open subset of X. Also, a *π-base* for X is a collection $\mathcal{B}$ of non-empty open sets such that every non-empty open subset of X contains some member of $\mathcal{B}$, and the *π-weight* $\pi w(X)$ of X is a least cardinal of a π-base for X. Finally, the *density* $d(X)$ is the least cardinal of a dense subset of X.

Proposition 2.1. *Assume $X = \Pi_{\alpha<\kappa} X_\alpha$, where all X_α are dense-in-themselves. Consider the following three conditions.*

(i) $\lambda = \inf_{\alpha<\kappa} \triangle(X_\alpha) \geq \sup_{\alpha<\kappa} \pi w(X_\alpha)$ *and* $2^\lambda \geq \kappa$*;*

(ii) *There is a very thin dense set in X;*

(iii) $\triangle(X_\alpha) \geq d(X_\beta)$ *for any $\alpha, \beta < \kappa$, $\alpha \neq \beta$.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof. *(i) $\Rightarrow$ (ii).* This proof is essentially a generalization of the proof of Theorem 8 in [S] for the countable case, as well as a mildly souped up version of the well-known Hewitt-Marczewski-Pondiczery Theorem (Theorem 2.3.15 in [RE]).

Since there is a one-to-one function from κ into the space 2^λ with the product topology, which has weight λ, there is a collection $\mathcal{C}$ of cardinality $\leq \lambda$ of subsets of κ such that, for any finite sequence $\alpha_0, \alpha_1, \ldots, \alpha_n$ of distinct points of κ, there are pairwise-disjoint sets $C_0, C_1, \ldots, C_n \in \mathcal{C}$ with $\alpha_i \in C_i$ for each i.

Let $\{B_{\alpha\beta} : \beta < \lambda\}$ index a π-base for X_α (repetitions are allowed), and consider the collection $\mathcal{P}$ of all finite sequences $\langle (C_i, \beta_i) \rangle_{i<k}$, where $\beta_i \in \lambda$ for each i, and $\{C_i\}_{i<k}$ is a pairwise-disjoint subcollection of $\mathcal{C}$.

Note that $|\mathcal{P}| = \lambda$; let

$$\{\langle (C_i(\delta), \beta_i(\delta)) \rangle_{i<k(\delta)} : \delta < \lambda\}$$

index $\mathcal{P}$. For each $\delta < \lambda$, choose a point $\vec{d_\delta} \in \Pi_{\alpha<\kappa} X_\alpha$ such that $d_\delta(\alpha) \in X_\alpha \setminus \{d_\gamma(\alpha) : \gamma < \delta\}$, and furthermore, if $\alpha \in C_i$, then $d_\delta(\alpha) \in B_{\alpha\beta_i(\delta)}$. It is straightforward to check that $D = \{\vec{d_\delta} : \delta < \lambda\}$ is dense and very thin.

(ii) $\Rightarrow$ (iii). Assume D is dense very thin set in X. Fix $\alpha, \beta < \kappa$, $\alpha \neq \beta$. Let D' be the projection of D onto the product $X_\alpha \times X_\beta$, i.e., $D' = \{\langle d(\alpha), d(\beta) \rangle : d \in D\}$. Then D' is very thin dense in $X_\alpha \times X_\beta$. Fix an open set $U \in X_\alpha$ with $\triangle(X_\alpha) = |U|$. Then D' is dense in $U \times X_\beta$, $|\mathrm{dom}(D' \cap U \times X_\beta)| = |\mathrm{rng}\,(D' \cap U \times X_\beta)|$, and $\mathrm{rng}\,(D' \cap U \times X_\beta)$ is dense in X_β, thus $\triangle(X_\alpha) \geq |U| \geq |\mathrm{rng}\,(D' \cap U \times X_\beta)| \geq d(X_\beta)$. $\square$

Remark 2.2. *For every Hausdorff space X with at least two points, there is a cardinal κ such that there is no dense very thin set in X^κ.*

Proof. We can assume that $\lambda = |X| > 1$. Let $\kappa = 2^{2^\lambda}$. Suppose D is a very thin dense set in X^κ. Then $|D| \le \lambda$, so $d(X^\kappa) < \lambda$. On the other hand, $2^\kappa \le |X^\kappa| \le 2^{2^{d(X)}} \le 2^{2^\lambda} = \kappa$ (see [RE, Theorem 1.5.3]), a contradiction. $\square$

Now we will work with dense thin sets. The examples from [Sz], [S] mentioned in the introduction which show that there is separable dense-in-itself Hausdorff space X with no dense thin set in X^2 cannot be metrizable, because metrizable dense-in-themselves separable spaces satisfy 2.1(i). However, we have an example of a metrizable dense-in-itself space X with a similar property.

Example 2.3. *There is a metrizable space X such that for every positive integer $n > 1$, X^n has no thin dense set, and X^ω has no very thin dense set.*

Proof. Let $X = \mathfrak{c} \times \mathbb{Q}$ be the product of the discrete space of size $\mathfrak{c}$ and the space $\mathbb{Q}$ of rationals with the Euclidean topology. Then $\triangle(X) = \omega$ and $d(X) = \mathfrak{c}$. Thus, by Theorem 2.1(iii), there is no dense very thin set in X^ω. Now fix a positive integer $n > 1$. Notice that X^n can be identify with the product $\mathfrak{c} \times \mathbb{Q}^n$. Suppose $D \subset X^n$ is a thin dense set in X^n. Then for each $\alpha < \mathfrak{c}$ there are rationals $q_{\alpha,1}, \dots, q_{\alpha,n}$ such that $\langle \alpha, q_{\alpha,1}, \dots, q_{\alpha,n}\rangle \in D$, and consequently, there are $q_1, \dots, q_n \in \mathbb{Q}$ for which the set $\{\alpha < \mathfrak{c} \colon \langle \alpha, q_1, \dots, q_n\rangle \in D\}$ has size $\mathfrak{c}$, contrary to the fact that D is assumed to be thin. $\square$

It is not very hard to observe that for X from the example above, there is a thin dense set in X^ω. Now we will show that this is nothing special.

Theorem 2.4. *Let $|X| = \lambda > 1$, and let κ be an infinite cardinal.*

(i) *If $\lambda < \kappa$, then there is a dense $D \subset X^\kappa$ such that, for any $d \ne d' \in D$, $d(\alpha) \ne d'(\alpha)$ for κ-many $\alpha < \kappa$;*

(ii) *If $\lambda \ge \kappa$, then there is a dense $D \subset X^\kappa$ such that, for any $d \ne d' \in D$, $d(\alpha) \ne d'(\alpha)$ for all but finitely many $\alpha < \kappa$.*

Proof. Clearly it suffices to prove this when $X = \lambda$ with the discrete topology. Let $\mathcal{F}$ be the family of all functions defined on finite subsets of κ with values in X.

If $\lambda < \kappa$, then $|\mathcal{F}| = \kappa \cdot \lambda = \kappa$. Set $\mathcal{F} = \{\varphi_\alpha \colon \alpha < \kappa\}$, and let $\mathcal{G} = \{g_\alpha \colon \alpha < \kappa\}$ list the characteristic functions of κ-many pairwise-disjoint subsets of κ, each of size κ. For every $\alpha < \kappa$ choose $f_\alpha \in X^\kappa$ such that $f_\alpha \restriction \mathrm{dom}\varphi_\alpha = \varphi_\alpha$ and f_α agrees with g_α otherwise. Let $D = \{f_\alpha : \alpha < \kappa\}$; then D is as required.

If $\lambda \ge \kappa$, then $|\mathcal{F}| = \kappa \cdot \lambda = \lambda$. Set $\mathcal{F} = \{\varphi_\alpha \colon \alpha < \lambda\}$. For every $\alpha < \lambda$ choose $f_\alpha \in X^\kappa$ such that

(1) $\varphi_\alpha \subset f_\alpha$;

(2) If $\xi \notin \mathrm{dom}\varphi_\alpha$ then $f_\alpha(\xi) \ne f_\beta(\xi)$ for all $\beta < \alpha$.

Clearly D is as required. $\square$

However, there are easy examples of infinite products of *different* spaces, even nice ones, which do not have thin dense sets.

Example 2.5. *There are countably many dense-in-themselves metrizable spaces whose product does not have a thin dense set.*

Proof. Let X_0 be the topological sum of more than continuum-many copies of the rationals $\mathbb{Q}$, and for $n > 0$, let $X_n = \mathbb{Q}$. Let $X = \Pi_{n\in\omega} X_n$. Since the density of X is greater than the continuum, any dense subset of X must contain at least two points that agree on $\Pi_{n>0} X_n$. $\square$

The above results lead to a natural question if there are spaces X, Y, Z such that the product of each pair has a (very) thin dense subset but $X \times Y \times Z$ does not. The next theorem answers this question under CH.

Example 2.6. *(CH) There is a countable regular space X such that X^2 has a thin dense subset, but X^3 does not.*

Proof. The set for X is ω, and we let $D = \{\langle n, n+1\rangle : n \text{ is even}\}$. Clearly D is thin. The plan is to inductively define a topology on X such that D is dense in X^2, killing potential thin dense subsets of X^3 as we go.

Call $E \subset X^3$ *suitable* if it is thin, all three coördinates of every $e \in E$ are distinct, and E satisfies one of the following conditions:

(a) No point $e \in E$ has coördinates $\{k, n, n+1\}$ where $k < n$ and n is even;

(b) There are $i, j < 3$ such that every point of E has coördinates $\{k, n, n+1\}$ where $k < n$ and n is even, and k is the i^{th} coördinate and n the j^{th} coördinate.

Let $\{E_\alpha : \alpha < \omega_1\}$ index all suitable subsets of X^3 with each appearing ω_1 times. We will inductively define $T_\alpha \subset \omega$ such that the topology generated by the T_α's and their complements gives the desired space, and we will denote basic (cl)open sets in this topology by $[\sigma]$, where σ is a function from a finite subset of ω_1 into 2, and $[\sigma] = \bigcap_{\alpha\in\mathrm{dom}(\sigma)} T_\alpha^{\sigma(\alpha)}$, where $T_\alpha^0 = T_\alpha$ and $T_\alpha^1 = \omega \setminus T_\alpha$. Further, we will make $\{T_\alpha\}_{\alpha<\omega_1}$ an independent family, so each $[\sigma]$ will be infinite; in particular, this ensures that X will have no isolated points.

If the T_β's for $\beta < \alpha$ have been defined, let τ_α denote the topology generated by them and their complements, and let $X_\alpha = (\omega, \tau_\alpha)$. It is easy to define an independent family T_n, $n \in \omega$, such that X_ω is a Hausdorff space with no isolated points and such that D is dense in X^2.

Suppose $\omega \le \alpha < \omega_1$ and suppose T_β has been defined for all $\beta < \alpha$ satisfying:

(i) $\{T_\gamma\}_{\gamma\le\beta}$ is an independent family;

(ii) D is dense in $X^2_{\beta+1}$;

(iii) $E_\beta \cap T^3_\beta$ is nowhere-dense in $X^3_{\beta+1}$.

The final space is $X_{\omega_1} = (\omega, \tau_{\omega_1})$. Let us first see that if we are able to carry out the inductive construction, then X_{ω_1} will have the desired properties. Clearly D will be dense in $X^2_{\omega_1}$ since it is so at every stage.

We need to see why $X^3_{\omega_1}$ has no thin dense set E. Suppose on the contrary, $E \subset X^3_{\omega_1}$ is thin and dense. W.l.o.g., each $e \in E$ has distinct coördinates. E maybe isn't suitable, but it is the union of no more than 7 suitable subsets (one satisfying condition (a) in the definition of suitable, 6 satisfying (b)). Label these suitable subsets $E_0, E_1, ..., E_6$ so that there are $\alpha_0 < \alpha_1 < ...\alpha_6 < \omega_1$ such that $E_i = E_{\alpha_i}$ for each i. By the construction, $E_i \cap T^3_{\alpha_i}$ is nowhere-dense in $X^3_{\alpha_i+1}$. It will follow that $E \cap \bigcap_{i<7} T^3_{\alpha_i}$ is nowhere-dense in $X^3_{\omega_1}$ as soon as we verify the following:

Claim. *If $\alpha < \omega_1$ and N is nowhere-dense in X^3_α, then N is nowhere-dense in X^3_β for any $\alpha \leq \beta \leq \omega_1$.*

To see this, let U be dense open in X^3_α with $U \cap N = \emptyset$. It suffices to show that U remains dense in X^3_β for $\beta > \alpha$. Suppose otherwise, and consider the least β such that U is not dense in X^3_β. It is easy to see that β is not a limit ordinal. Let $\beta = \gamma + 1$. There are σ_i, $i < 3$, with $\mathrm{dom}(\sigma_i)$ a finite subset S_i of β, such that $([\sigma_0] \times [\sigma_1] \times [\sigma_2]) \cap U = \emptyset$. Let $\sigma'_i = \sigma_i \restriction (S_i \setminus \{\gamma\})$. Since U is dense open in X^3_γ, there are extensions σ''_i of σ'_i (with $\mathrm{dom}(\sigma''_i) \subset \gamma$) such that $([\sigma''_0] \times [\sigma''_1] \times [\sigma''_2]) \subset U$. But then if $\tau_i = \sigma''_i \cup \sigma_i$, we have that $([\tau_0] \times [\tau_1] \times [\tau_2])$ is a non-empty open subset of $([\sigma_0] \times [\sigma_1] \times [\sigma_2]) \cap U = \emptyset$, contradiction.

To complete the inductive construction, suppose that T_β has been defined for $\beta < \alpha$, and we are given the suitable subset E_α of ω^3. Let $\langle U_n, V_n, i_n \rangle$, $n \in \omega$, index all triples $\langle U, V, i \rangle$ where $i \in \omega$ and U, V is a pair of nonempty open sets in a countable base for X_α, such that, for each such pair $\langle U, V \rangle$, each $i \in \omega$, and each $j < 4$, the set

$$M_j = \{m : \ \langle U_m, V_m \rangle = \langle U, V \rangle, \ \ i_m = i, \text{ and } m = j \bmod 4\}$$

is infinite.

We will define by induction disjoint finite sets F_k, G_k, and we promise to make $F_k \subset T_\alpha$ and $G_k \subset \omega \setminus T_\alpha$. Let $F_0 = G_0 = \emptyset$ to start, and suppose at step k, we have chosen F_k, G_k. We will choose at this step an even n with $\langle n, n+1 \rangle \in U_k \times V_k$, and distribute n and $n+1$ into F_k or G_k in some way depending on the value of $i_k \bmod 4$. Let us say that if $i_k = 0 \bmod 4$, we will put both n and $n+1$ in F_k to get F_{k+1}; so if $i_k \neq 0 \bmod 4$, at most one of $n, n+1$ will be added to F_k.

Since E_α is thin, the set $H_k = \{n : \ E_\alpha \cap (F_k \cup \{n\})^3 \setminus F^3_k \neq \emptyset\}$ is finite. If $i_k \neq 0 \bmod 4$, choose any even n such that $\langle n, n+1 \rangle \in U_k \times V_k$ and $n > \max(F_k \cup G_k \cup H_k)$ and distribute $n, n+1$ to F_k and G_k as follows: put $n \in F_k$, $n+1 \in G_k$ if $i_k = 1 \bmod 4$, the reverse if $i_k = 2 \bmod 4$, and both in G_k if $i_k = 3 \bmod 4$. Note that in this case, $E_\alpha \cap F^3_{k+1} = E_\alpha \cap F^3_k$.

Now suppose $i_k = 0 \bmod 4$; we will put both n and $n+1$ in F_k for some appropriate n. If possible, choose an even n larger than $\max(F_k \cup G_k \cup H_k)$ with $\langle n, n+1 \rangle \in U_k \times V_k$ such that $E_\alpha \cap F^3_{k+1} = E_\alpha \cap F^3_k$ (where $F_{k+1} = F_k \cup \{n, n+1\}$).

Note that this is always possible if E_α satisfies condition (a) of the definition of suitable. If this is not possible, then any even n larger than $\max(F_k \cup G_k \cup H_k)$ with $\langle n, n+1\rangle \in U_k \times V_k$ will do.

Let $T_\alpha = \bigcup_{k\in\omega} F_k$. By the construction, both T_α and its complement meet every basic clopen subset of X_α in an infinite set. It follows that $\{T_\beta : \beta \leq \alpha\}$ is an independent family. It also follows easily from the construction that each basic open set $[\sigma_0] \times [\sigma_1]$ in $X^2_{\alpha+1}$ meets D.

It remains to prove that $E_\alpha \cap T^3_\alpha$ is nowhere-dense in $X^3_{\alpha+1}$. Suppose not. Note that by the construction, if E_α satisfies condition (a) of the definition of suitable, then we could at any stage add what we needed to the set F_k so that $E_\alpha \cap F^3_{k+1} = E_\alpha \cap F_k$. So in this case, $E_\alpha \cap T^3_\alpha = \emptyset$. Hence we may assume E_α satisfies condition (b) of the definition of suitable, for some $i_0, j_0 < 3$. To ease the notation, let us assume $i_0 = 1$ and $j_0 = 2$, as the other cases are similar. Since we are assuming $E_\alpha \cap T^3_\alpha$ is somewhere-dense in $X^3_{\alpha+1}$, there are basic open sets B_i, $i < 3$, of X_α such that E_α contains a dense subset of $(B_0 \times B_1 \times B_2) \cap T^3_\alpha$. Let $e \in E_\alpha \cap (B_0 \times B_1 \times B_2) \cap T^3_\alpha$. Then $e \in F^3_{m+1} \setminus F^3_m$ for some m, and hence $e = \langle n_e + 1, k_e, n_e\rangle$, where $k_e \in F_m \subset n_e$, and $\langle n_e, n_e + 1\rangle \in U_m \times V_m$. It follows that

$$[(V_m \cap B_0) \times B_1 \times (U_m \cap B_2)] \cap T^3_\alpha$$

is open non-empty (it contains e), and thus so is

$$[(V_m \cap B_0) \times B_1 \times (U_m \cap B_2)] \cap (T_\alpha \setminus n_e + 1)^3.$$

There must be some $e' \in E_\alpha$ which is also in the above open set. Now $e' = \langle n_{e'} + 1, k_{e'}, n_{e'}\rangle$, where $k_{e'}, n_{e'} > n_e$. By the construction, since $e \in F^3_{m+1} \setminus F^3_m$, at step m it was not possible to add $n, n+1$ to F_m such that n is larger than $\max(F_m \cup G_m \cup H_m)$ with $\langle n, n+1\rangle \in U_m \times V_m$ and $E_\alpha \cap F^3_{m+1} = E_\alpha \cap F^3_m$. It follows that there is another point $e'' = \langle n_{e'} + 1, k, n_{e'}\rangle$ in E_α for some $k \in F_m$. But this contradicts thinness of E_α. $\square$

3 Slim dense sets

If X is the space defined in Example 2.3 then X^2 has no thin dense set, but it is not hard to show that X^2 has a slim dense set. In fact, we will prove that any product of metrizable spaces with no isolated points has a slim dense set. (See Corollary 4.3.) Thus *slim dense* does not imply *thin dense*.

Lemma 3.1. *Assume X and Y are topological spaces and $|X| = \kappa$. If no union of κ-many nowhere dense sets in Y is dense in Y, then there is no dense slim set in $X \times Y$.*

Proof. Suppose D is a slim dense set in $X \times Y$. Let $\mathrm{dom}(D)$ be the projection of D onto X, $\mathrm{rng}\,(D)$ be the projection of D onto Y, and for each $x \in \mathrm{dom}(D)$ let $D_x = \{y \in Y : \langle x, y\rangle \in D\}$ be the x-section of D. Then $\mathrm{rng}\,(D)$ is dense in Y and $\mathrm{rng}\,(D) = \bigcup\{D_x : x \in \mathrm{dom}(D)\}$. But each D_x is nowhere dense in Y, so $\mathrm{rng}\,(D)$ is not dense in Y, a contradiction. $\square$

Corollary 3.2. *There are two dense-in-themselves spaces, a metrizable separable space X and a regular space Y, with no slim dense set in $X \times Y$.*

Proof. Let $X = \mathbb{Q}$ be the space of rationals with the Euclidean topology, and let Y be the real line with the density topology. (Recall that the density topology is Tychonoff but not normal; see [GNN], or [O], the remark before Theorem 22.9.) Then X and Y satisfy assumptions of Lemma 3.1. $\square$

Recall that a space X is κ-resolvable if it can be decomposed into κ-many dense subsets.

Example 3.3. *Let $\mathbb{Q}$ be the space of rationals with the Euclidean topology, let Y be any ω-resolvable space in which no meager set is dense (e.g., let Y be the same as in Corollary 3.2), and let $X = \mathbb{Q} \oplus Y$. Then X^n has no slim dense set for any $n < \omega$ but X^ω does have a slim dense set.*

Proof. Notice that X is ω-resolvable, so the fact that X^ω has a slim dense set will follow from Corollary 3.7. Now, for a given $n < \omega$, suppose D is slim dense in X^n. Then $D \cap (\mathbb{Q}^{n-1} \times Y)$ is slim dense in $\mathbb{Q}^{n-1} \times Y$, contrary to Lemma 3.1. $\square$

Example 3.4. *(CH) There is a countable regular space X (generated by a maximal independent family on ω) such that X^2 has no slim dense subset.*

Proof. The set for X is ω. List all infinite subsets of ω in the sequence $\{B_\alpha \colon \alpha < \omega_1\}$ and all subsets of ω^2 in the sequence $\{E_\alpha \colon \alpha < \omega_1\}$ such that each set appears in this sequence ω_1 times. For $\alpha < \omega_1$ we will define 0-dimensional T_2 topology on ω generated by a countable independent family $\mathcal{A}_\alpha$ such that

- If $\beta < \alpha$ then $\mathcal{A}_\beta \subset \mathcal{A}_\alpha$, so $\tau_\beta \subset \tau_\alpha$.
- Either B_α or $\omega \setminus B_\alpha$ is not dense in $\tau_{\alpha+1}$.
- E_α is not slim dense in $\tau^2_{\alpha+1}$.

Let τ_0 be a 0-dimensional topology on ω generated by a countable independent family $\mathcal{A}_0$. Suppose $\mathcal{A}_\beta$ and τ_β are defined for $\beta \leq \gamma$ and $\alpha = \gamma + 1$. If B_γ is not dense or not co-dense in τ_γ then $\mathcal{A}'_\alpha = \mathcal{A}_\gamma$ and $\tau'_\alpha = \tau_\gamma$. Otherwise $\mathcal{A}_\gamma \cup \{B_\gamma\}$ is an independent family. Then set $\mathcal{A}'_\alpha = \mathcal{A}_\gamma \cup \{B_\gamma\}$ and define τ'_α as the topology generated by $\mathcal{A}'_\alpha$. Note that B_γ is not dense in τ'_α.

Next, if E_γ is not slim dense in $\tau'_\alpha \times \tau'_\alpha$ then put $\mathcal{A}_\alpha = \mathcal{A}'_\alpha$ and $\tau_\alpha = \tau'_\alpha$. Otherwise we will define $T_\alpha \subset \omega$ such that $\mathcal{A}'_\alpha \cup \{T_\alpha\}$ is an independent family and $T^2_\alpha \cap E_\gamma = \emptyset$. Let $\{U_n \colon n < \omega\}$ be a sequence of basis sets of τ'_α with each appearing infinite many times. Choose inductively two sequences a_n, b_n, $n < \omega$, such that

(i) $a_n, b_n \in U_n \setminus \{a_i, b_i \colon i < n\}$, $a_n \neq b_n$;

(ii) $\{a_i \colon i \leq n\}^2 \times E_\gamma = \emptyset$.

Such a choice is possible because the set $H_n = \{x \colon \langle x, a_i\rangle \in E_\gamma \text{ or } \langle a_i, x\rangle \in E_\gamma\}$ is nowhere dense (as E_γ is slim), so $U_n \setminus H_n$ is infinite. Let $T_\alpha = \{a_n \colon n < \omega\}$. Since $\{b_n \colon n \in \omega\} \subset \omega \setminus T_\alpha$, $\mathcal{A}_\alpha = \mathcal{A}'_\alpha \cup \{T_\alpha\}$ is an independent family. Let τ_α be the topology on ω generated by $\mathcal{A}_\alpha$. Then τ_α is a T_2 second countable 0-dimensional space without isolated points such that B_γ is not dense or not co-dense in τ_α and E_γ is not slim dense in τ_α^2.

Let $\tau = \bigcup_{\alpha<\omega_1} \tau_\alpha$ and $X = (\omega, \tau)$. Then τ is generated by the independent family $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$.

Observe that $\mathcal{A}$ is maximal. In fact, otherwise there is $B \subset \omega$ such that $\mathcal{A} \cup \{B\}$ is an independent family. Then $B = B_\gamma$ for some $\gamma < \omega_1$, and B is dense in τ. Thus B is dense in $\tau_{\gamma+1}$, a contradiction.

Finally we will verify that there is no dense slim set in X^2. Suppose E is such. W.l.o.g., we may assume each $e \in E$ has distinct coördinates. Since X is countable, every open set U in X (resp., X^2) is the union of countably many basic open sets, and hence is open in (X, τ_α) (resp., $(X, \tau_\alpha)^2$) for some $\alpha < \omega_1$. Since τ_α is a weaker topology, if U is also dense in X it is dense in the weaker topology. It follows that, since there are only countably many cross-sections to consider, E is also a slim dense set in some (X, τ_α). Let $\beta \geq \alpha$ such that $E_\beta = E$. Then by condition (iii) in the construction, E_β is not dense in $(X, \tau_{\beta+1})$, contradiction. □

Since *thin* implies *very thin* in X^2, it also implies *slim* in X^2. But for X^3, *thin* need not imply *slim*.

Example 3.5. *(CH) There is a countable regular space X such that X^2 has a (very) thin dense subset, and X^3 has a thin dense set, but X^3 has no slim (hence no very thin) dense subset.*

Proof. The set for X is ω. Let $D = \{\langle n, n+1\rangle : n \text{ even}\}$, and let $\{D_n\}_{n\in\omega}$ be a partition of D into infinite sets. Our goal will be to define a topology on X such that each D_n is dense in X^2. Thus X^2 has a (very) thin dense set. It also follows that $E = \{\langle n, m, m+1\rangle : \langle m, m+1\rangle \in D_n\}$ is a thin dense subset of X^3.

We need to take care that X^3 has no slim dense set. Let $\{E_\alpha : \alpha < \omega_1\}$ index all subsets of X^3 with each appearing ω_1 times, and such that each $e \in E_\alpha$ has distinct coördinates. We will define $T_\alpha \subset \omega$ such that the topology generated by the T_α's and their complements gives the desired space. We will denote basic (cl)open sets in this topology by $[\sigma]$, where σ is a function from a finite subset of ω_1 into 2, and $[\sigma] = \bigcap_{\alpha\in\mathrm{dom}(\sigma)} T_\alpha^{\sigma(\alpha)}$, where $T_\alpha^0 = T_\alpha$ and $T_\alpha^1 = \omega \setminus T_\alpha$.

We will define the T_α's inductively. If the T_β's for $\beta < \alpha$ have been defined, let τ_α denote the topology generated by them and their complements. To start, it is easy to define T_n, $n \in \omega$, such that (X, τ_ω) is a Hausdorff space with no isolated points and such that each D_n is dense.

Now suppose $\omega \leq \alpha < \omega_1$ and suppose T_β has been defined for all $\beta < \alpha$ satisfying:

(i) $(X, \tau_{\beta+1})$ has no isolated points;

(ii) Each D_n is dense in $(X, \tau_{\beta+1})^2$;

(iii) If E_β is a slim dense subset of $(X, \tau_\beta)^3$, then $E_\beta \cap T_\beta^3 = \emptyset$.

Conditions (i) and (ii) imply that $(X, \tau_\alpha)^2$ has no isolated points and each D_n is dense. Suppose that E_α is a slim dense subset of $(X, \tau_\alpha)^3$. Let $\langle U_n, V_n, i_n \rangle$, $n \in \omega$, index all triples $\langle U, V, i \rangle$ where $\langle U, V \rangle$ is a pair of nonempty open sets in a countable base for (X, τ_α) and $i \in \omega$, such that, for each such pair (U, V), each $i \in \omega$, and each $j < 4$, the set

$$M_j = \{m : \langle U_m, V_m \rangle = \langle U, V \rangle, \ i_m = i, \text{ and } m = j \bmod 4\}$$

is infinite.

We will define T_α in steps, taking care that conditions (i)–(iii) will hold for $\beta = \alpha$. Let $F_0 = G_0 = \emptyset$. Suppose at step n, we have disjoint finite sets F_n, G_n, we promise to make $F_n \subset T_\alpha$ and $G_n \subset \omega \setminus T_\alpha$, and we also have $F_n^3 \cap E_\alpha = \emptyset$ Since each $e \in E_\alpha$ has distinct coördinates, any $e \in E_\alpha \cap (\{l, m\} \cup F_n \cup G_n)^3$ has a coördinate in $F_n \cup G_n$. Hence, since E_α is slim, it follows that the set

$$H_n = \{\langle l, m \rangle : \ [(\{l, m,\} \cup F \cup G)^3 \setminus (F \cup G)^3] \cap E_\alpha \neq \emptyset\}$$

is nowhere dense in $(X, \tau_\alpha)^2$. Hence we can choose a_n, b_n, such that $\langle a_n, b_n \rangle \in D_{i_n} \cap [(U_n \times V_n) \setminus H_n]$ and $\{a_n, b_n\} \cap (F_n \cup G_n) = \emptyset$. Now, if $n = j \pmod 4$, put both a_n and b_n in F_n if $j = 0$, both in G_n if $j = 1$, a_n in F_n, $b_n \in G_n$ if $j = 2$, and a_n in G_n, b_n in F_n if $j = 3$.

Let $T_\alpha^0 = T_\alpha$ and $T_\alpha^1 = \omega \setminus T_\alpha$. By the way we indexed of the $\langle U, V, i \rangle$'s, for any fixed $\langle U, V \rangle$ and any i, and any $j, k < 2$, the set $(U \times V) \cap D_i \cap T_\alpha^j \times T_\alpha^k$ is infinite. It follows that $(X, \tau_{\alpha+1})$ has no isolated points and each D_n is dense in $(X, \tau_{\alpha+1})^2$, i.e., conditions (i) and (ii) hold.

In the inductive definition of T_α, note that if $e \in F_{n+1}^3 \cap E_\alpha$, then some coördinate of e is in F_n (since $|F_{n+1} \setminus F_n| \leq 2$ and all coördinates of e are distinct); but this contradicts the facts that $F_n^3 \cap E_\alpha = \emptyset$ and $\{a_n, b_n) \notin H_n$.

Let $\tau = \tau_{\omega_1}$; we will show that $X = (X, \tau)$ has the desired properties. We have already noted that X^2 and X^3 have dense thin sets, so it remains to show that there is no dense slim subset of X^3. Suppose E is such. This fact can be described by use only countably many sets from $\mathcal{A}$, thus there is $\alpha < \omega_1$ such that E is slim dense in $\tau_\beta \times \tau_\beta$ for each $\beta \geq \alpha$. (See the end of proof of Example 3.4.) Fix $\gamma \geq \alpha$ such that $E = E_\gamma$, then E is slim dense in $\tau_{\gamma+1} \times \tau_{\gamma+1}$, a contradiction. $\square$

Now we will consider the existence of slim dense subsets of infinite powers of a space X.

Proposition 3.6. *Suppose X admits a decreasing sequence D_n of dense sets such that $\bigcap_{n<\omega} D_n = \emptyset$. Then for any infinite cardinal κ, X^κ has a dense set which is both slim and thin.*

Proof. Let $x_0 \in X$, let θ be a one-to-one mapping of $\bigcup_{n\in\omega} D_n^n$ into $X \setminus \{x_0\}$, and let $\{K_\alpha\}_{\alpha<\kappa}$ be κ-many pairwise-disjoint subsets of κ of cardinality κ. Also, let $[\kappa]^{<\omega}$ denote the finite subsets of κ, and let $\nu : [\kappa]^{<\omega} \to \kappa$ be one-to-one.

For every $n > 0$, let E_n be the set of all points $e \in X^\kappa$ such that, for some subset F_e of κ of size n, we have:

(1) $e \restriction F_e$ is a one-to-one map of F_e into D_n ;

(2) For every $\beta \notin F_e$, if $\beta \in K_{\nu(F_e)}$ then $e(\beta) = x_0$ else $e(\beta) = \theta((e \restriction F_e) \circ s)$, where s is the order preserving bijection of n to F_e.

Then let $E = \bigcup_{n\in\omega} E_n$.

Claim 1. *E is dense.*

Clear.

Claim 2. *E is thin.*

Suppose $e \neq e' \in E$. If $F_e \neq F_{e'}$, then e and e' differ on $K_{\nu(F_e)} \cup K_{\nu(F_{e'})}$, while if $F_e = F_{e'}$, then $e \restriction F_e \neq e' \restriction F_{e'}$ (else $e = e'$), so e and e' differ on $\kappa \setminus (K_{\nu(F_e)} \cup F_e)$.

Claim 3. *E is slim.*

Let $p : A \to X$, where A is a nonempty proper subset of κ. Let $C(p) = \{f \in X^\kappa : f \restriction A = p\}$. Let us assume $E \cap C(p) \neq \emptyset$. We need to show $E \cap C(p)$ is nowhere dense in $C(p)$.

Case 1. A is infinite. Then there is $x \in \operatorname{rng}(\theta) \setminus \{x_0\}$ such that, for almost all $\alpha \in A$, $p(\alpha) \in \{x, x_0\}$. So $\theta^{-1}(x)$ is some point s in D_n^n for some unique n, and the projection of $E \cap C(p)$ on any coördinate is included in the finite set $\{x, x_0\} \cup \operatorname{rng}(s)$.

Case 2. A is finite. Then for some k, $\operatorname{rng}(p) \cap D_k = \emptyset$. Also, for each $x \in \operatorname{rng}(p)$, if $\theta^{-1}(x)$ is defined, it is some point in $D_{k_x}^{k_x}$ for some $k_x \in \omega$. Choose $m > k + \max\{k_x : x \in \operatorname{rng}(p)\}$. Then any point of $E \cap C(p)$ has no more than m distinct coördinates. So if U_i, $i < m+1$ are pairwise disjoint open sets in $X \setminus \{x_0\}$, and α_i, $i < m+1$ are distinct members of $\kappa \setminus A$, then

$$C(p) \cap \bigcap_{i<m+1} \pi_{\alpha_i}^{-1}(U_i)$$

is a dense open set in $C(p)$ which misses E. $\square$

Corollary 3.7. *If X is ω-resolvable then for any infinite cardinal κ, X^κ has a dense slim and thin set.*

Proof. Let $X = \bigcup_{n<\omega} X_n$ be the partition of X onto dense subsets. Then $D_n = \bigcup_{k>n} X_k$, $n < \omega$, satisfy assumptions of Proposition 3.6. $\square$

Corollary 3.8. *Assume X has a meager dense subset. Then for any infinite cardinal κ, X^κ has a dense slim and thin set.*

Proof. Let N_k, $k < \omega$ be a sequence of nowhere dense sets in X such that $\bigcup_{k<\omega} N_k$ is dense. Then $D_n = \bigcup_{k>n} N_k$, $n < \omega$, satisfy assumptions of Proposition 3.6. □

Corollary 3.9. *If X is a separable Hausdorff space with no isolated points then for any infinite cardinal κ, X^κ has a dense slim and thin set.*

Proof. If $D = \{d_n : n < \omega\}$ is dense in X, then $D_n = \{d_i : i > n\}$, $n < \omega$, satisfy the assumptions of Proposition 3.6. □

Proposition 3.6 is only true for powers, not the product of different spaces.

Example 3.10. *There are countably many dense-in-themselves spaces, each having a sequence of dense subsets as in Proposition 3.6, whose product has no slim or thin dense set.*

Proof. Let X_0 be any dense-in-itself ω-resovable space such that no union of $\mathfrak{c}$-many nowhere dense sets is dense. (Such an example is 2^κ, where $\kappa > \mathfrak{c}$, with the topology obtained by declaring to be open any intersection of less than κ-many sets open in the usual topology.) For each $n > 0$, let X_n be a copy of the rationals. Then for every dense set D in $\Pi_{n\in\omega} X_n$, there is some $\vec{v} = \langle v_1, v_2, \ldots\rangle \in \Pi_{n>0} X_n$ such that $\{x \in X_0 : \langle x, v_1, v_2, \ldots\rangle \in D\}$ is somewhere dense in X_0. Hence D is neither slim nor thin. □

Nevertheless, Corollary 3.9 can be generalized to products of different spaces.

Proposition 3.11. *Any infinite product of separable Hausdorff spaces with no isolated points has a slim and thin dense set.*

Proof. Let κ be an infinite cardinal, and suppose for each $\alpha < \kappa$, X_α is a Hausdorff space with no isolated points with a countable dense proper subset $D_\alpha \subset X_\alpha$, say $D_\alpha = \{d_{\alpha i} : i \in \omega\}$. We will show that $X = \Pi_{\alpha<\kappa} X_\alpha$ has a slim and thin dense set.

For each n, let $D_{\alpha n} = \{d_{\alpha i} : i \geq n\}$. W.l.o.g., there are countably infinite $K_{\alpha n}$, $n \in \omega$, which are disjoint from each other and from D_α, such that $K_\alpha = \bigcup_{n\in\omega} K_{\alpha n}$ is nowhere dense in X_α. Let $K_{\alpha n} = \{k_{\alpha n i} : i \in \omega\}$.

W.l.o.g., $D_\alpha \neq X_\alpha$, so we can choose $\vec{v} \in X$ such that $v(\alpha) \notin D_\alpha \cup K_\alpha$ for all α. Also let $\{H_\alpha\}_{\alpha<\kappa}$ be κ-many disjoint subsets of κ, each of size κ, and let $\nu : [\kappa]^{<\omega} \to \kappa$ be one-to-one, where $[\kappa]^{<\omega}$ denotes the set of all finite subsets of κ. Finally, let $\mu : \omega^{<\omega} \to \omega$ be one-to-one such that $\mu(\sigma) \geq \max \mathrm{rng}\,(\sigma)$.

Now for every $n > 0$, let E_n be the set of all points $e \in X$ such that, for some subset F_e of κ of size n, we have:

(1) $e(\alpha) \in D_{\alpha n}$ for every $\alpha \in F_e$;

(2) Let $\sigma_e \in \omega^n$ be such that $e(\alpha_j) = d_{\alpha_j \sigma_e(j)}$, where α_j is the j^{th} member (in the natural order on the ordinals) of F_e. Then for every $\beta \notin F_e$, if $\beta \in H_{\nu(F_e)}$ then $e(\beta) = v(\beta)$, else $e(\beta) = k_{\beta n \mu(\sigma_e)}$.

Note that the E_n's are pairwise disjoint. Let $E = \bigcup_{n\in\omega} E_n$.

Claim 1. *E is dense in X.*

This follows easily from the definition of E and the fact that each $D_{\alpha n}$ is dense in X_α.

Claim 2. *E is thin.*

Suppose $e \neq e' \in E$. If $F_e \neq F_{e'}$, then e and e' differ on $H_{\nu(F_e)} \cup H_{\nu(F_{e'})}$, while if $F_e = F_{e'}$, then $\sigma_e \neq \sigma_{e'}$ (else $e = e'$), so e and e' differ on $\kappa \setminus (H_{\nu(F_e)} \cup F_e)$.

Claim 3. *E is slim.*

Let $p \in \Pi_{\alpha\in A} X_\alpha$, where A is a nonempty proper subset of κ. Let $C(p) = \{f \in X : f \restriction A = p\}$. Let us assume $E \cap C(p) \neq \emptyset$. We need to show $E \cap C(p)$ is nowhere dense in $C(p)$.

Case 1. A is infinite. Then there are m, j such that $p(\beta) \in \{v(\beta), k_{\beta m j}\}$ for all but finitely many $\beta \in A$. It follows that if $e \in E \cap C(p)$, then $e(\gamma) \in \{v(\gamma), k_{\gamma m j}\}$ for all $\gamma \notin F_e$, and so for each $\alpha \in F_e$, if $e(\alpha) = d_{\alpha i}$, then $i \leq j$. Hence the projection of $E \cap C(p)$ on each coördinate is finite.

Case 2. A is finite. Then $\operatorname{rng}(p)$ is finite, so there is n_0 such that, for each $\alpha \in \operatorname{dom}(p)$, $p(\alpha) \notin \bigcup_{n\geq n_0} D_{\alpha n} \cup K_{\alpha n}$. It follows that if $e \in E_n \cap C(p)$, then $n < n_0$, and hence $e(\beta) \in K_\beta \cup \{v(\beta)\}$ for all but at most $n_0 - 1$ many β. Hence if we take a set $G \subset \kappa \setminus \operatorname{dom}(p)$ of size n_0, then

$$C(p) \cap \bigcap_{\alpha\in G} \pi_\alpha^{-1}(X_\alpha \setminus (K_\alpha \cup \{v(\alpha)\}))$$

is a dense open set in $C(p)$ which misses E. $\square$

Theorem 3.12. *Assuming V=L, every infinite power of any dense-in-itself space has a slim and thin dense set.*

Proof. Recall that every dense-in-itself space X has a dense subspace which is the union of two open sets U_0 and U_1, where U_0 is a Baire space and U_1 is meager. It is known that under V=L every dense-in-itself Baire space is ω-resolvable [Pa]. Thus U_0 can be decomposed into dense sets D_n, $n < \omega$. Since U_1 is meager, it can be decomposed into nowhere dense sets N_n, $n < \omega$. For each $n > 0$ put $E_n = D_n \cup N_n$. Moreover, let $E_0 = X \setminus \bigcup_{n>0} E_n$. Then $X = \bigcup_{n<\omega} E_n$ is decomposition of X onto ω many dense sets, so Corollary 3.7 gives a dense slim and thin set in X^κ. $\square$

We are unable to determine whether or not the assertion of Theorem 3.12 can be proved in ZFC. However, we will show that if X^ω has no slim dense set, then X contains a subspace which is strongly irresolvable and Baire. So the key to finding an infinite power of X with no slim dense set, if such exists, has to be to look at Baire strongly irresolvable spaces.

Recall that a space is *strongly irresolvable* if every dense-in-itself subset is irresolvable; Hewitt[H] showed that every irresolvable space contains an open strongly irresolvable subspace.[1] In a strongly irresolvable space, every dense set has nowhere dense complement. If the space is also Baire, it follows that every meager set is nowhere dense. It is known to be consistent, modulo large cardinals, that Baire (strongly) irresolvable spaces exist, though they don't exist if, e.g., $V = L$. There can be, e.g., a Hausdorff Baire strongly irresolvable topology on ω_1, and a regular Baire strongly irresolvable topology on 2^{ω_1}[KST], [KT].

Proposition 3.13. *If X^ω has no slim dense set, then X contains a subspace which is strongly irresolvable and Baire.*

Proof. Let U be the union of a maximal disjoint collection of open sets which have meager dense subsets, and let $V = X \setminus \overline{U}$. If V is empty then $X = \overline{U}$, thus U is nonempty and has a meager dense subset which is also a meager dense in X, and then, by Corollary 3.8, X^ω has a slim dense set. Thus we can assume that V is nonempty. From the definition of U, we see that any meager subset of V is nowhere dense; in particular V is Baire. Now observe that V cannot be ω-resolvable. In fact, else $U \cup V$ has a sequence of dense set with empty intersection, thus X is ω-resolvable, and by Corollary 3.7, X^ω has a slim dense set. It is known that a space is ω-resolvable iff it is n-resolvable for every positive integer n [I]. So, let n be the minimal integer for which V is not $(n+1)$-resolvable. Write V as the union of n many dense set $D_1, D_2, \ldots, D_n$. Then each of these sets must be irresolvable. In particular, D_1 is irresolvable. Each irresolvable space contains an open strongly irresolvable subspace, so let Y be an open strongly irresolvable subspace of D_1. From the fact that any meager subset of V is nowhere dense, it follows that any meager subset of Y is nowhere dense in Y, so Y is Baire strongly irresolvable. $\square$

4 Some conditions which imply the existence of slim dense sets

Let us define the following properties of a space X.

(NC) There is a pairwise disjoint collection $\mathcal{N}$ of nowhere dense sets in X such that, given any finite collection $\mathcal{U}$ of nonempty open sets in X, there is some $N \in \mathcal{N}$ which meets every $U \in \mathcal{U}$.

(GC) There is a pairwise disjoint collection $\mathcal{N}$ of nowhere dense sets in X such that every nonempty open set $U \subset X$ meets all but finitely many members of $\mathcal{N}$.

Moreover, for a given $k < \omega$ we will denote by (NC$_k$) a weaker version of the condition (NC) obtained by requiring $|\mathcal{U}| \leq k$ in the definition of (NC).

Notice that for every space X we have

[1] The authors of [KST] define strongly irresolvable to mean every open subset is irresolvable. It follows from Hewitt's result that the statements in [KST] are valid for either definition of strongly irresolvable.

- If $k < m$ then (NC_m) implies (NC_k).
- If X satisfies (GC) then it satisfies (NC).
- A collection $\mathcal{N}$ witnesses property (NC) iff it witnesses (NC_k) for every $k < \omega$.

In spite of this last statement, we don't know whether or not property (NC) holds iff (NC_k) holds for all $k < \omega$.

Observe also the following facts.

- In the definition of (GC) we may assume that the family $\mathcal{N}$ is countable. Also, for any partition of $\mathcal{N}$ into infinite subfamilies, their unions are disjoint dense sets; in particular, (GC) implies ω-resolvable.
- If a family $\mathcal{N}$ witnesses property (NC_2) then $\bigcup \mathcal{N}$ is dense in X. Moreover, every non-empty set $G \subset X$ meets infinitely many $N \in \mathcal{N}$, thus X is dense-in-itself.

Proposition 4.1. **(1)** *If X satisfies* (NC) *then every power of X has a slim dense set.*

(2) *If X satisfies* (NC_k) *for some $k < \omega$, then X^k has a slim dense set.*

(3) *Assume X_α, $\alpha < \kappa$, satisfy* (GC). *Then there is a slim dense set in $\Pi_{\alpha<\kappa} X_\alpha$.*

Proof. (1). Let $\mathcal{N} = \{N_\alpha \colon \alpha < \lambda\}$ be family fulfilling (NC). Fix a cardinal κ and define $D = \bigcup_{\alpha<\lambda} N_\alpha^\kappa$. It is clear that D is slim (because each cross-section of D is a product of nowhere dense sets), we will verify that D is dense in X^κ. Let B be a basic open set in X^κ and let A be the set of all $\xi < \kappa$ for which $\pi_\xi(B) \neq X$; A is finite. Thus there is $\alpha < \lambda$ such that N_α meets all $\pi_\xi(B)$ for $\xi \in A$ and consequently, $B \cap N_\alpha^\kappa \neq \emptyset$. The proof above works also for the statement (2).

(3). For $\alpha < \kappa$ let $\mathcal{N}_\alpha$ be a family witnessing (GC) in X_α. We may assume that all $\mathcal{N}_\alpha$ are countable, say $\mathcal{N}_\alpha = \{N_{\alpha,k} \colon k < \omega\}$. Then the set $D = \bigcup_{k<\omega} \Pi_{\alpha<\kappa} N_{\alpha,k}$ is slim in $X = \Pi_{\alpha<\kappa} X_\alpha$. We will verify that D is dense in X. Let B be a basic open set in X and let A be the set of all $\alpha < \kappa$ for which $\pi_\alpha(B) \neq X_\alpha$; A is finite. For each $\alpha \in A$ there is a positive integer k_α such that $\pi_\alpha(B) \cap N_{\alpha,n} \neq \emptyset$ if $n \geq k_\alpha$. Let $k = \max_{\alpha\in A} k_\alpha$. Then for every $\alpha < \kappa$ and $n > k$ we have $\pi_\alpha(B) \cap N_{\alpha,k} \neq \emptyset$, so $B \cap D \neq \emptyset$. □.

Proposition 4.2. *Every metrizable dense-in-itself space X satisfies* (GC).

Proof. Let X be metrizable and dense-in-itself. For each $n > 0$ choose a nowhere dense $\frac{1}{n}$-net E_n in X such that $E_n \cap E_m = \emptyset$ if $n \neq m$. Then the sequence E_n, $n < \omega$, fulfills the condition (GC). □

Corollary 4.3. *Every product of metrizable dense-in-themselves spaces has a slim dense set.*

In Example 4.6, we will show that the product of *different* spaces X_α satisfying (NC) may have no slim dense set. First a couple of results on spaces which satisfy (NC) or (GC) witnessed by a collection of finite sets.

Proposition 4.4. *Let X be a dense-in-itself space.*

(i) *If $\triangle(X) \geq \pi w(X)$, then X satisfies* (NC) *witnessed by a collection of finite sets;*

(ii) *If X is separable and $\pi w(X) = \omega$, then X satisfies* (GC) *witnessed by a collection of finite sets.*

Proof. (i) Let $\mathcal{B}$ be a π-base of X with $|\mathcal{B}| = \lambda \leq \triangle(X)$. List all finite subfamilies of $\mathcal{B}$ in the sequence $\{\mathcal{B}_\alpha \colon \alpha < \lambda\}$. Now, for each $\alpha < \lambda$, choose inductively a finite set N_α meeting every $B \in \mathcal{B}_\alpha$ and disjoint with $\bigcup_{\beta<\alpha} N_\beta$. Then $\mathcal{N} = \{N_\alpha \colon \alpha < \lambda\}$ witnesses (NC).

We omit the similar and easy proof of (ii). □

For the separable case, let us see that it is consistent to have the same conclusion as in Proposition 4.4(ii) as long as $\pi w(X) < \mathfrak{c}$. Recall that the cardinal $\mathfrak{p}$ is the least cardinal of a family of infinite subsets of ω with every finite intersection infinite, such that there is no infinite set almost contained in every member of the collection. It is well-known that $\mathfrak{p}$ is the least cardinal such that Martin's Axiom for σ-centered posets fails [B].

Proposition 4.5. *Let X be a separable space with $\pi w(X) < \mathfrak{p}$. Then X satisfies (GC) witnessed by a collection of finite sets.*

Proof. Let D be a countable dense subset of X and $\mathcal{B}$ a π-base of cardinality $< \mathfrak{p}$. Define the poset P to be all $p = \langle f^p, \mathcal{B}^p \rangle$ such that

(i) f^p is a function with domain $n^p \in \omega$;

(ii) For each $i < n^p$, $f^p(i)$ is a finite subset of D;

(iii) For $i \neq j < n^p$, $f^p(i) \cap f^p(j) = \emptyset$;

(iv) $\mathcal{B}^p$ is a finite subset of $\mathcal{B}$.

Also define $q \leq p$ if f^q extends f^p, $\mathcal{B}^q \supseteq \mathcal{B}^p$, and, for every $j \in n^q \setminus n^p$, $f^q(j)$ meets every member of $\mathcal{B}^p$. Note that if $f^p = f^q$, then p and q are compatible; hence P is σ-centered. A standard dense set argument (which we omit) shows that a suitably generic filter produces a function f with domain ω such that $\{f(i) : i \in \omega\}$ consists of disjoint finite sets, such that every $B \in \mathcal{B}$ meets all but finitely many of them. □

The above proposition shows, for example, that any countable dense-in-itself subset of 2^{ω_1} can in some models satisfy (GC), witnessed even by a collection of finite sets.

Example 4.6. *There are topological spaces X satisfying* (GC) *and Y satisfying* (NC) *such that $X \times Y$ has no dense slim set. Thus Y is a topological space which satisfies* (NC) *but does not satisfy* (GC).

Proof. The spaces X and Y are the same as in Corollary 3.2, i.e., $X = \mathbb{Q}$ and Y is the real line with the density topology. Observe that $\triangle(Y) = \mathfrak{c} = \pi\omega(Y)$; thus by Proposition 4.4, Y satisfies (NC). Since X satisfies (GC) and there is no slim dense set in $X \times Y$ (cf., Corollary 3.2), Y does not satisfy (GC). $\square$

On the other hand, it is not very hard to observe that if X satisfies (NC) with a countable family $\mathcal{N}$, then for every space Y satisfying (GC), there is a dense slim set in $X \times Y$.

Proposition 4.7. *If X,Y satisfy* (GC) *then $X \oplus Y$ satisfies* (GC) *too.*

Proof. Let $\mathcal{N}$ and $\mathcal{M}$ be a countable families of nowhere dense sets witnessing (GC) in X and Y, respectively. List $\mathcal{N} = \{N_k \colon k < \omega\}$, $\mathcal{M} = \{M_k \colon k < \omega\}$. Then all sets $K_k = N_k \cup M_k$ are pairwise disjoint, nowhere dense in $X \oplus Y$ and, clearly, each nonempty open set in $X \oplus Y$ meets almost all K_n. $\square$

It is easy to check that the topological sum of spaces X and Y from Example 4.6 does not satisfy (NC). Thus the topological sum of two spaces satisfying (NC) may not satisfy (NC), even if one of them satisfies (GC). However, if the condition (NC) in X is witnessed by a countable family $\mathcal{N}$, and Y satisfies (GC), then there is a slim dense set in $X \times Y$.

Proposition 4.8. *If all X_α, $\alpha < \kappa$ satisfies* (NC) *(respectively,* (NC_k)*, or* (GC)*), then the product $\Pi_{\alpha<\kappa} X_k$ has the same property.*

Proof. Let $\{X_\alpha \colon \alpha < \kappa\}$ be a family of topological spaces satisfying (NC) and let $X = \Pi_{\alpha<\kappa} X_\alpha$. For $\alpha < \kappa$ let $\mathcal{N}_\alpha$ be a family of pairwise disjoint nowhere dense sets in X_α which witnesses (NC). Then $\mathcal{N} = \{\Pi_{\alpha<\kappa} N_\alpha \colon N_\alpha \in \mathcal{N}_\alpha\}$ is a family of pairwise disjoint nowhere dense sets in X which witnesses (NC) in X. The same proof works in other cases ((GC) or (NC_k)). $\square$

Example 4.9. *(CH) There exists a topology τ on ω such that the space $X = (\omega, \tau)$ satisfies* (NC_2) *but X^3 has no slim dense sets. Hence X does not satisfy* (NC_3).

Proof. Let X be the space constructed in Example 3.5. Then the set $D = \{\langle n, n+1\rangle : n \text{ even}\}$ is dense in X^2, so the family $\mathcal{N} = \{\{n, n+1\} : n \text{ even}\}$ witnesses (NC_2) in X. On the other hand, there is no slim dense set in X^3, thus X does not satisfy (NC_3). $\square$

In the above example, we got our witness to (NC_2) easily from a very thin dense set whose elements were disjoint as unordered pairs. We now show that the existence of (very) thin countable dense set in X^2 always implies (NC_2). We don't know if "very thin" can be weakened to "slim" in this result.

Proposition 4.10. *If $k \in \omega$ and X^k has a countable very thin dense set, then X satisfies (NC_k) (witnessed by finite sets).*

Proof. Let $D \subset X^k$ be a countable very thin dense set. For each $\vec{d} \in D$, let $c(\vec{d})$ be the coördinates of d, and let $c(D) = \bigcup\{c(d) : \vec{d} \in D\} = \{x_n : n \in \omega\}$. (Note that the set $c(D)$ is dense in X, so X is separable.)

Let $H_0 = \{x_\emptyset\}$, and if the disjoint finite sets H_i, $i \leq n$, have been defined, let

$$H_{n+1} = \{x_{k_n}\} \cup \bigcup \left\{ c(\vec{d}) : \ \vec{d} \in D \text{ and } c(\vec{d}) \cap H_n \neq \emptyset \right\} \setminus \bigcup_{i \leq n} H_i,$$

where k_n is least such that $x_{k_n} \notin \bigcup_{i \leq n} H_i$. Then H_{n+1} is certainly disjoint from each H_i, $i \leq n$, and since D is very thin, it follows that H_{n+1} is finite.

Now for each infinite $A \subset \omega$, let $a_0, a_1, \ldots$ be its increasing enumeration, and let

$$\mathcal{N}_A = \left\{ \bigcup_{i \leq a_0} H_i, \ \bigcup_{i=a_0+1}^{a_1} H_i, \ \bigcup_{i=a_1+1}^{a_2} H_i, \ldots \right\}.$$

The following claim completes the proof.

Claim. *For some $A \subset \omega$, $\mathcal{N}_A$ witnesses* (NC_k). Suppose otherwise. Then for each infinite $A \subset \omega$, there are nonempty open sets $U(A, i)$, $i < k$, such that no member of $\mathcal{N}_A$ meets all of them. Let $\mathcal{A}$ be an uncountable almost disjoint family of subsets of ω. By separability of X, there are distinct $A, B \in \mathcal{A}$ such that $U(A,i) \cap U(B,i) \neq \emptyset$ for each $i < k$. There is $\vec{d} \in D \cap \Pi_{i<k} U(A,i) \cap U(B,i)$. Let n be the least number such that $c(\vec{d}) \cap H_n \neq \emptyset$. Note that by the construction, $c(\vec{d}) \subset H_n \cup H_{n+1}$. If n were not in A, it would follow that some $N \in \mathcal{N}_A$ contains $H_n \cup H_{n+1}$ hence would meet every $U(A,i)$. So $n \in A$, and similarly, $n \in B$. So we have that every $\vec{d} \in D \cap \Pi_{i<k} U(A,i) \cap U(B,i)$ has a coördinate in the finite set $\bigcup_{n \in A \cap B} H_n$. Since D is dense, there are infinitely many such $\vec{d}$, which contradicts D being very thin. □

As mentioned earlier, we don't know of any consistent example of a dense-in-itself space X such that X^ω has no slim dense set. Recall that, by Proposition 3.13, if such an example exists it must contain a strongly irresolvable Baire subspace. We now obtain some conditions on strongly irresolvable spaces which imply that (NC) or related properties do *not* hold.

Recall that an ideal $\mathcal{N}$ on a cardinal κ is *selective* if whenever $\mathcal{P}$ is a partition of κ by members of $\mathcal{N}$, then there is $A \notin \mathcal{N}$ such that $|A \cap P| \leq 1$ for every $P \in \mathcal{P}$. Also, $\mathcal{N}$ is a *normal* ideal if every regressive function $f : \kappa \to \kappa$ is constant on some set $A \notin \mathcal{N}$. It is known that normal ideals are selective, and the non-stationary ideal is contained in every normal ideal.

In the paper [KST], the authors show that, assuming enough large cardinals, there is a Hausdorff Baire strongly irresolvable topology on ω_1 for which the ideal of nowhere dense sets is selective, and there is also such a topology on a stationary subset S of ω_1 for which the ideal of subsets of ω_1 whose trace on S is nowhere dense in the topology on S is a normal ideal. Also, in the paper [DG], the authors construct under MA(σ-centered) a countable regular strongly irresolvable space for which the nowhere dense ideal is selective.

Observe that if X is a strongly irresolvable space with underlying set κ for which that ideal $\mathcal{N}$ of nowhere dense sets is selective, then the set A in the definition of selective can be taken to be an open set (since every somewhere dense set has non-empty interior).

Proposition 4.11. *Suppose X is a strongly irresolvable topology on κ for which that ideal $\mathcal{N}$ of nowhere dense sets is selective. Then X does not satisfy* (NC_2).

Proof. Suppose $\mathcal{P}$ is a partition of X into nowhere dense sets. By selectivity and strongly irresolvable, there is an open set U which meets each member of $\mathcal{P}$ in at most one point. Then if V and W are disjoint open subsets of U, no member of $\mathcal{P}$ meets both V and W. So $\mathcal{P}$ cannot satisfy (NC_2). $\square$

Proposition 4.12. *Suppose X is a strongly irresolvable topology on a stationary subset S of ω_1 such that the ideal*

$$\mathcal{N} = \{A \subset \omega_1 : A \cap S \textit{ is nowhere dense}\}$$

is normal. Then X^2 has no slim dense set.

Proof. Suppose $D \subset S^2$ is slim. W.l.o.g., D is symmetric, i.e., $\langle \alpha, \beta \rangle \in D \iff \langle \beta, \alpha \rangle \in D$. We may also assume no point with coördinate 0 or 1 is in D. Define $f : \omega_1 \to \omega_1$ by:

(1) $f(\alpha)$ is the least $\beta < \alpha$ such that $\langle \alpha, \beta \rangle \in D$, if there is such $\beta < \alpha$;

(2) If $\alpha \in S$ but there is no $\beta < \alpha$ with $\langle \alpha, \beta \rangle \in D$, then $f(\alpha) = 0$;

(3) If $\alpha \notin S$, then $f(\alpha) = 1$.

By normality, there is some β such that $f^{-1}(\beta)$ is not in the ideal. Clearly, $\beta \neq 1$. Suppose $\beta > 1$; then $\{\beta\} \times f^{-1}(\beta) \subset D$; but since D is slim, $f^{-1}(\beta)$ is nowhere dense, a contradiction. So $\beta = 0$. Let U be an open subset of $f^{-1}(0)$. There must be some $\beta < \alpha$ with $\langle \beta, \alpha \rangle \in D \cap U^2$. But then $f(\alpha) \neq 0$, a contradiction. $\square$

Lemma 4.13. *Let X be strongly irresolvable space satisfying* (NC_2)*, with a collection $\mathcal{N}$. Then the family $\mathcal{F}$ of all $\mathcal{M} \subset \mathcal{N}$ such that $\mathcal{M}$ witnesses* (NC_2) *in X is an ultrafilter on $\mathcal{N}$.*

Proof. **Claim 1.** *$\mathcal{M} \in \mathcal{F} \iff \cup\mathcal{M}$ is dense.*

The forward direction is easy. For the reverse, suppose $\cup\mathcal{M}$ is dense, but $\mathcal{M}$ does not satisfy (NC_2), say witnessed by the pair of open sets U, V. Note $Y = X \setminus \cup\mathcal{M}$ is nowhere dense. Let $U' = U \setminus \overline{Y}$, $V' = V \setminus \overline{Y}$. Then no member of $\mathcal{N}$ can meet both U' and V', a contradiction.

From Claim 1 and strong irresolvability, we have also that $\mathcal{M} \in \mathcal{F}$ iff $\cup\mathcal{M}$ has dense interior; this implies $\mathcal{F}$ is a filter.

Claim 2. *$\mathcal{F}$ is an ultrafilter.*

Suppose not. Then there is some $\mathcal{M}_0 \subset \mathcal{N}$ such that neither $\mathcal{M}_0$ nor its complement $\mathcal{M}_1$ is in $\mathcal{F}$. By Claim 1, for $i < 2$ there are open sets U_i such that $(\cup\mathcal{M}_i) \cap U_i = \emptyset$. But then the pair U_1, U_2 shows that $\mathcal{N}$ fails the (NC_2) property, a contradiction. $\square$

In Example 4.9 we showed that (NC_2) need not imply (NC_3) (consistently); the following shows that there are no irresolvable examples of this kind.

Proposition 4.14. *If X is strongly irresolvable space, then X satisfies* (NC) *iff it satisfies* (NC_2).

Proof. Let $\mathcal{N}$ be a family witnessing (NC_2) in X, and let $\mathcal{F}$ be as in Lemma 4.13. For each nonempty open set U let us put $\mathcal{F}(U) = \{N \in \mathcal{N} : N \cap U \neq \emptyset\}$. By (NC_2), $\bigcup \mathcal{F}(U)$ is dense, so by Claim 1 of Lemma 4.13, $\mathcal{F}(U) \in \mathcal{F}$. Thus for a given finite sequence of nonempty open sets $U_0, \ldots, U_k$, we have $\bigcap_{i \leq k} \mathcal{F}(U_i) \neq \emptyset$, and consequently, $\mathcal{N}$ witnesses (NC). $\square$

Theorem 4.15. *If X is strongly irresolvable Baire, and $|X|$ is less than the first measurable cardinal, then X cannot satisfy* (NC_2).

Proof. Suppose X satisfies the hypotheses, and $\mathcal{N}$ is a pairwise disjoint collection of nowhere dense sets witnessing (NC_2). Let $\mathcal{F}$ be the ultrafilter defined in Lemma 4.13. By Claim 1 of Lemma 4.13, $\mathcal{M} \in \mathcal{F}$ iff $\bigcup \mathcal{M}$ is dense in X. Thus it follows easily from the Baire property that $\mathcal{F}$ is countably complete (i.e., closed under countable intersections). Since $|\mathcal{N}|$ is less than the first measurable, this gives a contradiction. $\square$

The following example shows that Baireness is (consistently) essential in the previous result.

Example 4.16. *(CH) There is a countable regular space X generated by a maximal independent family on ω (thus there is also a strongly irresolvable space) which satisfies* (NC)[2].

Proof. Let $\mathcal{C}$ be the family of all finite subsets of $\mathbb{Q}$. For each $n < \omega$ let $A_n = \{C \in \mathcal{C} : |C| = n\}$. For each $q \in \mathbb{Q}$ let $E_q = \{C \in \mathcal{C} : |C \cap (-\infty, q)| \text{ is even}\}$. Note that the family E_q, $q \in \mathbb{Q}$, is independent, and generates the topology τ_0 on $\mathcal{C}$ (and consequently, on ω) such that for any $U \in \tau_0$, the set $\{n : |A_n \cap U| = \omega\}$ is cofinite.

Choose any ultrafilter $\mathcal{F}$ on ω. We are going to make a topology on ω such that the following condition holds:

$(*)$ For each nonempty open U, we have $F(U) = \{n < \omega : |A_n \cap U| = \omega\} \in \mathcal{F}$.

Thus τ_0 satisfies $(*)$. We will define an increasing sequence τ_α, $\alpha < \omega_1$, of topologies satisfying $(*)$; our final space will be (ω, τ), where $\tau = \bigcup_{\alpha < \omega_1} \tau_\alpha$.

Let B_α, $\alpha < \omega_1$, index all infinite subsets of ω. Now suppose we are at stage $\alpha < \omega_1$, and for every $\beta < \alpha$, we have defined a 0-dimensional topology τ_β with a countable base satisfying $(*)$ and:

(i) If $\beta = \gamma + 1$, either B_γ or $\omega \setminus B_\gamma$ is not dense in (ω, τ_β);

[2]Note that such a space cannot satisfy (GC).

(ii) $\beta < \beta'$ implies $\tau_\beta \subset \tau_{\beta'}$.

If α is a limit ordinal, let $\tau_\alpha = \bigcup_{\beta<\alpha} \tau_\beta$. Then (ω, τ_α) satisfies the required conditions.

Suppose $\alpha = \gamma + 1$. If B_γ is not dense and co-dense in (ω, τ_γ), we can let $\tau_\alpha = \tau_\gamma$. So, assume B_γ is dense and co-dense.

Case 1. For every $U \in \tau_\gamma$, we have $F(U \cap B_\gamma) = \{n\colon |A_n \cap U \cap B_\gamma| = \omega\} \in \mathcal{F}$. First, it is not difficult in this case to write B_γ as a disjoint union of sets $\tilde{B}_{\gamma,i}$, $i \in \omega$, each satisfying the condition of Case 1. To see this, let $U_0, U_1, \ldots$ index a countable base for τ_γ. Clearly it suffices to define $\tilde{B}_{\gamma,i}$ such that for any n, m and i,

$$|A_n \cap U_m \cap B_\gamma| = \omega \text{ iff } |A_n \cap U_m \cap \tilde{B}_{\gamma,i}| = \omega.$$

To this end, let $\langle m_i, n_i \rangle$, $i \in \omega$, index all pairs $\langle m, n \rangle \in \omega^2$, each appearing infinitely often. Then at step k choose

$$x_{k,0}, x_{k,1}, \ldots, x_{k,k} \in U_{m_k} \cap B_\gamma \cap A_{n_k} \setminus \{x_{i,j}\colon i < k, j \le i\}.$$

Finally, for $i \ge 1$ let $\tilde{B}_{\gamma,i} = \{x_{j,i}\colon j \ge i\}$ and let $\tilde{B}_{\gamma,0} = B_\gamma \setminus \bigcup_{i\ge 1} \tilde{B}_{\gamma,i}$. It is easy to check that this works.

Now, let $\mathcal{A} = \{A_m\colon m < \omega\}$ be an independent family of subsets of ω. Let Φ denote the family of all functions $\varphi\colon C \to 2$, $C \in [\omega]^{<\omega}$. For each $\varphi \in \Phi$, let $\varphi[\mathcal{A}] = \bigcap\{A_m^{\varphi(m)}\colon m \in \mathrm{dom}(\varphi)\}$, where $A^0 = \omega \setminus A$, $A^1 = A$. For each $i \in \omega$ put $B_i = \bigcup_{m \in A_i} \tilde{B}_{\gamma,m}$. Let $\mathcal{B}_\gamma = \{B_i\colon i < \omega\}$. Observe that for each $\varphi \in \Phi$ the set $\varphi[\mathcal{B}_\gamma]$ satisfies the condition of Case 1. In fact, since $\varphi[\mathcal{A}] \ne \emptyset$, there is $i \in \varphi[\mathcal{A}]$ and $\tilde{B}_{\gamma,i} \subset \varphi[\mathcal{B}_\gamma]$. Then $\{n\colon |A_n \cap U \cap \varphi[\mathcal{B}_\gamma]| = \omega\} \supset \{n\colon |A_n \cap U \cap \tilde{B}_{\gamma,i}| = \omega\} \in \mathcal{F}$.

Let τ_α be the topology generated by the sets $U \cap \varphi[\mathcal{B}_\gamma]$, $U \in \tau_\gamma$, $U \ne \emptyset$, $\varphi \in \Phi$. Then τ_α is a 0-dimensional topology with the countable base satisfying $(*)$ in which $\omega \setminus B_\gamma$ is nowhere dense.

Case 2. Not Case 1. Then for some clopen set U in τ_γ, we have $F(U \cap B_\gamma) \notin \mathcal{F}$. Since $F(V) \in \mathcal{F}$ for any non-empty open V, it follows that for any non-empty open subset V of U, $F(V \cap (\omega \setminus B_\gamma)) \in \mathcal{F}$. Hence $B_\gamma^* = [U \cap (\omega \setminus B_\gamma)] \cup [\omega \setminus U]$ satisfies the condition of Case 1 with B_γ replaced by B_γ^*. Hence there is a 0-dimensional topology τ_α with a countable base satisfying $(*)$ such that $\omega \setminus B_\gamma^*$ is nowhere dense. Note that this makes B_γ fail to be dense in τ_α (its intersection with U is nowhere dense).

This completes the inductive construction. Let $X = (\omega, \tau)$, where $\tau = \bigcup_{\alpha<\omega_1} \tau_\alpha$, and let $\mathcal{B} = \bigcup_\gamma \mathcal{B}_\gamma = \{B_{\gamma,i}\colon \gamma < \omega_1, i < \omega\}$. Then $\mathcal{B}$ is an independent family generating τ. Observe that $\mathcal{B}$ is maximal. In fact, otherwise there is $D \subset \omega$ which is dense and co-dense in τ. Then there is $\alpha < \omega_1$ such that $D = B_\alpha$ and D is dense in co-dense in $\tau_{\alpha+1}$, a contradiction. Hence X is irresolvable.

X satisfies the condition $(*)$ since it does so at each stage. It follows that the A_n's witness property (NC) for X.

Finally, the existence of a strongly irresolvable space with these properties follows simply from the fact that every irresolvable space has an open strongly irresolvable subspace. □

5 Applications

Let us recall one of the oldest and most significant generalizations of continuity, called *near continuity*. (The history of this concept one can find e.g in [R].) Let X and Y be topological spaces. A function $f\colon X \to Y$ is nearly continuous if $f^{-1}(V) \subset \operatorname{int}(\overline{f^{-1}(V)})$ for every open set $V \subset Y$. Now, for a function $f\colon \Pi_{\alpha<\kappa} X_\alpha \to Y$ it is natural to consider the problem of relations between *separate near continuity* versus *joint near continuity*. It is known that for real functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ neither separate near continuity implies joint near continuity, nor vice versa [N]. Now, using notions of small dense sets considered in the paper, we obtain a wide class of examples of jointly nearly continuous but not separately nearly continuous functions. Indeed, for $D \subset \Pi_{\alpha<\kappa} X_\alpha$ let χ_D be the characteristic function of D. It is easy to observe that if D is dense in $\Pi_{\alpha<\kappa} X_\alpha$, then χ_D is jointly nearly continuous. Moreover, if D is thin then each codimension 1 cross-section of χ_D is not nearly continuous (however it is Baire one); and if D is slim then each cross-section of χ_D is not nearly continuous. In the both cases, χ_D is not separately nearly continuous.

6 Open questions

1. Can there be a dense-in-itself Hausdorff space X such that X^ω has no slim dense set? (Consistently, no; see Theorem 3.12.)
2. Does X strongly irresolvable Baire imply X^2 and/or X^ω has no slim dense set?
3. Is there in ZFC a separable Hausdorff space X such that X^2 has no slim dense set? (Yes under CH, see Example 3.4.)
4. Is there in ZFC a separable space X which does not satisfy (NC)? (Yes under CH, see Example 3.4.)
5. Is there a space X in ZFC such that X^2 has a thin dense set but X^3 does not? (Yes under CH, see Example 2.6.)
6. Is there in ZFC a space X satisfying (NC_2) but not (NC_3)? (Yes under CH, see Example 4.9.)
7. Is there a space X such that there is a slim dense set in X^2 but X does not satisfy (NC_2)?
8. Is there a space X that satisfies (NC_k) for every $k < \omega$ but does not satisfy (NC)?
9. Can there be a space X such that X^k has a slim (resp., very thin) dense set for every $k < \omega$, but X^ω does not?
10. Do there exist spaces X, Y with (NC) witnessing by countable families $\mathcal{N}_X$, $\mathcal{N}_Y$, respectively, with no slim dense set in $X \times Y$?

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