ON A QUESTION CONCERNING SHARP BASES

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Abstract. A sharp base \( B \) is a base such that whenever \((B_i)_{i<\omega}\)

is an injective sequence from \( B \) with \( x \in \bigcap_{i<\omega} B_i \), then \( \{ \bigcap_{i\leq n} B_i : n < \omega \} \) is a base at \( x \). Alleche, Arhangel’skii and Calbrix asked: if \( X \) has a sharp base, must \( X \times [0, 1] \) have a sharp base? Good, Knight and Mohamad claimed to construct an example of a Tychonoff space \( P \) with a sharp base such that \( P \times [0, 1] \) does not have a sharp base. However, the space was not regular. We show how to modify the construction to make \( P \) Tychonoff.

1. Introduction

A sharp base is a base \( B \) such that whenever \((B_i)_{i<\omega}\) is an injective sequence from \( B \) with \( x \in \bigcap_{i<\omega} B_i \), then \( \{ \bigcap_{i\leq n} B_i : n < \omega \} \) is a base at \( x \). In a \( T_1 \) space, \( \bigcap_{i<\omega} B_i = \{ x \} \).

In [AAC], Alleche, Arhangel’skii and Calbrix defined sharp bases and asked if there is a topological space with a sharp base whose product with \([0, 1]\) does not have a sharp base. Good, Knight and Mohamad [GKM] claimed to have a Tychonoff counterexample, but it turns out that their space is not regular. It is not regular because they added a closed discrete set \( L \) to the Baire metric space \( ^\omega \mathbb{N} \), in such a way to make the new space \( P \) pseudocompact. Such \( P \) cannot be regular: for if it is, one may find a neighborhood of \( p \in ^\omega \mathbb{N} \) whose closure misses \( L \). That neighborhood can be assumed to come from a clopen basis for \( ^\omega \mathbb{N} \), and would then be homeomorphic to \( ^\omega \mathbb{N} \) and be pseudocompact, a contradiction.

In this paper we give a modification of the Good, Knight, Mohamad space which makes the space Tychonoff. The space we construct is pseudocompact but not compact, hence not metrizable; we also show it is not developable. Our space has no isolated points and a sharp base, and for \( T_1 \) spaces a sharp base is always weakly uniform. Since Heath and Lindgren show that a \( T_2 \) space with a weakly uniform base

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has a $G_δ$-diagonal [HL], our space has one also. In [AJRS], it is shown that a pseudocompact space with a $G_δ$-diagonal is Čech-complete, and that if a space with not more than $ω_1$ isolated points has a sharp base, then it has a point countable base. Therefore, the space we construct is a counterexample for these three other questions:

*Is every pseudocompact Tychonoff space with a sharp base metrizable?* [AJRS]

*Is every pseudocompact space $X$ with a $G_δ$-diagonal and a point-countable base developable?* [A]

*Is every Čech-complete pseudocompact space with a point-countable base metrizable?* [A]

We have borrowed much of our notation from the paper [GKM].

2. The Example

2.1. The Construction of space $\mathcal{P}$. Let $B = \omega^c$, and for $σ ∈ \omega^c$ define $[σ] = \{g ∈ B : σ ⊆ g\}$. We also denote $σ ∈ n+1c$ by $(α_0, α_1, \cdots, α_n)$, where $σ(i) = α_i$. For $σ = (α_0, α_2, \cdots, α_n)$, we denote $(α_0, α_2, \cdots, α_n, δ)$ by $σ^−(δ)$. By $σ_1 ⊥ σ_2$ we mean that $σ_1$ and $σ_2$ are incompatible (i.e. the two finite partial functions disagree at a point in both domains).

Define $\mathcal{S}$ to be the collection of elements of $ω(ω^c)$ subject to these two conditions:

1. For all $S ∈ \mathcal{S}$ there exists a $k_s < ω$ and a $ρ_s ∈ \omega^c$ such that whenever $σ ∈ S$, $σ|k_s = ρ_s$. This $ρ_s$ will be called the root of $S$.

2. Whenever $σ_1$ and $σ_2$ are distinct elements of $S$, $σ_1(k_s) ≠ σ_2(k_s)$.

Let $\mathcal{S} = \{S_α : α < c\}$, and let the root of $S_α$ be $ρ_α$.

Define $T_α ∈ ω(ω^c)$ so that $T = \{T_α : α < c\}$ has these three properties:

(i) for $i ≠ j$, $T_α(i) ⊥ T_α(j)$

(ii) if $β, α < c$, $β ≠ α$, with $T_α$ and $T_β$ defined, then $\text{ran}T_β ∩ \text{ran}T_α = \emptyset$, and

(iii) for $β, α < c$, $β ≠ α$, $T_α$ and $T_β$ defined, if $T_α(i) ⊇ T_β(j)$, then whenever $j′ ≠ j$, $T_α(i′) ⊥ T_β(j′)$ for all $i′ < ω$.

Assume for $α < γ$ we have either constructed a $T_α ∈ ω(ω^c)$ subject to the conditions above or we have not constructed a $T_α$ at all. Now we define $T_γ$. Choose a $δ ∈ c$ not in $∪\{\text{ran}T_α(j) : α < γ, j ∈ ω\}$. Then for each $i ∈ ω$ let $S_γ(i) = S_γ(i)−(δ)$. The sequence $(T_γ(i))_{i < ω}$ will be a subsequence of $(S_γ(i))_{i < ω}$, so the fact that no previous $T_α$ contains a finite partial function with $δ$ in the range will yield property (ii) for $T_γ$. 


In addition, the fact that the elements of $S'_\gamma$ are pairwise incompatible will make the elements of $T_\gamma$ also incompatible, satisfying property (i). We need to construct our subsequence $T_\gamma$ of $S'_\gamma$ to make property (iii) hold at step $\gamma$.

Case 1. Suppose there exists some $\alpha < \gamma$ for which $T_\alpha$ was defined, such that for infinitely many $j$ there is some $\iota \in \omega$ with $S_\gamma(\iota) \supseteq T_\alpha(j)$. If this is the case, do not define $T_\gamma$.

Case 2. If for each $\alpha < \gamma$ there are at most finitely many $j$ for which $S_\gamma(\iota) \supseteq T_\alpha(j)$ for some $\iota$, we will define a $T_\gamma$.

Suppose that for $i \leq k$ we have already selected a sequence of natural numbers $0 = n_0 < n_1 < \cdots < n_k$ and defined $T_\gamma(i) = S'_\gamma(n_i)$. There are at most finitely many different finite partial functions $f$ such that $f \subseteq T_\gamma(i)$ for some $i \leq k$. The second induction condition implies that there are at most finitely many $\alpha < \gamma$ with such an $f$ in the range of $T_\alpha$. List these as $\alpha(0), \ldots, \alpha(m)$. We have assumed that for each $\alpha < \gamma$, there are at most finitely many $j$ for which $S'_\gamma(n_j)$ extends $T_\alpha(j)$ for some $\iota$. Using this fact, we see that for each $\alpha(p)$ there is a $j_p$ such that for all $j \geq j_p$, $S'_\gamma(n_j)$ does not extend any $T_{\alpha(p)}(j)$. Then define $n_{k+1} = \max\{j_p : p \leq m\} \cup \{n_k + 1\}$ and $T_\gamma(k + 1) = S'_\gamma(n_{k+1})$. To check property (iii), suppose that $\beta < \gamma$ and $T_\gamma(\iota) \supseteq T_\beta(j)$ for some $j, k < \omega$. (Note that $T_\beta(j) \nsubseteq T_\gamma(k)$ since $\delta \in \text{ran}T_\gamma(k) \setminus \text{ran}T_\beta(j)$.) Assume that $k$ is the least possible for which there exists such a $j$. Then $\beta = \alpha(p)$ for some $p \leq m$ in the above construction. Since $n_{k+1}, n_{k+2}, \ldots$ are all greater than $j_p$, $T_\gamma(\iota)$ cannot extend $T_\beta(j')$ for any $j' \neq j$ and any $\iota$, so we have property (iii). Indeed from (iii) together with what we noted above and conditions (1) and (2) of $S \in \mathcal{S}$, we have the following.

(iv) If $\rho_\alpha = \rho_\beta$, then $T_\alpha(j)$ and $T_\beta(\iota)$ are compatible for at most one pair $(i, j)$ in $\omega \times \omega$.

Choose $L$ disjoint from $B$ such that $L = \{s_\alpha : T_\alpha$ is defined$\}$. Let the root of $s_\alpha$ refer to $\rho_\alpha$. Let $P = B \cup L$.

For $\sigma \in \mathcal{O} \epsilon$, let $B(\sigma) = [\sigma] \cup \{s_\beta : \rho_\beta \supseteq \sigma\}$ and let $B_n(\sigma) = \{s_\alpha\} \cup \bigcup_{m \geq n} ([T_\alpha(m)] \cup \{s_\beta : \rho_\beta \supseteq T_\alpha(m)\})$. These will be the basic open sets for $P$, and call the collection of them $\mathcal{B}$.

2.2. Verifying Properties of $\mathcal{P}$. First, we will observe some properties of $\mathcal{B}$.

(a) For $\sigma_1, \sigma_2 \in \mathcal{O} \epsilon$, $\sigma_1 \perp \sigma_2$ iff $B(\sigma_1) \cap B(\sigma_2) = \emptyset$ and if $\rho_\alpha \perp \rho_\beta$ then $B_n(\sigma_\alpha) \cap B_m(\sigma_\beta) = \emptyset$.

(b) $\sigma_1 \subseteq \sigma_2$ iff $B(\sigma_2) \subseteq B(\sigma_1)$, and if $\rho_\alpha \supseteq \sigma$, then for each $\iota < \omega$, $B_n(\sigma_\alpha) \subseteq B(\sigma)$. 

(c) Suppose $B(\sigma) \cap B_n(s_\alpha) \neq \emptyset$. Then $\sigma \subseteq \rho_\alpha$ or $\rho_\alpha \subseteq \sigma$. If $\sigma \subseteq \rho_\alpha$ then $B(\sigma) \cap B_n(s_\alpha) = B_n(s_\alpha)$. If $\sigma \supseteq \rho_\alpha$, then the intersection is either $B(\sigma)$ or $B(T_\alpha(m))$ for some $m \geq n$. Finally, if $B(\sigma) \subseteq B_n(s_\alpha)$ then for some $m \geq n$ we have $B(\sigma) \subseteq B(T_\alpha(m))$.

(d) If $B_n(s_\alpha) \cap B_{n'}(s_{\alpha'}) \neq \emptyset$ and $\rho_{\alpha'} \subseteq \rho_\alpha$, then the intersection is either $B_n(s_\alpha)$ or a set of form $B(\sigma)$, for some $\sigma \in \{T_\alpha(m), T_{\alpha'}(m') : m \geq n, m' \geq n'\}$. In particular, the latter holds if $\rho_{\alpha'} = \rho_\alpha$.

Proof of (a) - (d):
(a) Suppose that $\sigma_1 \perp \sigma_2$; then there is no point of $B$ nor any finite partial function that could extend both $\sigma_1$ and $\sigma_2$. If $s_\alpha \in L$ is in $B(\sigma_1) \cap B(\sigma_2)$ then $\rho_\alpha$ extends both, contradiction. Suppose for the reverse, that $B(\sigma_1) \cap B(\sigma_2) = \emptyset$; then since $[\sigma_1] \cap [\sigma_2]$ is contained in this set, it is clear that $\sigma_1 \perp \sigma_2$.

Now if the roots of $s_\alpha$ and $s_\beta$ are incompatible then each pair of extensions of the roots will be incompatible, hence $B(T_\alpha(n')) \cap B(T_\beta(m')) = \emptyset$ for each $n' \geq n$ and $m' \geq m$. Further, $s_\alpha \in B_m(s_\beta)$ implies that $\rho_\alpha$ extends $\rho_\beta$, which has been assumed to be not the case. So $B_n(s_\alpha) \cap B_m(s_\beta) = \emptyset$.

(b) Clear from the definition of $B(\sigma)$ and $s_\alpha$.

(c) Suppose that $B(\sigma) \cap B_n(s_\alpha) \neq \emptyset$. Since $B_n(s_\alpha) \subseteq B(\rho_\alpha)$ we have $\sigma \not\subseteq \rho_\alpha$, by (a). If $\sigma \subseteq \rho_\alpha$, then for each $m \geq n$, $\sigma \subseteq T_\alpha(m)$ and $s_\alpha \in B(\sigma)$, so $B_n(s_\alpha) \subseteq B(\sigma)$.

Suppose $\sigma \not\supseteq \rho_\alpha$; then for some $m \geq n$, $B(\sigma) \cap B(T_\alpha(m)) \neq \emptyset$, while property (i) of $T$ implies that $B(\sigma) \cap B(T_\alpha(k)) = \emptyset$ for $k \neq m$. By (a) and (b), one of $B(\sigma)$ and $B(T_\alpha(m))$ is contained in the other, and the intersection is simply the contained set. This implies the last sentence of (c).

(d) Suppose $B_n(s_\alpha) \cap B_{n'}(s_{\alpha'}) \neq \emptyset$, where $s_\alpha \neq s_{\alpha'}$. If $\rho_{\alpha'} \subseteq \rho_\alpha$ then $s_{\alpha'} \not\in B_n(s_\alpha)$ and $[T_\alpha(j)] \cap [\rho_\alpha] \neq \emptyset$ for at most one $j \in \omega$. Therefore, $B_n(s_\alpha) \cap B_{n'}(s_{\alpha'}) = B(T_{\alpha'}(j)) \cap B_n(s_\alpha)$ for some $j \geq n'$. Now the rest follows from (c).

If $\rho_\alpha = \rho_{\alpha'}$, then the conclusion follows from condition (iv).

$B$ is a clopen base for $P$. Notice that the properties show immediately that $B$ is a base. To see that $B_n(s_\alpha)$ is closed, consider $s_\gamma \in L \setminus B_n(s_\alpha)$. Suppose that $B_j(s_\gamma)$ meets $B_n(s_\alpha)$, where $j$ is sufficiently large that $s_\alpha \notin B_j(s_\gamma)$. Then by (d) the intersection is one of $B(T_\alpha(n'))$ for some $n' \geq n$, $B(T_\gamma(j'))$ for some $j' \geq j$, $B_j(s_\gamma)$ or $B_n(s_\alpha)$.

Since $s_\gamma \notin B_n(s_\alpha)$ and $s_\alpha \notin B_j(s_\gamma)$, we know that the intersection cannot be $B_j(s_\gamma)$ or $B_n(s_\alpha)$. If the intersection is $B(T_\gamma(j'))$ then $B_{j' + 1}(s_\gamma)$ misses $B_n(s_\alpha)$. So without loss of generality, the intersection is some
$B(T_{\alpha}(n'))$. Then $B(T_{\gamma}(j')) \supseteq B(T_{\alpha}(n'))$ for some $j'$. So $B_{j'+1}(s_\gamma) \cap B_n(s_\alpha) = \emptyset$.

To see that each limit point of $B_n(s_\alpha)$ in $B$ is in $B(s_\alpha)$, suppose that $p$ is a limit point of $B_n(s_\alpha)$ contained in $B \setminus B_n(s_\alpha)$. Clearly, $p \supseteq \rho_\alpha$. Choose $k < \omega$ so that $p|k \not\subseteq \rho_\alpha$. Then by property (c), $B(p|k) \cap B_n(s_\alpha) = B(T_\alpha(m))$ for some $m \geq n$. Then for $k' < \omega$ with $k' > |T_\alpha(m)|$, we have $B(p|k') \cap B_n(s_\alpha) = \emptyset$.

Lastly, we observe that $B(\sigma)$ is clopen. Since $B$ is dense and the subspace base is clopen, we only need to turn our attention to limit points of $B(\sigma)$ in $L$. Suppose then that $s_\alpha \in L$ is a limit point of $B(\sigma)$, not in $B(\sigma)$; then for all $n < \omega$, $B_n(s_\alpha)$ meets $B(\sigma)$. If $\rho_\alpha \perp \sigma$, then clearly $B(\sigma) \cap B_n(s_\alpha) = \emptyset$. If $\rho_\alpha \supseteq \sigma$, then $s_\alpha$ is in $B(\sigma)$ which is contrary to our assumptions. So assume that $\sigma \supseteq \rho_\alpha$, then there is at most one $T_\alpha(m)$ that extends $\sigma$ or is extended by $\sigma$. Then $B_{m+1}(s_\alpha) \cap B(\sigma) = \emptyset$.

$B$ is sharp. Let the injective sequence $(B(\sigma_i))_{i < \omega}$ come from $B$. If $p \in B$ is contained in every $B(\sigma_i)$ then $p \supseteq \sigma_i$, so since $|\sigma_i|$ must be unbounded, it is clear that $\{\bigcap_{i \leq n} B(\sigma_i) : n < \omega\}$ is a base at $p$. If $s_\alpha$ is in every $B(\sigma_i)$, then $\rho_\alpha$ extends every $\sigma_i$, but since $|\rho_\alpha|$ is finite, this is not possible.

Now consider an injective sequence $(B_{n_i}(s_{\alpha_i}))_{i < \omega}$, with nonempty intersection. If there is an infinite subset $J$ of $\omega$ such that the $\rho_{\alpha_i}$, $i \in J$, are distinct, then it is easy to see that $\{B(\rho_{\alpha_i}) : i \in J\}$ is a base for a unique point $p \in B$. Hence, so is $\{\bigcap_{i \leq j} B_{n_i}(s_{\alpha_i}) : j < \omega\}$, since for each $i \in J$ we have $B_{n_i}(s_{\alpha_i}) \subseteq B(\rho_{\alpha_i})$.

Next, suppose that $s_{\alpha_i} = s_\alpha$ for all $i$ in an infinite subset $J$ of $\omega$. Then $\{B_{n_i}(s_\alpha) : i < \omega\}$ is a base at $s_\alpha$, therefore $\bigcap_{i \leq j} B_{n_i}(s_\alpha) : j < \omega\}$ is a base at $s_\alpha$ too.

The final case, without loss of generality, is when the $s_{\alpha_i}$’s are distinct, but $\rho_{\alpha_i} = \rho$ for all $i < \omega$. Then by (d), pairwise intersections have the form $B(\sigma)$ for some $\sigma$ in the range of the corresponding pair from $T$. By property (ii) of $T$, $\{B_{n_i}(s_{\alpha_i}) \cap B_{n_{i+1}}(s_{\alpha_{i+1}}) : i \text{ is even}, i < \omega\}$ consists of distinct $B(\sigma)$’s. Therefore, this must be a base at some $p \in B$, and $\bigcap_{i \leq j} B_{n_i}(s_{\alpha_i}) : j < \omega\}$ is as well.

$P$ is not compact. Consider $C_0 = \{s_\alpha \in L : \rho_\alpha = \emptyset\}$. Note that $P \setminus C_0 = \bigcup_{\alpha < \gamma} B((\alpha))$. We intend to show that the closed set $C_0$ is infinite and discrete. To see that this is a discrete set, notice that for $s_\alpha \in C_0$, the set $B_1(s_\alpha) \cap C_0$ can only contain $s_\alpha$. Examine $\{s_{\alpha_i} : \gamma < c\}$, where $\alpha_i = \alpha_i \gamma$ iff both $i = i'$ and $\gamma = \gamma'$, Call this collection $S_0$; then this is a subset of $\mathcal{S}$. Note, that for each $S_\alpha \in S_0$ and $i < \omega$,
we have that the length of \( S_\alpha(i) \) is exactly one. Also, each \( T_\alpha(j) \) is constructed to have length at least 2. Therefore, during the induction that defined \( T \), for each \( S_\alpha \in S_0 \), Case 1 does not hold. Therefore, a corresponding \( T_\alpha \) is constructed for each \( S_\alpha \in S_0 \).

\( P \) is not perfect, hence not developable. Let \( U = P \setminus C_0 \). We show that \( U \) is not \( F_\sigma \), and hence \( P \) is not developable. Suppose that \( \{ F_j \}_{j<\omega} \) is a collection of closed sets so that \( \bigcup_{j<\omega} F_j = U \). By the Baire property of \( B \), each \([\alpha]\) is Baire. So for all \( \alpha < \omega \) there is an \( n_\alpha \) and an \( \bar{\alpha} = [(\alpha, \beta_1, \cdots, \beta_{n_\alpha})] \subseteq F_{n_\alpha} \). Choose \( n_0 \) so that \( \{ \alpha : [\bar{\alpha}] \subseteq F_{n_0} \} \) is infinite. Order \( \{ \alpha_i \}_{i<\omega} \subseteq \{ \alpha : [\bar{\alpha}] \subseteq F_{n_0} \} \), then \( S = ([\bar{\alpha}_i])_{i<\omega} \in S \), and has the empty set as its root. So an \( s \in L \) was defined as a limit point of \( S \), and \( \sigma \) the root of \( s \) is also the empty set. Therefore, \( s \) is a limit point of the closed set \( F_{n_\sigma} \). This implies that \( s \in P \setminus C_0 \), contradicting that \( s \) has the empty root.

\( P \) is pseudocompact. Suppose that \( \varphi \) is an unbounded continuous real valued function on \( P \). Since \( B \) is dense, for each \( n \in \omega \) there is an \( x_n \) such that \( \varphi(x_n) > n \). Let \( D = \{ x_n : n \in \omega \} \) and let’s note that \( D \) is closed discrete, hence not compact. If \( p \) were a cluster point of \( D \), then every open neighborhood of \( p \) contains infinitely many elements of \( D \). This implies that \( \varphi \) increases unboundedly over every neighborhood of \( p \), contradicting the continuity of \( \varphi \).

Since \( D \) is closed and not compact we can find a \( k < \omega \) such that \( \{ x_n | k : x_n \in D \} \) is infinite. Choose the minimum such \( k \). Then there is a \( \sigma \in \prec \omega \) and an infinite subset \( A \) of \( \omega \), such that \( x_n | (k-1) = \sigma \) for \( n \in A \), and \( x_n(k-1) \) is different for these infinitely many \( n \in A \).

Let \( D^* = \{ x_n : n \in A \} \). Since \( \varphi(x_n) > n \) by continuity of \( \varphi \) there exists \( j_n > k \) so that \( \varphi(B(x_n | j_n)) > n \). Then for some \( \alpha < \omega \), \( \{ x_n | j_n : x_n \in D^* \} \) is \( S_\alpha \) and \( \rho_\alpha = \sigma \). If \( s_\alpha \) was not defined then for some \( \beta < \alpha \), \( T_\beta(j) \subseteq S_\alpha(n) = x_n | k \) for infinitely many \( j \). Then each basic open neighborhood of \( s_\beta \) contains infinitely many of the sets \( B(x_n | k) \). So \( \varphi \) takes on arbitrarily large values over every neighborhood of \( s_\beta \) contradicting continuity. If \( s_\alpha \) was defined, then \( T_\alpha(i) \) was chosen so that \( T_\alpha(i) \supseteq x_n | j_n \) for each \( i \in \omega \), so \( B(T_\alpha(i)) \subseteq B(x_n | j_n) \). So again, \( \varphi \) takes on large values over every open set containing \( s_\alpha \), contradicting the continuity of \( \varphi \).

The following lemma, which was suggested by the referee, is essentially due to [GKM].

**Lemma 1.** Let \( X \) be a Tychonoff, pseudocompact, non-compact space which partitions into \( B \cup L \), and has a sharp base \( B \). If

(a) \( B = B_1 \cup B_2 \) where \( B_1 \) is a \( \sigma \)-point finite base for \( B \)
(b) for all \( x \in L \) there is a local base \( \{ B_n(x) : n < \omega \} \) so that \( n < m \) implies \( B_m(x) \subset B_n(x) \) and \( \mathcal{B}_2 = \{ B_n(x) : n < \omega, x \in L \} \)

(c) for \( x \neq y \in L \), \( n, m \in \omega \), \( B_n(x) \neq B_m(y) \).

Then \( X \times [0,1] \) does not have a sharp base.

Proof. Assume, by way of contradiction, that \( \mathcal{W} \) is a sharp base for \( X \times [0,1] \). Let \( \mathcal{C} \) be a countable base for \([0,1]\). For each \( x \in L \), choose \( W^x_n \in \mathcal{W} \), \( B^x_n \in \mathcal{B} \) and \( C^x_n \in \mathcal{C} \) so that \( (x, \frac{1}{3}) \in B^x_n \times C^x_n \subseteq W^x_n \subseteq B_n(x) \times [0,1] \). Let \( \mathcal{B}_C = \{ B \in \mathcal{B} : \text{for some } n \in \omega \text{ and } x \in L, B = B^x_n \text{ and } C = C^x_n \} \).

We claim that \( \mathcal{B}_C \) is point-finite. Suppose not; then there exists an infinite collection \( (B_j)_{j<\omega} \) from \( \mathcal{B}_C \) that has nonempty intersection. Let \( y \in \bigcap_{j<\omega} B_j \); then there are \( x_j \in L \) and \( n_j \in \omega \) so that \( B_j = B^x_{n_j} \) and \( C = C^x_{n_j} \). Then \( \{ y \} \times C \subseteq \bigcap_{j<\omega} (B^x_{n_j} \times C^x_{n_j}) \subseteq \bigcap_{j<\omega} W^x_{n_j} \). If \( x_j \neq x_k \) then \( B_n(x_j) \neq B_n(x_k) \).

There are two cases to consider.

Case 1. There is an infinite \( J \subseteq \omega \) so that \( x_j \neq x_k \) whenever \( j \neq k \) with \( j, k \in J \). Then \( \{ W^x_{n_j} : j \in J \} \) is infinite. Suppose not; then some \( W \) is contained in infinitely many different \( B_{n_j}(x_j) \times [0,1] \). The sharpness of \( \mathcal{B} \) implies that \( \bigcap_{j<\omega} B_{n_j}(x_j) \) is at most a singleton; it must be \( \{ y \} \), implying \( W \subseteq \{ y \} \times [0,1] \), which is impossible. Hence \( \{ W^x_{n_j} : j \in J \} \) is infinite, and so \( \{ y \} \times C \subseteq \bigcap_{j<\omega} W^x_{n_j} \) is a single point, a contradiction.

Case 2. There is an infinite \( K \subseteq \omega \) so that \( x_j = x_k = x \) for \( j, k \in K \). Then the set \( \{ n_k : k \in K \} \) is infinite, since the \( B^x_{n_k} \) are distinct. Again, \( \{ y \} \times C \subseteq \bigcap_{k \in K} (B^x_{n_k} \times C^x_{n_k}) \subseteq \bigcap_{k \in K} W^x_{n_k} = \bigcap_{k \in K} W^x_{n_k} \). Once again, this is simply one point, so we have the same contradiction as in Case 1.

Therefore, \( \mathcal{B}_C \) is point finite. Let \( \mathcal{B}' = \bigcup_{C \in \mathcal{C}} \mathcal{B}_C \); then \( \mathcal{B}_1 \cup \mathcal{B}' \) is a \( \sigma \)-point finite base for \( X \). All pseudocompact spaces with \( \sigma \)-point finite bases are metrizable [U]. However, all metrizable pseudocompact spaces are also compact, contradiction.

\( P \times [0,1] \) does not have a sharp base. We use the above lemma. Let \( \mathcal{B}_1 = \bigcup_{n<\omega} \{ B(\sigma) : |\sigma| = n \} \) and \( \mathcal{B}_2 = \{ B_n(s_\alpha) : s_\alpha \in L, n < \omega \} \).

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