ON A QUESTION CONCERNING SHARP BASES

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Abstract. A sharp base $\mathcal{B}$ is a base such that whenever $(B_i)_{i<\omega}$ is an injective sequence from $\mathcal{B}$ with $x \in \bigcap_{i<\omega} B_i$, then $\{\bigcap_{i<n} B_i : n < \omega\}$ is a base at x. Alleche, Arhangel'skiĭ and Calbrix asked: if X has a sharp base, must $X \times [0,1]$ have a sharp base? Good, Knight and Mohamad claimed to construct an example of a Tychonoff space P with a sharp base such that $P \times [0,1]$ does not have a sharp base. However, the space was not regular. We show how to modify the construction to make P Tychonoff.

1. Introduction

A *sharp base* is a base $\mathcal{B}$ such that whenever $(B_i)_{i<\omega}$ is an injective sequence from $\mathcal{B}$ with $x \in \bigcap_{i<\omega} B_i$, then $\{\bigcap_{i\leq n} B_i : n < \omega\}$ is a base at x. In a T$_1$ space, $\bigcap_{i<\omega} B_i = \{x\}$.

In [AAC], Alleche, Arhangel'skiĭ and Calbrix defined sharp bases and asked if there is a topological space with a sharp base whose product with [0, 1] does not have a sharp base. Good, Knight and Mohamad [GKM] claimed to have a Tychonoff counterexample, but it turns out that their space is not regular. It is not regular because they added a closed discrete set L to the Baire metric space ${}^{\omega}\mathfrak{c}$, in such a way to make the new space P pseudocompact. Such P cannot be regular: for if it is, one may find a neighborhood of $p \in {}^{\omega}\mathfrak{c}$ whose closure misses L. That neighborhood can be assumed to come from a clopen basis for ${}^{\omega}\mathfrak{c}$, and would then be homeomorphic to ${}^{\omega}\mathfrak{c}$ and be pseudocompact, a contradiction.

In this paper we give a modification of the Good, Knight, Mohamad space which makes the space Tychonoff. The space we construct is pseudocompact but not compact, hence not metrizable; we also show it is not developable. Our space has no isolated points and a sharp base, and for T$_1$ spaces a sharp base is always weakly uniform. Since Heath and Lindgren show that a T$_2$ space with a weakly uniform base

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has a G$_\delta$-diagonal [HL], our space has one also. In [AJRS], it is shown that a pseudocompact space with a G$_\delta$-diagonal is Čech-complete, and that if a space with not more than ω_1 isolated points has a sharp base, then it has a point countable base. Therefore, the space we construct is a counterexample for these three other questions:

Is every pseudocompact Tychonoff space with a sharp base metrizable? [AJRS]

Is every pseudocompact space X with a G_δ-diagonal and a point-countable base developable? [A]

Is every Čech-complete pseudocompact space with a point-countable base metrizable? [A]

We have borrowed much of our notation from the paper [GKM].

2. The Example

2.1. The Construction of space P. Let $B = {}^{\omega}\mathfrak{c}$, and for $\sigma \in {}^{<\omega}\mathfrak{c}$ define $[\sigma] = \{g \in B : \sigma \subseteq g\}$. We also denote $\sigma \in {}^{n+1}\mathfrak{c}$ by $(\alpha_0, \alpha_1, \cdots, \alpha_n)$, where $\sigma(i) = \alpha_i$. For $\sigma = (\alpha_0, \alpha_2, \cdots, \alpha_n)$, we denote $(\alpha_0, \alpha_2, \cdots, \alpha_n, \delta)$ by $\sigma^\frown(\delta)$. By $\sigma_1 \perp \sigma_2$ we mean that σ_1 and σ_2 are incompatible (i.e. the two finite partial functions disagree at a point in both domains).

Define $\mathcal{S}$ to be the collection of elements of ${}^{\omega}({}^{<\omega}\mathfrak{c})$ subject to these two conditions:

1. For all $S \in \mathcal{S}$ there exists a $k_s < \omega$ and a $\rho_s \in {}^{<\omega}\mathfrak{c}$ such that whenever $\sigma \in S$, $\sigma \restriction k_s = \rho_s$. This ρ_s will be called the *root* of S.
2. Whenever σ_1 and σ_2 are distinct elements of S, $\sigma_1(k_s) \neq \sigma_2(k_s)$.

Let $\mathcal{S} = \{S_\alpha : \alpha < \mathfrak{c}\}$, and let the root of S_α be ρ_α.

Define $T_\alpha \in {}^{\omega}({}^{<\omega}\mathfrak{c})$ so that $\mathcal{T} = \{T_\alpha : \alpha < \mathfrak{c}\}$ has these three properties:

(i) for $i \neq j$, $T_\alpha(i) \perp T_\alpha(j)$

(ii) if $\beta, \alpha < \mathfrak{c}$, $\beta \neq \alpha$, with T_α and T_β defined, then $\mathtt{ran} T_\beta \cap \mathtt{ran} T_\alpha = \emptyset$, and

(iii) for $\beta, \alpha < \mathfrak{c}$, $\beta \neq \alpha$, T_α and T_β defined, if $T_\alpha(i) \supseteq T_\beta(j)$, then whenever $j' \neq j$, $T_\alpha(i') \perp T_\beta(j')$ for all $i' < \omega$.

Assume for $\alpha < \gamma$ we have either constructed a $T_\alpha \in {}^{\omega}({}^{<\omega}\mathfrak{c})$ subject to the conditions above or we have not constructed a T_α at all. Now we define T_γ. Choose a $\delta \in \mathfrak{c}$ not in $\bigcup\{\mathtt{ran} T_\alpha(j) : \alpha < \gamma, j \in \omega\}$. Then for each $i \in \omega$ let $S'_\gamma(i) = S_\gamma(i)^\frown(\delta)$. The sequence $(T_\gamma(i))_{i<\omega}$ will be a subsequence of $(S'_\gamma(i))_{i<\omega}$, so the fact that no previous T_α contains a finite partial function with δ in the range will yield property (ii) for T_γ.

In addition, the fact that the elements of S'_γ are pairwise incompatible will make the elements of T_γ also incompatible, satisfying property (i). We need to construct our subsequence T_γ of S'_γ to make property (iii) hold at step γ.

Case 1. Suppose there exists some $\alpha < \gamma$ for which T_α was defined, such that for infinitely many j there is some $i \in \omega$ with $S_\gamma(i) \supseteq T_\alpha(j)$. If this is the case, do not define T_γ.

Case 2. If for each $\alpha < \gamma$ there are at most finitely many j for which $S_\gamma(i) \supseteq T_\alpha(j)$ for some i, we will define a T_γ.

Suppose that for $i \leq k$ we have already selected a sequence of natural numbers $0 = n_0 < n_1 < \cdots < n_k$ and defined $T_\gamma(i) = S'_\gamma(n_i)$. There are at most finitely many different finite partial functions f such that $f \subseteq T_\gamma(i)$ for some $i \leq k$. The second induction condition implies that there are at most finitely many $\alpha < \gamma$ with such an f in the range of T_α. List these as $\alpha(0), \ldots, \alpha(m)$. We have assumed that for each $\alpha < \gamma$, there are at most finitely many j for which $S'_\gamma(i)$ extends $T_\alpha(j)$ for some i. Using this fact, we see that for each $\alpha(p)$ there is a j_p such that for all $j \geq j_p$, $S'_\gamma(i)$ does not extend any $T_{\alpha(p)}(j)$. Then define $n_{k+1} = \max(\{j_p : p \leq m\} \cup \{n_k + 1\})$ and $T_\gamma(k+1) = S'_\gamma(n_{k+1})$. To check property (iii), suppose that $\beta < \gamma$ and $T_\gamma(k) \supseteq T_\beta(j)$ for some $j, k < \omega$. (Note that $T_\beta(j) \not\supseteq T_\gamma(k)$ since $\delta \in \mathtt{ran} T_\gamma(k) \setminus \mathtt{ran} T_\beta(j)$.) Assume that k is the least possible for which there exists such a j. Then $\beta = \alpha(p)$ for some $p \leq m$ in the above construction. Since $n_{k+1}, n_{k+2}, \ldots$ are all greater than j_p, $T_\gamma(i)$ cannot extend $T_\beta(j')$ for any $j' \neq j$ and any i, so we have property (iii). Indeed from (iii) together with what we noted above and conditions (1) and (2) of $S \in \mathcal{S}$, we have the following.

(iv) If $\rho_\alpha = \rho_\beta$, then $T_\alpha(j)$ and $T_\beta(i)$ are compatible for at most one pair (i, j) in $\omega \times \omega$.

Choose L disjoint from B such that $L = \{s_\alpha : T_\alpha \text{ is defined}\}$. Let *the root of* s_α refer to ρ_α. Let $P = B \cup L$.

For $\sigma \in {}^{<\omega}\mathfrak{c}$, let $B(\sigma) = [\sigma] \cup \{s_\beta : \rho_\beta \supseteq \sigma\}$ and let $B_n(s_\alpha) = \{s_\alpha\} \cup \bigcup_{m \geq n}([T_\alpha(m)] \cup \{s_\beta : \rho_\beta \supseteq T_\alpha(m)\})$. These will be the basic open sets for P, and call the collection of them $\mathcal{B}$.

2.2. **Verifying Properties of P.** First, we will observe some properties of $\mathcal{B}$.

(a) For $\sigma_1, \sigma_2 \in {}^{<\omega}\mathfrak{c}$, $\sigma_1 \perp \sigma_2$ iff $B(\sigma_1) \cap B(\sigma_2) = \emptyset$ and if $\rho_\alpha \perp \rho_\beta$ then $B_n(s_\alpha) \cap B_m(s_\beta) = \emptyset$.

(b) $\sigma_1 \subseteq \sigma_2$ iff $B(\sigma_2) \subseteq B(\sigma_1)$, and if $\rho_\alpha \supseteq \sigma$, then for each $n < \omega$, $B_n(s_\alpha) \subseteq B(\sigma)$.

(c) Suppose $B(\sigma) \cap B_n(s_\alpha) \neq \emptyset$. Then $\sigma \subseteq \rho_\alpha$ or $\rho_\alpha \subseteq \sigma$. If $\sigma \subseteq \rho_\alpha$ then $B(\sigma) \cap B_n(s_\alpha) = B_n(s_\alpha)$. If $\sigma \supsetneq \rho_\alpha$, then the intersection is either $B(\sigma)$ or $B(T_\alpha(m))$ for some $m \geq n$. Finally, if $B(\sigma) \subseteq B_n(s_\alpha)$ then for some $m \geq n$ we have $B(\sigma) \subseteq B(T_\alpha(m))$.

(d) If $B_n(s_\alpha) \cap B_{n'}(s_{\alpha'}) \neq \emptyset$ and $\rho_{\alpha'} \subseteq \rho_\alpha$, then the intersection is either $B_n(s_\alpha)$ or a set of form $B(\sigma)$, for some $\sigma \in \{T_\alpha(m), T_{\alpha'}(m') : m \geq n, m' \geq n'\}$. In particular, the latter holds if $\rho_{\alpha'} = \rho_a$.

Proof of (a) - (d):

(a) Suppose that $\sigma_1 \perp \sigma_2$; then there is no point of B nor any finite partial function that could extend both σ_1 and σ_2. If $s_\alpha \in L$ is in $B(\sigma_1) \cap B(\sigma_2)$ then ρ_α extends both, contradiction. Suppose for the reverse, that $B(\sigma_1) \cap B(\sigma_2) = \emptyset$; then since $[\sigma_1] \cap [\sigma_2]$ is contained in this set, it is clear that $\sigma_1 \perp \sigma_2$.

Now if the roots of s_α and s_β are incompatible then each pair of extensions of the roots will be incompatible, hence $B(T_\alpha(n')) \cap B(T_\beta(m')) = \emptyset$ for each $n' \geq n$ and $m' \geq m$. Further, $s_\alpha \in B_m(s_\beta)$ implies that ρ_α extends ρ_β, which has been assumed to be not the case. So $B_n(s_\alpha) \cap B_m(s_\beta) = \emptyset$.

(b) Clear from the definition of $B(\sigma)$ and s_α.

(c) Suppose that $B(\sigma) \cap B_n(s_\alpha) \neq \emptyset$. Since $B_n(s_\alpha) \subseteq B(\rho_\alpha)$ we have $\sigma \not\perp \rho_\alpha$, by (a). If $\sigma \subseteq \rho_\alpha$, then for each $m \geq n$, $\sigma \subseteq T_\alpha(m)$ and $s_\alpha \in B(\sigma)$, so $B_n(s_\alpha) \subseteq B(\sigma)$.

Suppose $\sigma \nsubseteq \rho_\alpha$; then for some $m \geq n$, $B(\sigma) \cap B(T_\alpha(m)) \neq \emptyset$, while property (i) of $\mathcal{T}$ implies that $B(\sigma) \cap B(T_\alpha(k)) = \emptyset$ for $k \neq m$. By (a) and (b), one of $B(\sigma)$ and $B(T_\alpha(m))$ is contained in the other, and the intersection is simply the contained set. This implies the last sentence of (c).

(d) Suppose $B_n(s_\alpha) \cap B_{n'}(s_{\alpha'}) \neq \emptyset$, where $s_\alpha \neq s_{\alpha'}$. If $\rho_{\alpha'} \subsetneq \rho_\alpha$ then $s_{\alpha'} \notin B_n(s_\alpha)$ and $[T_{\alpha'}(j)] \cap [\rho_\alpha] \neq \emptyset$ for at most one $j \in \omega$. Therefore, $B_n(s_\alpha) \cap B_{n'}(s_{\alpha'}) = B(T_{\alpha'}(j)) \cap B_n(s_\alpha)$ for some $j \geq n'$. Now the rest follows from (c).

If $\rho_\alpha = \rho_{\alpha'}$, then the conclusion follows from condition (iv).

$\mathcal{B}$ is a clopen base for P. Notice that the properties show immediately that $\mathcal{B}$ is a base. To see that $B_n(s_\alpha)$ is closed, consider $s_\gamma \in L \setminus B_n(s_\alpha)$. Suppose that $B_j(s_\gamma)$ meets $B_n(s_\alpha)$, where j is sufficiently large that $s_\alpha \notin B_j(s_\gamma)$. Then by (d) the intersection is one of $B(T_\alpha(n'))$ for some $n' \geq n$, $B(T_\gamma(j'))$ for some $j' \geq j$, $B_j(s_\gamma)$ or $B_n(s_\alpha)$.

Since $s_\gamma \notin B_n(s_\alpha)$ and $s_\alpha \notin B_j(s_\gamma)$, we know that the intersection cannot be $B_j(s_\gamma)$ or $B_n(s_\alpha)$. If the intersection is $B(T_\gamma(j'))$ then $B_{j'+1}(s_\gamma)$ misses $B_n(s_\alpha)$. So without loss of generality, the intersection is some

$B(T_\alpha(n'))$. Then $B(T_\gamma(j')) \supseteq B(T_\alpha(n'))$ for some j'. So $B_{j'+1}(s_\gamma) \cap B_n(s_\alpha) = \emptyset$.

To see that each limit point of $B_n(s_\alpha)$ in B is in $B_n(s_\alpha)$, suppose that p is a limit point of $B_n(s_\alpha)$ contained in $B \setminus B_n(s_\alpha)$. Clearly, $p \supseteq \rho_\alpha$. Choose $k < \omega$ so that $p \restriction k \not\subseteq \rho_\alpha$. Then by property (c), $B(p \restriction k) \cap B_n(s_\alpha) = B(T_\alpha(m))$ for some $m \geq n$. Then for $k' < \omega$ with $k' > |T_\alpha(m)|$, we have $B(p \restriction k') \cap B_n(s_\alpha) = \emptyset$.

Lastly, we observe that $B(\sigma)$ is clopen. Since B is dense and the subspace base is clopen, we only need to turn our attention to limit points of $B(\sigma)$ in L. Suppose then that $s_\alpha \in L$ is a limit point of $B(\sigma)$, not in $B(\sigma)$; then for all $n < \omega$, $B_n(s_\alpha)$ meets $B(\sigma)$. If $\rho_\alpha \perp \sigma$, then clearly $B(\sigma) \cap B_n(s_\alpha) = \emptyset$. If $\rho_\alpha \supseteq \sigma$, then s_α is in $B(\sigma)$ which is contrary to our assumptions. So assume that $\sigma \supsetneq \rho_\alpha$, then there is at most one $T_\alpha(m)$ that extends σ or is extended by σ. Then $B_{m+1}(s_\alpha) \cap B(\sigma) = \emptyset$.

$\mathcal{B}$ is sharp. Let the injective sequence $(B(\sigma_i))_{i<\omega}$ come from $\mathcal{B}$. If $p \in B$ is contained in every $B(\sigma_i)$ then $p \supseteq \sigma_i$, so since $|\sigma_i|$ must be unbounded, it is clear that $\{\bigcap_{i \leq n} B(\sigma_i) : n < \omega\}$ is a base at p. If s_α is in every $B(\sigma_i)$, then ρ_α extends every σ_i, but since $|\rho_\alpha|$ is finite, this is not possible.

Now consider an injective sequence $(B_{n_i}(s_{\alpha_i}))_{i<\omega}$, with nonempty intersection. If there is an infinite subset J of ω such that the ρ_{α_i}, $i \in J$, are distinct, then it is easy to see that $\{B(\rho_{\alpha_i}) : i \in J\}$ is a base for a unique point $p \in B$. Hence, so is $\{\bigcap_{i \leq j} B_{n_i}(s_{\alpha_i}) : j < \omega\}$, since for each $i \in J$ we have $B_{n_i}(s_{\alpha_i}) \subseteq B(\rho_{\alpha_i})$.

Next, suppose that $s_{\alpha_i} = s_\alpha$ for all i in an infinite subset J of ω. Then $\{B_{n_i}(s_{\alpha_i}) : i < \omega\}$ is a base at s_α, therefore $\{\bigcap_{i \leq j} B_{n_i}(s_{\alpha_i}) : j < \omega\}$ is a base at s_α too.

The final case, without loss of generality, is when the s_{α_i}'s are distinct, but $\rho_{\alpha_i} = \rho$ for all $i < \omega$. Then by (d), pairwise intersections have the form $B(\sigma)$ for some σ in the range of the corresponding pair from $\mathcal{T}$. By property (ii) of $\mathcal{T}$, $\{B_{n_i}(s_{\alpha_i}) \cap B_{n_{i+1}}(s_{\alpha_{i+1}}) : i \text{ is even}, i < \omega\}$ consists of distinct $B(\sigma)$'s. Therefore, this must be a base at some $p \in B$, and $\{\bigcap_{i \leq j} B_{n_i}(s_{\alpha_i}) : j < \omega\}$ is as well.

P is not compact. Consider $C_0 = \{s_\alpha \in L : \rho_\alpha = \emptyset\}$. Note that $P \setminus C_0 = \bigcup_{\alpha < \mathfrak{c}} B((\alpha))$. We intend to show that the closed set C_0 is infinite and discrete. To see that this is a discrete set, notice that for $s_\alpha \in C_0$, the set $B_1(s_\alpha) \cap C_0$ can only contain s_α. Examine $\{(\alpha_i^\gamma)_{i<\omega} : \gamma < \mathfrak{c}\}$, where $\alpha_i^\gamma = \alpha_{i'}^{\gamma'}$ iff both $i = i'$ and $\gamma = \gamma'$. Call this collection $\mathcal{S}_0$; then this is a subset of $\mathcal{S}$. Note, that for each $S_\alpha \in \mathcal{S}_0$ and $i < \omega$,

we have that the length of $S_\alpha(i)$ is exactly one. Also, each $T_\gamma(j)$ is constructed to have length at least 2. Therefore, during the induction that defined $\mathcal{T}$, for each $S_\alpha \in \mathcal{S}_0$, Case 1 does not hold. Therefore, a corresponding T_α is constructed for each $S_\alpha \in \mathcal{S}_0$.

P is not perfect, hence not developable. Let $U = P \setminus C_0$. We show that U is not F_σ, and hence P is not developable. Suppose that $\{F_j\}_{j<\omega}$ is a collection of closed sets so that $\bigcup_{j<\omega} F_j = U$. By the Baire property of B, each $[(\alpha)]$ is Baire. So for all $\alpha < \mathfrak{c}$ there is an n_α and an $[\tilde{\alpha}] = [(\alpha, \beta_1, \cdots, \beta_{n_\alpha})] \subseteq F_{n_\alpha}$. Choose n_0 so that $\{\alpha : [\tilde{\alpha}] \subseteq F_{n_0}\}$ is infinite. Order $\{\alpha_i\}_{i<\omega} \subseteq \{\alpha : [\tilde{\alpha}] \subseteq F_{n_0}\}$, then $S = ((\tilde{\alpha_i}))_{i<\omega} \in \mathcal{S}$, and has the empty set as its root. So an $s \in L$ was defined as a limit point of S, and σ the root of s is also the empty set. Therefore, s is a limit point of the closed set F_{n_0}. This implies that $s \in P \setminus C_0$, contradicting that s has the empty root.

P is pseudocompact. Suppose that φ is an unbounded continuous real valued function on P. Since B is dense, for each $n \in \omega$ there is an x_n such that $\varphi(x_n) > n$. Let $D = \{x_n : n \in \omega\}$ and let's note that D is closed discrete, hence not compact. If p were a cluster point of D, then every open neighborhood of p contains infinitely many elements of D. This implies that φ increases unboundedly over every neighborhood of p, contradicting the continuity of φ.

Since D is closed and not compact we can find a $k < \omega$ such that $\{x_n{\restriction}k : x_n \in D\}$ is infinite. Choose the minimum such k. Then there is a $\sigma \in {}^{<\omega}\mathfrak{c}$ and an infinite subset A of ω, such that $x_n{\restriction}(k-1) = \sigma$ for $n \in A$, and $x_n(k-1)$ is different for these infinitely many $n \in A$.

Let $D^* = \{x_n : n \in A\}$. Since $\varphi(x_n) > n$ by continuity of φ there exists $j_n > k$ so that $\varphi(B(x_n{\restriction}j_n)) > n$. Then for some $\alpha < \mathfrak{c}$, $\{x_n{\restriction}j_n : x_n \in D^*\}$ is S_α and $\rho_\alpha = \sigma$. If s_α was not defined then for some $\beta < \alpha$, $T_\beta(j) \subseteq S_\alpha(n) = x_n{\restriction}k$ for infinitely many j. Then each basic open neighborhood of s_β contains infinitely many of the sets $B(x_n{\restriction}k)$. So φ takes on arbitrarily large values over every neighborhood of s_β contradicting continuity. If s_α was defined, then $T_\alpha(i)$ was chosen so that $T_\alpha(i) \supseteq x_{n_i}{\restriction}j_{n_i}$ for each $i \in \omega$, so $B(T_\alpha(i)) \subseteq B(x_{n_i}{\restriction}j_{n_i})$. So again, φ takes on large values over every open set containing s_α, contradicting the continuity of φ.

The following lemma, which was suggested by the referee, is essentially due to [GKM].

Lemma 1. *Let X be a Tychonoff, pseudocompact, non-compact space which partitions into $B \cup L$, and has a sharp base $\mathcal{B}$. If*

(a) *$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ where $\mathcal{B}_1$ is a σ-point finite base for B*

(b) *for all* $x \in L$ *there is a local base* $\{B_n(x) : n < \omega\}$ *so that* $n < m$ *implies* $B_m(x) \subsetneq B_n(x)$ *and* $\mathcal{B}_2 = \{B_n(x) : n < \omega, x \in L\}$

(c) *for* $x \neq y \in L$, n, $m \in \omega$, $B_n(x) \neq B_m(y)$.

Then $X \times [0,1]$ *does not have a sharp base.*

Proof. Assume, by way of contradiction, that $\mathcal{W}$ is a sharp base for $X \times [0,1]$. Let $\mathcal{C}$ be a countable base for $[0,1]$. For each $x \in L$, choose $W_n^x \in \mathcal{W}$, $B_n^x \in \mathcal{B}$ and $C_n^x \in \mathcal{C}$ so that $(x, \frac{1}{2}) \in B_n^x \times C_n^x \subseteq W_n^x \subseteq B_n(x) \times [0,1]$. Let $\mathcal{B}_C = \{B \in \mathcal{B}$: for some $n \in \omega$ and $x \in L$, $B = B_n^x$ and $C = C_n^x\}$.

We claim that $\mathcal{B}_C$ is point-finite. Suppose not; then there exists an infinite collection $(B_j)_{j<\omega}$ from $\mathcal{B}_C$ that has nonempty intersection. Let $y \in \bigcap_{j\in\omega} B_j$; then there are $x_j \in L$ and $n_j \in \omega$ so that $B_j = B_{n_j}^{x_j}$ and $C = C_{n_j}^{x_j}$. Then $\{y\} \times C \subset \bigcap_{j\in\omega}(B_{n_j}^{x_j} \times C_{n_j}^{x_j}) \subseteq \bigcap_{j\in\omega} W_{n_j}^{x_j}$. If $x_j \neq x_k$ then $B_n(x_j) \neq B_{n'}(x_k)$.

There are two cases to consider.

Case 1. There is an infinite $J \subseteq \omega$ so that $x_j \neq x_k$ whenever $j \neq k$ with $j, k \in J$. Then $\{W_{n_j}^{x_j} : j \in J\}$ is infinite. Suppose not; then some W is contained in infinitely many different $B_{n_j}(x_j) \times [0,1]$. The sharpness of $\mathcal{B}$ implies that $\bigcap_{j<\omega} B_{n_j}(x_j)$ is at most a singleton; it must be $\{y\}$, implying $W \subseteq \{y\} \times [0,1]$, which is impossible. Hence $\{W_{n_j}^{x_j} : j \in J\}$ is infinite, and so $\{y\} \times C \subseteq \bigcap_{j\in\omega} W_{n_j}^{x_j}$ is a single point, a contradiction.

Case 2. There is an infinite $K \subseteq \omega$ so that $x_j = x_k = x$ for $j, k \in K$. Then the set $\{n_k : k \in K\}$ is infinite, since the $B_{n_k}^{x_k}$ are distinct. Again, $\{y\} \times C \subseteq \bigcap_{k\in K}(B_{n_k}^{x_k} \times C_{n_k}^{x_k}) \subseteq \bigcap_{k\in K} W_{n_k}^{x_k} = \bigcap_{k\in K} W_{n_k}^{x}$. Once again, this is simply one point, so we have the same contradiction as in Case 1.

Therefore, $\mathcal{B}_C$ is point finite. Let $\mathcal{B}' = \bigcup_{C\in\mathcal{C}} \mathcal{B}_C$; then $\mathcal{B}_1 \cup \mathcal{B}'$ is a σ-point finite base for X . All pseudocompact spaces with σ-point finite bases are metrizable [U]. However, all metrizable pseudocompact spaces are also compact, contradiction. □

$P \times [0,1]$ *does not have a sharp base.* We use the above lemma. Let $\mathcal{B}_1 = \bigcup_{n<\omega}\{B(\sigma) : |\sigma| = n\}$ and $\mathcal{B}_2 = \{B_n(s_\alpha) : s_\alpha \in L, n < \omega\}$.

□

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