MONOTONICALLY COMPACT T_2-SPACES ARE METRIZABLE

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Abstract. We answer a question of M. Matveev by showing that every monotonically compact Hausdorff space is metrizable.

A space X is *monotonically compact* (mC) if one can assign to each open cover $\mathcal{U}$ a finite open refinement $r(\mathcal{U})$ such that $r(\mathcal{V})$ refines $r(\mathcal{U})$ whenever $\mathcal{V}$ refines $\mathcal{U}$. It is easily seen that compact metrizable spaces are mC. Matveev[3] asked whether every mC Hausdorff space is metrizable.

In [1], we proved that every mC Hausdorff space having property K is metrizable, where a space X has *property* K if every uncountable collection $\mathcal{U}$ of nonempty open sets contains an uncountable subcollection $\mathcal{V}$ such that $V_1, V_2 \in \mathcal{V}$ implies $V_1 \cap V_2 \neq \emptyset$. In this brief note, using this result as a starting point, we complete the solution to Matveev's problem by showing that the class of mC Hausdorff spaces coincides with the class of compact metrizable spaces.

In the proof, we use two theorems from partition calculus, Erdös' theorem $\omega_1 \to (\omega, \omega_1)^2$ and Ramsey's theorem $\omega \to (\omega)^r_n$. The first says that if the unordered pairs of an uncountable set A are divided into two pots, say Pot I and Pot II, then there is either an infinite subset of A all pairs from which are in Pot I, or an uncountable subset of A all pairs from which are in Pot II. In Ramsey's theorem, r and n are positive integers and it says that if the r-element subsets of a countably infinite set W_1 are divided into n pots, then there is an infinite subset W of W_1 such that all r-element subsets of W are in the same pot. See, e.g., Appendix 4 of [2] for more information on these and other partition calculus theorems.

Theorem 0.1. *Monotonically compact Hausdorff spaces are metrizable.*

Proof. Let X be Hausdorff mC with mC operator r, and suppose X is not metrizable. Then X does not have property K [1], i.e., there are nonempty open sets U_α, $\alpha < \omega_1$, such that, for any uncountable $W \subset \omega_1$, there are $\alpha, \beta \in W$ with $U_\alpha \cap U_\beta = \emptyset$.

Claim 1. For any uncountable $A \subset \omega_1$, there is an infinite $B \subset A$ such that $\{U_\alpha : \alpha \in B\}$ is pairwise-disjoint. To see this, for $\beta < \alpha \in A$, put $\{\beta, \alpha\}$ in Pot I if $U_\alpha \cap U_\beta = \emptyset$, and in Pot II otherwise. Since there is no uncountable subset of A all pairs from which are in Pot II, by Erdös' theorem, there must be an infinite $B \subset A$ all pairs from which are in Pot I. Then $\{U_\alpha : \alpha \in B\}$ is pairwise-disjoint.

Now for each $\alpha < \omega_1$, pick $p_\alpha \in U_\alpha$, and let $\mathcal{U}_\alpha = \{X \setminus \{p_\alpha\}, U_\alpha\}$.

Claim 2. For each $W \subset \omega_1$, there is a finite $W' \subset W$ such that if $V \in r(\mathcal{U}_\alpha)$, $\alpha \in W$, then there is $\beta \in W'$ such that $V \subset U_\beta$ or $p_\beta \notin V$.

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To see this, suppose $W \subset \omega_1$. Let $\mathcal{U}_W = \bigcup_{\alpha \in W} \mathcal{U}_\alpha$. For each $O \in r(\mathcal{U}_W)$, there is $\beta(O) \in W$ with $O \subset U_{\beta(O)}$ or $O \subset X \setminus \{p_{\beta(O)}\}$. Let $W' = \{\beta(O) : O \in r(\mathcal{U}_W)\}$.

Suppose $\alpha \in W$ and $V \in r(\mathcal{U}_\alpha)$. Note that $\mathcal{U}_\alpha$ refines $\mathcal{U}_W$, hence $r(\mathcal{U}_\alpha)$ refines $r(\mathcal{U}_W)$. So there is some $O \in r(\mathcal{U}_W)$ with $V \subset O$. Then $O \subset U_{\beta(O)}$ or $O \subset X \setminus \{p_{\beta(O)}\}$. Since $V \subset O$, the same is true of V. This proves Claim 2.

There is an uncountable $W_0 \subset \omega_1$ and positive integer k such that $|r(\mathcal{U}_\alpha)| = k$ for every $\alpha \in W_0$. Say $r(\mathcal{U}_\alpha) = \{V_{\alpha,i} : i < k\}$.

By Claim 1, there is an infinite subset W_1 of W_0 such that $\{U_\alpha : \alpha \in W_1\}$ is pairwise-disjoint. W.l.o.g., W_1 has order type ω. For $\beta < \alpha \in W_1$ and $j < k$, put $\{\beta, \alpha\}$ in Pot j iff j is least such that $p_\beta \in V_{\alpha,j}$.

By Ramsey's theorem, there is an infinite subset W of W_1 and $j_0 < k$ such that $\beta < \alpha \in W$ implies $\{\beta, \alpha\}$ is in Pot j_0. Suppose W' is the finite subset of W guaranteed by Claim 2. Let $\alpha \in W$ with $\alpha > max(W')$ and $|\alpha \cap W| \geq 2$. Then $\{p_\beta : \beta \in W'\} \subset V_{\alpha,j_0}$ and V_{α,j_0} contains at least two points of $\{p_\beta : \beta \in W\}$. Since the U_β's for $\beta \in W$ are disjoint, it follows that V_{α,j_0} is not contained in any U_β, $\beta \in W'$. Since V_{α,j_0} also contains p_β for every $\beta \in W'$, Claim 2 is contradicted with $V = V_{\alpha,j_0}$. □

Remark. It may be interesting to note that in the proof of the theorem (both the one in this paper and the property K result it relies upon from [1]) , it was only necessary to apply the defining property of the mC operator r, i.e., "$r(\mathcal{V})$ refines $r(\mathcal{U})$ whenever $\mathcal{V}$ refines $\mathcal{U}$", in the case where $\mathcal{V}$ is a 2-element open cover, $\mathcal{U}$ is a union of 2-element open covers, and $\mathcal{V} \subset \mathcal{U}$.

References

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