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ARE STRATIFIABLE SPACES M_1?

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The Bing-Nagata-Smirnov characterization of metrizable spaces as the regular spaces having a σ-locally finite (or σ-discrete) base was one of the seminal results of the early 1950's in general topology . Then the late 1950's saw E. Michael's characterizations of paracompactness in regular spaces via σ-locally finite, σ- closure-preserving, and other related types of refinements. Clearly motivated by these now classical results, in 1961 Michael's student J. Ceder[Ced61] introduced the following class of spaces as a natural generalization of metrizable spaces:

Definition 0.1. *A regular space X is an M_1-space if it has σ-closure-preserving base*[1].

M_1-spaces are paracompact by one of Michael's theorems, and it is easy to see that closed sets are G_δ, so they are also perfectly normal. An important subclass of M_1-spaces is the class of closed images of metrizable spaces [Sla73].

However, Ceder could not show that M_1-spaces are hereditary, even for closed subspaces. To see the problem, note that the trace of a closure-preserving collection on a closed subset need not be closure preserving (there are easy examples in the plane illustrating this). Nor could he show that they are preserved by nice mappings such as closed or even perfect mappings. Thus he also considered two formally larger classes, which he called M_2-spaces and M_3-spaces, respectively. These classes had more technical definitions, but otherwise, they had essentially the same topological properties and they had the advantage of being preserved by arbitrary subspaces as well as closed mappings.

Definition 0.2. *A collection $\mathcal{B}$ is a quasi-base for X if whenever $x \in U$, U open, there is $B \in \mathcal{B}$ with $x \in B^o \subset B \subset U$. A regular space X is an M_2-space if it admits a σ-closure-preserving quasi-base $\mathcal{B}$ (which may be taken to consist of closed sets).*

Note that M_2-spaces are hereditary, since the trace of a closure-preserving collection of *closed* sets on a subspace is closure-preserving in the subspace.

Recall that B is a regular closed set if $B = \overline{B^o}$. If $\mathcal{B}$ is a σ-closure-preserving quasi-base of regular closed sets, it is easy to check that the interiors form a σ-closure-preserving base. So if M_2 is really more general than M_1, it comes from allowing members of the quasi-base to have nonempty "outliers" $B \backslash \overline{B^o}$. Note that such outliers can help make a collection closure-preserving; e.g., a collection $D_0, D_1, \ldots$ of disks in the plane converging to a point p is not closure-preserving, but $\{D_n \cup \{p\} : n \in \omega\}$ is closure-preserving.

The M_3-spaces were defined by Ceder as the regular spaces having a σ-cushioned pair-base, though the following characterization of Borges[Bor66], who showed that

[1]Recall that a collection $\mathcal{U}$ is *closure-preserving* if $\overline{\cup \mathcal{U}'} = \cup\{\overline{U} : U \in \mathcal{U}'\}$ for any subcollection $\mathcal{U}'$ of $\mathcal{U}$, and is *σ-closure-preserving* if it is a countable union of closure-preserving collections.

M_3-spaces have many other good properties (e.g., they satisfy the Dugundji Extension Theorem), and renamed them "stratifiable spaces", provides a more elegant definition:

Definition 0.3. *A T_1-space X is an M_3-space (or stratifiable) iff one can assign to each closed set H a decreasing sequence $U_n(H)$, $n \in \omega$, of open sets satisfying:*

(1) $H = \bigcap_{n\in\omega} U_n(H) = \bigcap_{n\in\omega} \overline{U_n(H)}$;
(2) $H \subset K \Rightarrow U_n(H) \subset U_n(K)$.

Since the first condition characterizes perfect normality, stratifiable spaces can be thought of as the class of "monotonically perfectly normal" spaces. They are also exactly the monotonically normal σ-spaces (σ-spaces are spaces having a σ-discrete network).

Ceder didn't know if any of these classes were in fact different. In the mid-1970's, the author [Gru77] and Junnila[Jun78] independently proved that stratifiable and M_2-spaces are the same. But to this day, it is not known if stratifiable and M_1-spaces are the same.

Problem 1. *Are stratifiable (equivalently, M_2-) spaces M_1?*

Since stratifiable spaces have turned out to be one of the most useful and important classes of generalized metrizable spaces, an answer to the problem would be of great interest, and if positive, would render many papers on the subject obsolete.

1. Equivalent questions

As mentioned in the introduction, Ceder was led to define M_2- and M_3-spaces because he could not show that M_1-spaces were preserved by some basic topological operations. In fact, certain preservation statements are equivalent to Problem 1:

Theorem 1.1. *The following statements are equivalent:*

(1) *Stratifiable spaces are M_1;*
(2) *Every (closed) subspace of an M_1-space is M_1;*
(3) *Perfect (closed) images of M_1-spaces are M_1.*

The above equivalences follow immediately from the the fact that stratifiable spaces are preserved by subspaces and closed images, along with the following very pretty result of Heath and Junnila[HJ81]:

Theorem 1.2. *Every stratifible space X is a closed subspace of an M_1-space Z such that $Z\backslash X$ consists of isolated points and there is a perfect retraction $r : Z \to X$.*

Here are some other equivalences:

Theorem 1.3. *The following statements are equivalent:*

(1) *Stratifiable spaces are M_1;*
(2) *Every point of every stratifiable space has a (σ-)closure-preserving local base;*
(3) *Every closed subset of every stratifiable space has a (σ-)closure-preserving outer base.*

Here, an *outer base* for a subset H of X is a collection $\mathcal{U}$ of open supersets of H such that every open superset of H contains a member of $\mathcal{U}$. That these statements are equivalent follow fairly easily from the fact that stratifiable spaces are paracompact σ-spaces, preservation under closed mappings, and using the following recent and important result of Mizokami[Miz04]:

Theorem 1.4. *Every closed subset of an M_1-space has a closure-preserving outer base.*

For some time the class $\mathcal{P}$ of M_1-spaces in which every closed subset has a closure-preserving outer base was studied; by Theorem 1.4, every M_1-space is in $\mathcal{P}$.

2. Related classes and partial results

One of the most important early partial results on Problem 1 was the following result of Ito[Itō85]:

Theorem 2.1. *The following are equivalent for a stratifiable space X:*
(1) *Every closed subset of X has a closure-preserving outer base (and hence X is M_1);*
(2) *Every point of X has a closure-preserving local base.*

Using the fact the stratifiable spaces are paracompact σ-spaces, it is easy to see that if every closed subset of a stratifiable space X has a closure-preserving outer base, then X is M_1 (by Theorem 1.4, the converse also holds). So Ito's result says it suffices that every point have a closure-preserving local base. E.g., first-countable stratifiable spaces are M_1.

Mizokami, Shimane, and Kitamura[MSK01], extending a result of the first two of these authors [MS00], have improved the first countable result to sequential spaces and more:

Theorem 2.2. *Suppose X is stratifiable and has the following property:*
(δ) *Whenever U is dense open in X and $x \in X \backslash U$, there is a closure-preserving collection $\mathcal{F}$ of closed subsets of X that is a network at x, such that $\overline{F \cap U} = F$ for every $F \in \mathcal{F}$.*

Note that this result extends Ito's, for if $\mathcal{B}$ is a closure-preserving local base at x, and U is dense open, then $\mathcal{F} = \{\overline{B} : B \in \mathcal{B}\}$ witnesses property (δ). It is easy to observe that every Fréchet space satisfies (δ); less obvious is that sequential stratifiable spaces satisfy (δ)[MS00]. More generally, a stratifiable space satisfies (δ) (see [MSK01])if it has the following property, which has been called *weak approximation by points (WAP)*[Sim94]:

(WAP) If A is not closed, there exists $B \subset A$ such that $\overline{B} \backslash A$ is exactly one point.

There are classes of spaces formally stronger than M_1 for which it is as yet undetermined whether every M_3-(or sometimes even every M_1-) space belongs to the class. The most pertinent of these classes, so it seems at present, is the class of μ-spaces, introduced by Nagami[Nag71] for dimension-theoretic reasons.

Definition 2.3. *A space X is F_σ-metrizable if it is a countable union of closed metrizable subspaces, and X is a μ-space if it is homeomorphic to a subspace of a countable product of paracompact F_σ-metrizable spaces.*

I showed [Gru80] that stratifiable F_σ-metrizable spaces are M_1. The following two results extend this:

Theorem 2.4. (1) *Stratifiable μ-spaces are M_1;*
(2) *A stratifiable space is M_1 if it is a countable union of closed M_1 subspaces.*

The second result follows from Mizokami's theorem in [Miz84] that a stratifiable space which is a countable union of closed subspaces in the class $\mathcal{P}$ is M_1, together with his more recent result mentioned earlier that every M_1-space is in $\mathcal{P}$.

Theorem 2.4(1) is due to Mizokami[Miz84]; Junnila and Mizokami[JM85] subsequently show that a couple of other subclasses of stratifiable spaces that had been studied in the literature are μ-spaces. Tamano[Tam85] obtained the following useful internal characterization of μ-spaces:

Theorem 2.5. *The following are equivalent:*
(1) *X is a stratifiable μ-space;*
(2) *X has a base $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$, where each $\mathcal{B}_n$ is mosaical, i.e., there is a σ-discrete cover $\mathcal{F}_n$ of X such that $F \cap B \neq \emptyset \iff F \subset B$ for every $F \in \mathcal{F}_n$ and $\mathcal{B} \in \mathcal{B}_n$.*

There are spaces having a countable network which are not μ-spaces[Tam01][TT05]; but we don't know the answer to:

Problem 2. *Is every stratifiable space a μ-space?*

Since the class of μ-spaces is hereditary, by the Heath-Junnila theorem it is equivalent to ask if every M_1-space is a μ-space. Obviously a positive answer to Problem 2 settles Problem 1. An important partial result is that spaces having a σ-closure-preserving clopen base, which are called *M_0-spaces*, are μ-spaces[Itō84b]. The class of M_0-spaces turns out to coincide with the class of stratifiable μ-spaces X with dim $X = 0$; also, every stratifiable μ-space is a perfect image of an M_0-space[Miz84].

Consider the following string of containments, where $\mathcal{M}_i$ denotes the class of M_i-spaces, $\mathcal{S}$ ($\mathcal{S}\mu$) is the class of stratifiable (stratifiable μ-) spaces, and $\mathcal{PM}_0$ ($\mathbf{C}\mathcal{M}_0$) is the class of perfect (closed) images of M_0-spaces:

$$\mathcal{M}_0 \subset \mathcal{S}\mu \subset \mathcal{PM}_0 \subset \mathbf{C}\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{S}.$$

It is not known if any of these containments other than the leftmost are strict. Indeed, parts of this line could collapse, maybe all the way from $\mathcal{S}$ to $\mathcal{S}\mu$. But it could also happen, e.g., that $\mathcal{S} = \mathcal{M}_1 \neq \mathbf{C}\mathcal{M}_0$. That every space in $\mathbf{C}\mathcal{M}_0$ is M_1, in fact hereditarily M_1, follows from the observation that M_0 spaces are hereditary, and the following result of Ito[Itō84a]:

Proposition 2.6. *If every closed subset of an M_1-space X is M_1, then every closed image of X is M_1.*

The class $\mathcal{PM}_0$ would equal $\mathcal{S}\mu$ if the following old question of Nagami[Nag71] had a positive answer:

Problem 3. *Are μ-spaces preserved by perfect mappings?*

This seems to be open even for closed mappings. A partial result is that the closed image of a stratifiable F_σ-metrizable space is a μ-space[JM85].

Another interesting subclass of stratifiable spaces was introduced by Oka[Oka83]:

Definition 2.7. *A stratifiable space X is in the class $\mathcal{EM}_3$ if there is a σ-closure-preserving collection $\mathcal{E}$ satisfying:*

(∗) *Whenever $x \in U$, U open, there is $\mathcal{F} \subset \mathcal{E}$ such that $\cup\mathcal{F}$ is closed, and $x \in X \setminus \cup \mathcal{F} \subset U$.*

Oka's motivation for defining $\mathcal{EM}_3$ was dimension-theoretic; he proved the following:

Proposition 2.8. (1) *dim X = Ind X for every $X \in \mathcal{EM}_3$;*
(2) *$\mathcal{EM}_3$ is the class of perfect (or closed) images of (strongly) 0-dimensional stratifiable spaces;*
(3) *$\mathcal{EM}_3$ is hereditary, countably productive, and preserved by closed maps.*

It follows that the class $\mathcal{EM}_3$ fits between $\mathbf{C}\mathcal{M}_0$ and $\mathcal{S}$; but it is not known if it is equal to either one or both, nor is its relation to $\mathcal{M}_1$ known. If $\mathcal{EM}_3 = \mathcal{M}_1$, then it would follow from the Heath-Junnila theorem that $\mathcal{S} = \mathcal{M}_1$. Also note that $\mathcal{S} = \mathcal{EM}_3$ iff every stratifiable space is the closed (or perfect) image of a (srongly) 0-dimensional stratifiable space. The following dimension theoretic questions are also open:

Problem 4. (1) *Let X be strongly 0-dimensional. If $X \in \mathcal{M}_1$, must $X \in \mathcal{M}_0$? What if $X \in \mathcal{PM}_0$?*
(2) *Is every M_1-space the perfect (or closed) image of a (strongly) 0-dimensional M_1-space?*
(3) *For an M_1-space X, is it true that Ind$(X) \leq n$ iff X has a σ-closure-preserving base $\mathcal{B}$ such that for every $B \in \mathcal{B}$, Ind$(\partial B) \leq n-1$?*
(4) *Does dim X=Ind X for all stratifiable X? What if X is separable?*

If the answer to (2) is positive, then by the Heath-Junnila result, every stratifiable space is also the closed image of a 0-dimensional M_1-space, so it would follow that $\mathcal{S} = \mathcal{EM}_3$. (3) is known to hold for stratifiable μ-spaces[Miz84]. (4) is known to consistently fail (e.g., under CH) for spaces having a countable network ([DW00]; see also [DH]).

3. Function spaces and a possible counterexample

Gartside and Reznichencko[GR00] investigated stratifiability of function spaces, and in particular, proved that the space $C_k(X)$ of all real-valued continuous functions on X with the compact-open topology is stratifiable whenever X is Polish (complete separable metric). They show this first when X is the space $\mathbb{P}$ of irrationals, and then use the fact that any Polish space Y is the continuous image of $\mathbb{P}$, and hence $C_k(Y)$ embeds in $C_k(\mathbb{P})$. It is quite interesting that their proof of stratifiability of $C_k(\mathbb{P})$ gives no clue as to its M_1-ness, and no one has yet been able to determine if $C_k(\mathbb{P})$ is M_1 or not.

Problem 5. [GR00] *Is $C_k(\mathbb{P})$ an M_1-space?*

It is also not known if $\mathbf{C}_k(\mathbb{P})$ is a μ-space or in $\mathcal{EM}_3$. A negative answer to Problem 5 of course solves Problem 1 in the negative. There are unpublished results of Balogh and Gruenhage, Gartside, Nyikos, and Tamano showing that collections built in some ways from standard basic open sets won't work; e.g., no collection of sets consisting of finite unions of standard basic open sets of $C_k(\mathbb{P})$ can form either a σ-closure-preserving or σ-mosaical base. In the positive direction, the author and Tamano[GT05] have shown that $C_k(X)$ is a μ-space whenever X is σ-compact Polish.

Possibly, one could show that $C_k(\mathbb{P})$ is M_1 by showing it has property (δ). However, while it is known that $C_k(\mathbb{P})$ is not sequential[Pol74], the following is open even for σ-compact Polish spaces:

Problem 6. *If X is Polish, does $C_k(X)$ have the WAP property?*

Gartside and Reznichenko asked if a converse of their result is true:

Problem 7. *If X is separable metzrizable, and $C_k(X)$ is stratifiable, must X be Polish?*

This is still unsettled, but Nyikos[Nyi] has shown that $C_k(X)$ is not stratifiable for any separable metric X which contains a 0-dimensional closed subspace with no uncountable compact sets (e.g., a closed subspace homeomorphic to the rationals); a corollary is that the answer to Problem 7 is positive for coanalytic subsets of $\mathbb{R}$.

4. Some final remarks

We close with a few more remarks about Problem 1 and some suggestions for further reading. A brief survey, with proofs, of basic results on stratifiable and M_1-spaces is included in [Gru84]. Much more extensive and highly recommended is Tamano's survey[Tam89], which includes among other things proofs of Ito's theorems as well as most of the results we mentioned on μ-spaces and the class $\mathcal{EM}_3$. Also discussed there are some classes of M_1-spaces that fall between stratifiable μ-spaces and hereditarily M_1-spaces that are defined in terms of special bases.

For stratifiable spaces, separability and Lindelöfness, as well as the hereditary versions, are equivalent, and these are in turn equivalent to having a countable network. So Problem 1 would seem to split naturally into two cases, the countable network case and the σ-discrete but uncountable network case. However, there seems to be no evidence that these cases will turn out any differently or that the countable network case is any easier. Indeed, $C_k(\mathbb{P})$, which presently the only specific space known to be stratifiable that is not known to be M_1, has a countable network.

Anyone hoping to prove that stratifiable implies M_1 should become familiar with techniques in [MSK01] and/or [MS00], many of which were also used in the important paper [Miz04]. Large parts of these arguments involve fattening up closure-preserving collections of closed sets to collections which have certain combinatorial and regularity properties (with the goal of building closure-preserving collections of regular closed sets). It is difficult to characterize these techniques briefly, so we only mention some tools that are common to not only these arguments but many that preceded these. Monotone normality is heavily exploited. Any stratifiable space has a weaker metrizable topology; constructing weaker metrizable topologies having certain close relations to the given topology is frequently useful. Another important tool is the following key lemma in Ito's proof of Theorem 2.1: given a closure-preserving collection $\mathcal{B}$ of closed sets, there is a σ-discrete set D such that, for every $B \in \mathcal{B}$, $D \cap B$ is dense in B. Also, building networks with special properties can be useful; oft-used here is the result in [SN68] that any closure-preserving collection $\mathcal{B}$ of closed sets in a stratifiable space is mosaical (see 2.5(2) for the meaning of mosaical).

Finally, we remark that it seems doubtful that the answer will turn out to be independent of ZFC; the only known consistency result in the area is due to N.

Zhong[Zho94], who showed that that stratifiable spaces of cardinality less than $\mathfrak{b}$ are M_1.

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