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FRÉCHET-URYSOHN FOR FINITE SETS, II

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Abstract. We continue our study [6] of several variants of the property of the title. We answer a question in [6] by showing that a space defined in a natural way from a certain Hausdorff gap is a Fréchet α_2 space which is not Fréchet-Urysohn for 2-point sets (FU_2), and answer a question of Hrusak by showing that under MA_{ω_1}, no such "gap space" is FU_2. We also introduce versions of the properties which are defined in terms of "selection principles", give examples when possible showing that the properties are distinct, and discuss relationships of these properties to convergence in product spaces, to the α_i-spaces of A.V. Arhangel'skii, and to topological games.

1. Introduction

For a space X and a point $x \in X$, a family $\mathcal{P}$ of subsets of X is said to be a *π-network at x* iff for each open U containing x, there is $P \in \mathcal{P}$ such that $P \subseteq U$. We will say that a sequence P_n, $n \in \omega$, *converges to x*, and write $P_n \to x$, iff every neighborhood of x contains P_n for all but finitely many n. We will also say that a countably infinite family $\mathcal{P}$ of subsets of X converges to x iff the sequence formed by any one-to-one enumeration of its elements converges to x; equivalently, for each open U containing x, the set $\{P \in \mathcal{P} : P \not\subseteq U\}$ is finite.

A space X is said to be *Fréchet-Urysohn for finite sets* (respectively, *n-sized sets*), which we will denote by FU_{fin} (respectively, FU_n), if for each $x \in X$ and each $\mathcal{P} \subset [X]^{<\aleph_0}$ (resp, $\mathcal{P} \subset [X]^n$), if $\mathcal{P}$ forms a π-network at x, then $\mathcal{P}$ contains a subfamily that converges to x. We also say that X is *boundedly FU_{fin}* if X is FU_n for every $n \in \omega$.

Though the concept appeared earlier (without being named), Reznichenko and Sipacheva [15] were the first to undertake a detailed investigation of the FU_{fin} property. A primary motivation was the problem due to Malychin whether there could be in ZFC a separable Fréchet topological group which is not metrizable. They showed that if there is a countable FU_{fin} space which is not first countable, then there is such a group. Whether or not there is in ZFC such a group or a countable FU_{fin} space which is not first countable is still an open question. In [6], we continued their investigation of FU_{fin} spaces, and also introduced the related FU_n and boundedly FU_{fin} spaces. Sipacheva's paper [18] contains some characterizations of these properties in terms of Fréchetness of products, and shows that standard Fréchetness in groups and some other spaces with structure often implies some of the stronger Fréchetness with respect to finite set properties.

This paper is a natural continuation of [6]. In Section 2, we consider a class of countable spaces obtained in a natural way from Hausdorff gaps. Motivated by a

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connection with a certain topological game, we had asked in [6] if there is a Fréchet α_2-space which is not FU_{fin}. (See below for the definition of the α_i-spaces.) Here we show that there is a gap space that is a Fréchet α_2 space which is not even FU_2. While all such gap spaces are Fréchet α_2 in ZFC, we show that in fact none are FU_2 under MA_{ω_1}. This answers a question of M. Hrusak, who had asked if there is in ZFC a gap space which is FU_{fin}. On the other hand, we show that the generic gap added by Hechler forcing always produces a FU_{fin} space.

In [6] , we showed that the following "selection principle" versions of the FU_{fin} notion were equivalent to FU_{fin}: whenever $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \dots$ is a sequence of π-nets of finite sets at the point x, there exist $F_n \in \mathcal{P}_n$ for every (respectively, infinitely many) $n \in \omega$ such that the F_n's converge to x.

However, for the other FU-properties, the selection versions are not necessarily the same. Before defining these, we recall the definition of the α_i-spaces of Arhangel'skii [1]:

Let X be a space, and $x \in X$. Suppose that for any countable family $\{A_n\}_{n\in\omega}$ of sequences converging to x, there is a sequence A converging to x such that:

(1) $|A_n \setminus A| < \omega$ for every $n \in \omega$, then x is an α_1-point;
(2) $|A_n \cap A| \neq \emptyset$ for every $n \in \omega$, then x is an α_2-point;
(3) $|A_n \cap A| = \omega$ for infinitely many $n \in \omega$, then x is an α_3-point;
(4) $|A_n \cap A| \neq \emptyset$ for infinitely many $n \in \omega$, then x is an α_4-point.

X is an *α_i-space* if every point is an α_i-point. While these definitions make sense in any space, we only consider them in Fréchet spaces; an α_i-space which is Fréchet is called an *α_i-FU space.* Note that α_2-FU (resp., α_4-FU) spaces are equivalent to the following selection versions of the Fréchet property:

$$x \in \overline{A_n} \text{ for all } n \text{ implies: for all } n \text{ (resp., for } \infty\text{-many } n) \ \exists x_n \in A_n \text{ with } x_n \to x.$$

Now it is natural to make the following definition:

Definition. Given a π-net $\mathcal{P}$ (at x) of finite sets with property $*$, we write:
(a) *FU_** to mean "for every π-net $\mathcal{P}$ with property $*$, there are $P_n \in \mathcal{P}$ with $P_n \to x$;
(b) *(α_4-) α_2-FU_** to mean "if $\mathcal{P}_n$,$n \in \omega$, is a sequence of π-nets having property $*$, then for every (for infinitely many) n in ω, there are $P_n \in \mathcal{P}_n$ with $P_n \to x$.

So, e.g., (α_4-) α_2-FU_5 is: Given π-nets $\mathcal{P}_n$, $n \in \omega$, where each $P \in \mathcal{P}_n$ has cardinality ≤ 5, then for every (for infinitely many) n there are $P_n \in \mathcal{P}_n$ with $P_n \to x$. Also, (α_4-) α_2-boundedly-FU_{fin} is: Given π-nets $\mathcal{P}_n$, $n \in \omega$, where $\{|P| : P \in \mathcal{P}_n\}$ is bounded for each n, then for every (infinitely many) n, there are $P_n \in \mathcal{P}_n$ with $P_n \to x$. Of course, (α_4-) α_2-FU_1 is equivalent to (α_4-) α_2-FU.

In Section 3 we discuss the inter-relationships of these properties, giving examples separating the properties when possible. In several cases, we can give consistent examples but do not know if there may be ZFC examples. In Section 4 the relationships of these properties to a certain game, and to convergence in product spaces and the α_i-properties are given.

2. Gap spaces

An example due to J. Isbell, appearing in [14], produces two countable Fréchet α_2-spaces whose product is not Fréchet. The assumption $2^{\aleph_0} < 2^{\aleph_1}$ is used in [14] in describing Isbell's example, and it is only claimed that the spaces are countably bi-sequential (=Fréchet α_4). But P. Nyikos [13] noticed that the examples are α_2-spaces, and that what is needed to construct the examples is a Hausdorff gap, so they exist in ZFC.

Recall that an ω_1-sequence $(a^0_\alpha, a^1_\alpha : \alpha < \omega_1)$ of pairs of infinite subsets of ω is a *Hausdorff gap* if

(a) $a^0_\alpha \subset^* a^0_\beta \subset^* a^1_\beta \subset^* a^1_\alpha$ for all $\alpha < \beta < \omega_1$;
(b) There is no c such that $a^0_\alpha \subset^* c \subset^* a^1_\alpha$ for all $\alpha < \omega_1$.

(Recall $a \subset^* b$ means $|a \setminus b| < \omega$.)

Given a Hausdorff gap as above, let

$$\mathcal{I}^0 = \{a \subset \omega : |a \cap a^0_\alpha| < \omega \text{ for all } \alpha < \omega_1\},$$
$$\mathcal{I}^1 = \{a \subset \omega : |a \cap (\omega \setminus a^1_\alpha)| < \omega \text{ for all } \alpha < \omega_1\}.$$

Then Nyikos's observation is that the spaces $X_e = \omega \cup \{\infty\}$, where neigborhoods of ∞ are complements of members of the ideal $\mathcal{I}^e$, are the same as Isbell's spaces and are Fréchet α_2. We call a space obtained from (either the left or right side of) a Hausdorff gap g in this way a *gap space*, and the corresponding filter $\mathcal{F}_g$ on ω a *gap filter*. (We will also say that the gap filter has a certain covergence property iff the corresponding gap space does.) It is not hard to show that the product $X_0 \times X_1$ of the above gap spaces is not Fréchet. (See [14] and [13], or Example 2.4 in [5].)

We asked in [6] if there is a Fréchet α_2-space which is not FU_{fin}. (See Section 4 for a game theoretic motivation for this question.) We will show that the Isbell-Nyikos example can be modified to produce, in ZFC, a gap space which is not FU_2.

Example 2.1. *There is a gap space $X = \omega \cup \{\infty\}$ which is Fréchet α_2 but not FU_2.*

Proof. Let X_0 and X_1 be the gap spaces as described above. Let Y_1 be the space X_1 using a disjoint copy ω' of ω, and let X be the space obtained by identifying the points ∞ of X_0 and Y_1. Note that X is also a gap space via the Hausdorff gap $(a^0_\alpha \cup (\omega \setminus a^1_\alpha)', a^1_\alpha \cup (\omega \setminus a^0_\alpha)' : \alpha < \omega_1)$ in $\omega \cup \omega'$. (Here c' denotes the copy in ω' of a subset c of ω.) So X is Fréchet α_2.

Let $\mathcal{P} = \{\{n, n'\} : n \in \omega\}$. Any neighborhood U of ∞ has to almost contain every a^0_α, so there is $\beta < \omega_1$ such that $U \setminus a^1_\beta$ is infinite. Then $(U \setminus a^1_\beta)'$ is convergent. Hence we can find $n \in U \setminus a^1_\beta$ such that $n' \in U$, and so $\{n, n'\} \in [U]^2 \cap \mathcal{P}$. Thus $\mathcal{P}$ is a π-net at ∞.

But no infinite subcollection $\mathcal{C}$ of $\mathcal{P}$ converges to ∞. Suppose otherwise, and consider the set $c = \{n : \langle n, n' \rangle \in \mathcal{C}\}$. Then c and c' are convergent. Now, c' convergent implies $c \cap (\omega \setminus a^1_\alpha)$ is infinite for some α. But then c is not convergent, contradiction. □

In the other direction, noting that it is not difficult to construct gap spaces that are FU_{fin} under CH, M. Hrusak asked if there could be in ZFC a gap space which is FU_{fin}. We will show that the answer is no: under MA_{ω_1}, no gap space is FU_2.

Theorem 2.2. *MA_{ω_1} implies that no gap space is FU_2.*

Proof. Given a gap $\overline{g} = (a_\alpha, b_\alpha : \alpha \in \omega_1)$, let $\mathbb{P}_g$ be the set of pairs (p, F) such that

(1) $p \in Fn(\omega, \omega)$ and p is one-to-one and the domain and range of p are disjoint.
(2) $F \in [\omega_1]^{<\omega}$

The ordering on $\mathbb{P}_g$ is defined by

3. $(p, F) < (q, G)$ if p extends q, $F \supseteq G$, and for each $n \in dom(p) \setminus dom(q)$ and each $\alpha \in G$,
   (a) $n \in a_\alpha$ if and only if $p(n) \notin b_\alpha$.
   (b) $n \in b_\alpha$ if and only if $p(n) \notin a_\alpha$.

It is a standard argument to show that $\mathbb{P}_g$ is σ-centered.

For each $\alpha \in \omega_1$ and each $n \in \omega$, let

$$D_{\alpha,n} = \{(p, F) : \alpha \in F \text{ and } \exists m \in (dom(p) \cap a_{\alpha+1}) \setminus (a_\alpha \cup n)\},$$

and let

$$E_{\alpha,n} = \{(p, F) : \alpha \in F \text{ and } \exists m \in (dom(p \cap b_\alpha) \setminus (b_{\alpha+1} \cup n))\}.$$

Lemma 2.3. *$D_{\alpha,n}$ and $E_{\alpha,n}$ are dense in $\mathbb{P}_g$ for every α and n.*

Proof. We present the proof for $D_{\alpha,n}$. By the symmetry in the definition of the order on $\mathbb{P}_g$, density of $E_{\alpha,n}$ follows.

Fix $(p, F) \in \mathbb{P}_g$. Without loss of generality $\{\alpha, \alpha + 1\} \subseteq F$. Fix N large enough so that $n \cup dom(p) \cup ran(p) \subseteq N$ and $\{a_\xi, b_\xi : \xi \in F\}$ is totally ordered by $\subseteq_N$, where $a \subseteq_N b \iff a \setminus N \subseteq b \setminus N$.

Choose $m \in a_{\alpha+1} \setminus N$ such that $m \notin a_\xi$ for each $\xi \in F \cap \alpha + 1$. And choose $k \in \omega \setminus (b_{\alpha+1} \cup N)$ such that $k \in b_\xi$ for each $\xi \in F \cap (\alpha + 1)$. Then by choice of N we have for every $\xi \in F$

(1) $m \in b_\xi$.
(2) $k \notin a_\xi$.
(3) $m \in a_\xi$ if and only if $\xi > \alpha$
(4) $k \in b_\xi$ if and only if $\xi \leq \alpha$

Thus $(p \cup \{(m, k)\}, F) \in D_{\alpha,n}$ and, by definition of the order on $\mathbb{P}_g$, we have that $(p \cup \{(m, k)\}, F) < (p, F)$. □

If $G \subseteq \mathbb{P}_g$ is $\{D_{\alpha,n}, E_{\alpha,n} : n, \alpha\}$-generic and if $\Gamma : dom(\Gamma) \to ran(\Gamma)$ is the generic function, then both $g_d = g \restriction dom(\Gamma)$ and $g_r = g \restriction ran(\Gamma)$ are pregaps, i.e., satisfy condition (a) of the definition of a gap. Moreover, the function Γ defines an isomorphism between the pregap g_d and the complement of the pregap g_r. Either of these pregaps is a gap if and only if the other is a gap. Indeed, the pregap g_d is filled by a set $x \subseteq dom(\Gamma)$ if and only if g_r if filled by $ran(\Gamma) \setminus \Gamma(x)$. Moreover, if $dom(\Gamma) \cup ran(\Gamma) = \omega$, then both pregaps must in fact be gaps.

Let $D_n = \{(p, F) : n \in dom(p) \cup ran(p)\}$. If we could prove that the sets D_n are dense, then our theorem would easily follow. However, these sets need not be dense, so our proof is a bit more complicated.

The rest of the proof depends on the following property of gaps.

Lemma 2.4. *For every gap $(a_\alpha, b_\alpha : \alpha \in \omega_1)$ there are $\alpha < \beta$ such that both $a_\alpha \setminus b_\beta$ and $a_\beta \setminus b_\alpha$ are not empty.*

Proof. Let

$$C = \{n \in \omega : \exists \alpha_n \in \omega_1 [\forall \alpha \geq \alpha_n (n \in b_\alpha)]\}.$$

Since $C \subset^* b_\alpha$ for all α, $\omega \setminus C$ is infinite.

Claim 1. $\{(a_\alpha \setminus C, b_\alpha \setminus C) : \alpha < \omega_1\}$ is also a Hausdorff gap.

Proof of Claim 1. Suppose $f \subset \omega$ fills this gap. Then $f \cup C$ is easily seen to fill the original gap.

By Claim 1, there is $k \in \omega \setminus C$ such that $A_k = \{\alpha < \omega_1 : k \in a_\alpha\}$ is uncountable. Since $k \notin C$, the set $B_k = \{\beta < \omega_1 : k \notin b_\beta\}$ is also uncountable. Note that for any $\alpha \in A_k$ and $\beta \in B_k$, we have $a_\alpha \setminus b_\beta \neq \emptyset$. So the proof is complete once the following claim is established.

Claim 2. $\exists \alpha \in A_k \exists \beta \in B_k[(\alpha \neq \beta) \wedge (a_\beta \setminus b_\alpha \neq \emptyset)]$.

Proof of Claim 2. Suppose not. Then for each $\alpha \in A_k$ and $\beta \in B_k \setminus \{\alpha\}$, we have $a_\beta \setminus b_\alpha = \emptyset$, i.e., $a_\beta \subset b_\alpha$. Consider the set $A = \bigcup_{\beta \in B_k} a_\beta$. We aim for a contradiction by showing that A fills the gap. Clearly $A\ ^* \supset a_\alpha$ for all α. Now fix $\alpha \in \omega_1$. Then $a_\alpha \subset^* b_\alpha$ and $\bigcup_{\beta \in B_k \setminus \{\alpha\}} a_\beta \subset b_\alpha$, so

$$A \subset a_\alpha \cup \bigcup_{\beta \in B_k \setminus \{\alpha\}} a_\beta \subset^* b_\alpha,$$

which completes the proof.

We now need the following lemma:

Lemma 2.5. *If $G \subseteq \mathbb{P}_g$ is $\mathbb{V}$-generic and $\Gamma : \omega \to \omega$ is the generic function then $\mathbb{P}_g$ forces that both of the following sequences are gaps:*

$$(a_\alpha \cap dom(\Gamma), b_\alpha \cap dom(\Gamma) : \alpha \in \omega_1) \text{ and } (a_\alpha \cap ran(\Gamma), b_\alpha \cap ran(\Gamma) : \alpha \in \omega_1).$$

Proof. They are both pregaps by Lemma 2.3 and the comments after its proof. To see that they are not filled, by symmetry it suffices to prove that the pregrap restricted to the domain of Γ is not filled. Suppose not and let τ be a name for a subset of ω that fills the gap. For each α, fix r_α and n_α such that

$$r_\alpha \Vdash a_\alpha \cap dom(\Gamma) \setminus n_\alpha \subseteq \tau \setminus n_\alpha \subseteq b_\alpha$$

There is an uncountable set $X \subseteq \omega_1$ and a $p \in Fn(\omega, \omega)$ and $n \in \omega$ such that for each $\alpha \in X$, $p_\alpha = (p, F_\alpha)$ and $n_\alpha = n$. Moreover, we may assume $\alpha \in F_\alpha$ for each α and that the F_α form a Δ-system with root F. So $\{p_\alpha : \alpha \in X\}$ is a centred family.

Let N be large enough so that $dom(p) \cup ran(p) \cup n \subseteq N$.

Consider the following gap $(A_\alpha, B_\alpha : \alpha \in X)$ defined as follows: Let

1. $A = \bigcup\{a_\xi : \xi \in F\}$.
2. $B = \bigcap\{b_\xi : \xi \in F\}$.
3. $A_\alpha = \bigcap\{a_\xi : \xi \in F_\alpha \setminus F\} \cap (B \setminus (A \cup N))$.
4. $B_\alpha = \bigcup\{b_\xi : \xi \in F_\alpha \setminus F\} \cap (B \setminus (A \cup N))$.

Clearly, $(A_\alpha, B_\alpha : \alpha \in X)$ is a gap. So by Lemma 2.4, we may fix $\alpha < \beta$ in X, $m \in A_\alpha \setminus B_\beta$, and $k \in A_\beta \setminus B_\alpha$.

Note that m and k satisfy the following for each $\xi \in F_\alpha \cup F_\beta$:

(a) $m \in b_\xi \iff \xi \in F_\alpha$
(b) $m \in a_\xi \iff \xi \in F_\alpha \setminus F$
(c) $k \in b_\xi \iff \xi \in F_\beta$
(d) $k \in a_\xi \iff \xi \in F_\beta \setminus F$.

Fix $\xi \in F_\alpha \cup F_\beta$. Note that if (b) and (c) we have that $m \in a_\xi$ if and only if $k \notin b_\xi$. Also, by (a) and (d) we have that $m \in b_\xi$ if and only if $k \notin a_\xi$.

Thus $r = (p \cup \{(m, k)\}, F_\alpha \cup F_\beta)$ extends both r_α and r_β. Clearly, $\alpha \in F_\alpha$ and $\beta \in F_\beta$ so we have that $m \in a_\alpha \setminus b_\beta$. But we have that

$$r \Vdash m \in dom(\Gamma) \cap a_\alpha \setminus N$$

But since r extends r_α

$$r \Vdash m \in \tau \setminus N$$

and since r extends r_β we have that

$$r \Vdash m \in b_\beta$$

But this contradicts $m \in a_\alpha \setminus b_\beta$. □

Next we need some basic results about indestructibility of (ω_1, ω_1^*) gaps from [7].

Theorem 2.6. *(Kunen) Assume that g is an (ω_1, ω_1^*) pregap, such that $a_\alpha \subseteq b_\alpha$; then*

(e) *g is a gap in every ccc forcing extension if for all $\alpha < \beta$ $a_\alpha \setminus b_\beta \neq \emptyset$.*
(f) *If g is a gap, let $\mathbb{Q}$ be the set of finite subsets s of ω_1 such that $a_\alpha \setminus b_\beta \neq \emptyset$ for each $\alpha < \beta$ in s. Then $\mathbb{Q}$ is ccc.*
(g) *If g is a gap, then there is $\alpha < \omega_1$ so that for each $s \in \mathbb{Q}$ such that $\alpha < min(s)$,*

$$\{\beta > max(s) : s \cup \{\beta\} \in \mathbb{Q}\} \text{ is uncountable.}$$

The poset $\mathbb{Q}$ described in the above theorem makes any gap indestructible. Hence, MA_{ω_1} implies that all (ω_1, ω_1^*) gaps are indestructible.

Now we are ready to apply MA_{ω_1} to prove the theorem:

We apply MA_{ω_1} to the iteration of two ccc-posets: $\mathbb{P}_g * \mathbb{Q}$, where $\mathbb{P}_g$ is the poset described above, and $\mathbb{Q}$ is a $\mathbb{P}_g$-name for the *ccc* poset that makes $g \upharpoonright dom(\Gamma)$ indestructible. We assume that $\mathbb{Q}$ is the subposet that satisfies (g) of Kunen's theorem. Without loss of generality we may assume that for all $(p, q) \in \mathbb{P}_g * \mathbb{Q}$,

(h) there is a finite subset s of ω_1 such that $q = \check{s}$, and
(i) if $p = (r_p, F_p)$, and $\alpha < \beta$ are elements of s, then $dom(r_p) \cap a_\alpha \setminus b_\beta \neq \emptyset$

(Clearly, the set of such conditions is dense). Now we fix a subset $G \subseteq \mathbb{P}_g * \mathbb{Q}$ that is generic for the family of dense sets $D'_{\alpha,n}$, E'_{α_n} and C_α, where

$$C_\alpha = \{(p, q) : \exists \beta > \alpha(\beta \in q)\}.$$

and $D'_{\alpha,n} = \{(p, q) : p \in D_{\alpha,n}\}$ and $E'_{\alpha,n}$ is defined similarly.

If Γ is the generic function defined from the first coordinate, we first verify that $g \upharpoonright dom(\Gamma)$ is a gap. Density of $D'_{\alpha,n}$ and $E'_{\alpha,n}$ imply that it is a pregap. To verify that it is a gap, it suffices to verify that Kunen's condition (e) holds. Let $X = \bigcup\{q \in [\omega_1]^{<\omega} : \exists p(p, q) \in G\}$. Density of the C_α's imply that X is uncountable. It is easy to see that $dom(\Gamma) \cap a_\alpha \setminus b_\beta \neq \emptyset$ for each $\alpha < \beta$ in X. Indeed this follows from item (i) above. So $g \upharpoonright dom(\Gamma)$ and thus $g \upharpoonright ran(\Gamma)$ are both gaps.

Now, consider the family $H = \{\{n, \Gamma(n)\} : n \in dom(\Gamma)\}$.

Claim 1. H is a π-net for the gap filter $\mathcal{F}_g$.

Proof. Suppose X is a subset of ω which is in the filter for the gap. Thus, $a_\alpha \subseteq^* X$ for all α. Since $g \upharpoonright dom(\Gamma)$ is a gap, there are $\alpha < \beta$ such that $Y = X \cap dom(\Gamma) \cap (b_\alpha \setminus b_\beta)$ is infinite. Since $\Gamma(b_\alpha \setminus b_\beta) \subseteq^* a_\beta$, it follows that $\Gamma(Y) \subseteq^* a_\beta$. Thus, since $a_\beta \subseteq^* X$, there is $n \in Y$ such that $\Gamma(n) \in X$. Thus $\{n, \Gamma(n)\} \subseteq X$. So H is a π-net.

Claim 2. For each α, $\{n : \{n, \Gamma(n)\} \subseteq a_\alpha\}$ is finite. Hence, no subset of H converges in $\mathcal{F}_g$.

Proof. Fix $(p, F) \in \mathbb{P}_g$ such that $\alpha \in F$ and $((p, F), r) \in G$. Then for each $n \in dom(\Gamma) \setminus dom(p)$, if $n \in a_\alpha$, then $\Gamma(n) \notin b_\alpha$. Thus, for all $n \in dom(\Gamma) \setminus domp$ we have that $\{n, \Gamma(n)\} \not\subseteq a_\alpha$ as required.

Thus, $\mathcal{F}_g$ is not FU_2, completing the proof of Theorem 2.2. □

Theorem 2.7. *If g is any gap added generically by Hechler's poset Then $\mathcal{F}_g$ is a FU_{fin} filter.*

Proof. Let $I = \omega_1 \times \{0, 1\}$ have the order $<$ making it order isomorphic to $\omega_1 + \omega_1^*$. Recall that Hechler's poset $\mathbb{P}$ is the set of all finite partial functions $p : I \to 2^{<\omega}$ such that

(a) For all α, $(\alpha, 0) \in dom(p)$ iff $(\alpha, 1) \in dom(p)$. For any such α we also require that $p(\alpha, 0)(k) = 1$ implies that $p(\alpha, 1)(k) = 1$.
(b) There is $n \in \omega$ (called the height of p and denoted $ht(p)$) such that $p(i) \in 2^n$ for all $i \in dom(p)$.

The ordering $\leq$ on $\mathbb{P}$ is defined by $p \leq q$ if

(c) $ht(q) \leq ht(p)$
(d) $dom(q) \subseteq dom(p)$
(e) $q(i) \subseteq p(i)$ for all $i \in domq$
(f) For all $i < j$ in the domain of q, $p(i)(k) = 1$ implies that $p(j)(k) = 1$ for every $ht(q) \leq k < ht(p)$.

If G is $\mathbb{P}$-generic over some model M, in $M[G]$ we define sets a_α and b_α as follows: $k \in a_\alpha$ if $p(\alpha, 0)(k) = 1$ for some $p \in G$. And $k \in b_\alpha$ if $p(\alpha, 1)(k) = 1$ for some $p \in G$. It is clear from the definition of the order on $\mathbb{P}$ and by a standard density argument that $(a_\alpha, b_\alpha : \alpha < \omega_1)$ is a pre-gap. The fact that it is a gap is also standard. For now, let a_α and b_α also denote the $\mathbb{P}$-names for these elements.

We claim that the filter $\mathcal{F}_g$ determined by this generic gap is always FU$_{fin}$. To see this, suppose that σ is a $\mathbb{P}$-name for a subset of $[\omega]^{<\omega}$. And suppose that $p \in \mathbb{P}$ and p forces that σ is a π-net at ∞. I.e.,

$$p \Vdash \forall\alpha\forall n\exists y \in \sigma \text{ such that } y \subseteq b_\alpha \setminus n.$$

We now claim that p also forces that there is some α such that for every n some element of σ is a subset of $a_\alpha \setminus n$. I.e., we claim that

$$p \Vdash \exists\alpha\forall n\exists y \in \sigma \text{ such that } y \subseteq a_\alpha \setminus n.$$

Assuming this is the case, it is easy to recursively define in $M[G]$ a sequence $y_n \in \sigma_G$ such that $y_n \subseteq a_\alpha \setminus n$. Then $\{y_n : n \in \omega\}$ is necessarily infinite and converges to ∞. Proving that the filter is FU_{fin}.

So, by way of contradiction assume not. I.e., there is some extension $q \leq p$ such that for all α,

$$q \Vdash \exists n\forall y \in \sigma(y \not\subseteq (a_\alpha \setminus n)).$$

For each α fix $q_\alpha < q$ and $n_\alpha \in \omega$ such that

$$q_\alpha \Vdash \forall y \in \sigma(y \not\subseteq (a_\alpha \setminus \check{n}_\alpha)).$$

Without loss of generality we may assume that $\alpha \in dom(q_\alpha)$ for each α.

$\mathbb{P}$ is *ccc*, so without loss of generality there is a $\beta < \omega_1$ such that for every q that appears in the name σ, $dom(q) \subseteq \beta \times \{0, 1\}$. In particular, if for some $k \in [\omega]^{<\omega}$ and some $q \in \mathbb{P}$ we have that $q \Vdash k \in \sigma$, then $q \restriction (dom(q) \cap \beta \times \{0, 1\})$ also forces $k \in \sigma$. Henceforth, we will abbreviate restrictions of conditions like $q \restriction (dom(q) \cap \beta \times \{0, 1\})$ as just $q \restriction \beta$.

Now, choose $\alpha > \beta$. Without loss of generality assume that $n_\alpha = 0$. Let $n > ht(q_\alpha)$. Since $q_\alpha < p$, we have that

$$q_\alpha \Vdash \exists y \in \sigma \text{ such that } y \subseteq b_\alpha \setminus n.$$

Thus we may fix $r \leq q_\alpha$ and a $y \in [\omega \setminus n]^{<\omega}$ such that

$$r \Vdash \check{y} \in \sigma \text{ and } \check{y} \subseteq b_\alpha$$

In particular r forces that y is an element of σ. Thus, $r' = r \restriction \beta$ also forces that $y \in \sigma$. Note that the height of r and r' are the same and is greater than the maximum of y which of course is greater than n, the height of q_α.

We also have that $r' \leq q_\alpha \restriction \beta$ and that r' and q_α are compatible (r is a common extension). We now define a possibly different common extension q. We require that $dom(q) = dom(r') \cup dom(q_\alpha)$ and that $m = ht(q)$ is equal to the height of r'. We define each $q(i)$ as follows:

(g) For $i \in dom(r')$ let $q(i) = r'(i)$.

Let i_0 the the $<$-largest element of $dom(q_\alpha)$ below $(\beta, 0)$. Thus, i_0 is also in the domain of r' and so $q(i_0)$ is defined by item (g).

(h) If $i \in dom(q_\alpha) \setminus dom(r')$ then $(\beta, 0) < i < (\beta, 1)$ and we let $q(i) \restriction n = q_\alpha(i)$.
(i) For $i \in dom(q_\alpha) \setminus dom(r')$ and for $k \in m \setminus n$ let $q(i)(k) = 1$ if either $k \in y$ or $q(i_0)(k) = 1$.

Items (g), (h) and (i) completely define the condition q.

Claim 1: $q \Vdash \check{y} \subseteq a_\alpha \setminus n_\alpha$

Proof. Fix $k \in y$. Since $(\alpha, 0) \in dom(q_\alpha) \setminus dom(r')$, $q(\alpha, 0)(k)$ is defined to be 1 by clause (i) in the definition of q. Thus for each $k \in y$ we have that $q \Vdash k \in a_\alpha$.

Claim 2: $q \leq q_\alpha$

Proof. Certainly (c), (d) and (e) in the definition of $\leq$ are satisfied. To verify (f) we need to fix $i < j$ in the domain of q_α and fix k in the interval $[ht(q_\alpha), ht(q))$.We need to check that $q(i)(k) = 1 \Rightarrow q(j)(k) = 1$. Assume that $q(i)(k) = 1$.

(1) If both $i, j \in dom(r')$. Then by clause (g) in the definition of q and since $r' < q_\alpha \restriction \beta$ we have that $q(j)(k) = 1$.
(2) Both $i, j \in dom(q_\alpha) \setminus dom(r')$. Then $i_0 < i$. So either $k \in y$, or $q(i_0)(k) = 1$. In either case $q(j)(k) = 1$.
(3) If $i \leq i_0$ and $(\beta, 0) < j < (\beta, 1)$. Then $q(i)(k) = 1$ implies that $q(i_0)(k) = 1$ since $r' < q_\alpha \restriction \beta$. Thus $q(j)(k) = 1$ by (i).
(4) If $(\beta, 0) < i < (\beta, 1) < j$. In the case that $q(i_0)(k) = 1$, note that $r'(i_0) = q(i_0)$ and that $j \in dom(r')$. Thus $r'(j) = q(j)$. Also, $r'(i_0)(k) = 1$. Thus, the fact that $r' < q_\alpha \restriction \beta$ allows us to to conclude that $r'(j)(k) = 1$. Thus $q(j)(k) = 1$ as required.
In the case that $k \in y$, we need to use the facts that $(\alpha, 1) < j$ and that $r \Vdash \check{y} \subseteq b_\alpha$. Thus, since $k \in y$ we have that $r(\alpha, 1)(k) = 1$. Also, since both $(\alpha, 1)$ and j are in the domain of q_α and $r < q_\alpha$ we may conclude that $r(j)(k) = 1$. But $q(j) = r'(j) = r(j)$ since $j \in \beta \times \{0, 1\}$. Thus $q(j)(k) = 1$ as required.

Claim 3: $q \leq r'$

Proof. Trivial since $q \restriction dom(r') = r'$.

We now finish the proof of the theorem: by Claim 3, we have that $q \Vdash y \in \sigma$. So Claims 1 and 2 give us a contradiction since q_α forces there is no such y in σ. □

3. Selection principle versions

The table below shows the relationships among the properties we are considering (see introduction for the definitions).

$$
\begin{array}{l}
\mathrm{FU}_{fin} \iff \alpha_2\text{-}\mathrm{FU}_{fin} \iff \alpha_4\text{-}\mathrm{FU}_{fin} \\
\quad \Downarrow^1 \\
\alpha_2\text{-bddly-}\mathrm{FU}_{fin} \iff \alpha_4\text{-bddly-}\mathrm{FU}_{fin} \\
\quad \Downarrow^2 \\
\forall n(\alpha_2\text{-}\mathrm{FU}_n) \Rightarrow^3 \text{bddly-}\mathrm{FU}_{fin} = \forall n(\mathrm{FU}_n) \iff \forall n(a_4 - \mathrm{FU}_n) \\
\qquad\quad \Downarrow^4 \qquad\qquad \Downarrow^5 \\
\qquad \alpha_2\text{-}\mathrm{FU}_n \qquad \mathrm{FU}_{n+1} \\
\qquad\qquad \Downarrow^6 \qquad \Downarrow^7 \\
\qquad\qquad \alpha_4\text{-}\mathrm{FU}_n \\
\qquad\qquad\quad \Downarrow^8 \\
\qquad\qquad\quad \mathrm{FU}_n
\end{array}
$$

Of course, first-countability implies FU_{fin} and hence all of these properties. Sipacheva [18] noted that X is FU_n at x iff X^n is Fréchet at $(x, x, ..., x)$. It follows that the class bisequential spaces, being Fréchet and countably productive [8], are boundedly FU_{fin}. Bisequential also implies α_3-FU [2].

All but one of the implications in the chart are trivial or proven in [6]. The exception is the equivalence of α_2-boundedly FU_{fin} and α_4-boundedly FU_{fin}.

Theorem 3.1. *A space X is α_2-boundedly FU_{fin} iff it is α_4-boundedly FU_{fin}.*

Proof. Since the other direction is trivial, we need only show that if X is α_4-boundedly FU_{fin}, then it is is α_2-boundedly FU_{fin}. Suppose $\mathcal{P}_n$, $n \in \omega$, is a sequence of π-nets at $x \in X$, such that each $\mathcal{P}_n$ consists of sets bounded in cardinality by some $k_n \in \omega$. Let $\mathcal{P}_n^* = \{\cup_{i \le n} P_i : P_i \in \mathcal{P}_i\}$. It is easy to check that each $\mathcal{P}_n^*$ is a π-net at x consisting of sets bounded in cardinality by $\Sigma_{i \le n} k_i$. By the α_4-boundedly FU_{fin} property, there are $P_n \in \mathcal{P}_n^*$ for each n in some infinite set $A \subset \omega$ such that $P_n \to x$. Let $P_n = \cup_{i \le n} P_{ni}$. If $\{n_0, n_1, \dots\}$ is the increasing enumeration of A, then $Q_m = P_{n_m m} \in \mathcal{P}_m$ for each m, and $Q_m \to x$. □

Next we present a series of examples showing that implications in the chart need not reverse. Example 3.12 below shows in ZFC that implication 3 does not reverse. All of the other examples are consistent ones, usually constructed using CH. Example 3.5 gets implication 8, Example 3.6 gets 6 and 7, Example 3.11 gets 1, and Example 3.9 gets 2. Also, Example 3.8 shows α_2-FU_n need not imply FU_{n+1}.

Corollary 3.4 below shows that implication 8 consistently reverses. On the other hand, not only do we not know ZFC examples showing 1,2,4,6,7 do not reverse, but for all we know, substantial portions of the chart could collapse in some models.

Question 1. *Is there in ZFC an FU_2-space, or even an α_2-FU_2-space, which is not boundedly FU_{fin}? A α_2-FU_2 space which is not FU_{fin}?*

The reader may notice that all of our examples are of one of two types. Most are built from almost-disjoint families on a countable set; in fact the only exception is Example 3.8, which is a gap space as defined in the previous section.[1] It is relevant to note that the examples from almost disjoint families also yield compact examples in a natural way. Given an almost disjoint family $\mathcal{A}$ of subsets of (say) ω, we associate a corresponding space $\omega \cup \{\infty\}$, where ω is the set of isolated points and the complements of the members of $\mathcal{A}$ form a local subbase at ∞. This is essentially equivalent to the following. Consider the locally compact space $\omega \cup \{x_A : A \in \mathcal{A}\}$, where neighborhoods of x_A have the form $\{x_A\} \cup (A \setminus F)$ for some finite $F \subset A$. This is called the ψ-space corresponding to $\mathcal{A}$. If we denote it by $\psi(\mathcal{A})$, and its one point compactification by $\psi(\mathcal{A})^*$, the topology we just defined on $\omega \cup \{\infty\}$ is the same as its subspace topology in $\psi(\mathcal{A})^*$, where ∞ is the compactifying point. Indeed, it is easy to check that this subspace will have one of the convergence properties we are considering iff the whole space $\psi(\mathcal{A})^*$ does.

It follows from results of Arhangel'skii and Nogura that every compact α_2-FU space is boundedly FU_{fin} (see Proposition 3.7). But the other parts of Question 1 are unsolved even in the realm of compact spaces (including the special case of spaces generated from almost-disjoint families).

We also point out that the following special kind of almost-disjoint family always yields a bisequential, hence boundedly FU_{fin}, space. For each x in some subset Y of a compact first-countable space K, choose a sequence S_x of points of K converging to x. Then the space obtained as above from the almost-disjoint family $\{S_x : x \in Y\}$ (where the set of isolated points is $S = \bigcup_{x\in Y} S_x$) is bisequential. This seems to be a folklore result that we neglected to observe in [6]. For the sake of completeness, we give its easy proof here.

Proposition 3.2. *Let K be compact first-countable, and $Y \subset K$. Suppose for each $y \in Y$, we have chosen a sequence S_y of points of K converging to y. Let $S = \bigcup_{y\in Y} S_y$, and let $X = S \cup \{\infty\}$, where the points of S are isolated and a neighborhood of ∞ has the form $\{\infty\} \cup [S \setminus (F \cup \bigcup_{y\in G} S_y)]$ for some finite $F \subset S$ and finite $G \subset Y$. Then X is bisequential (and hence α_3-FU and boundedly FU_n).*

Proof. Recall that a space is bisequential at p iff for every ultrafilter $\mathcal{F}$ which clusters at p, there is a sequence $A_0, A_1, \dots$ of members of $\mathcal{F}$ converging to p. Then suppose $\mathcal{F}$ is an ultrafilter clustering at the point ∞ in the space X. We may view $\mathcal{F}$ as an ultrafilter on S. $\mathcal{F}$ converges in the compact space K to a unique point p. Let U_n, $n \in \omega$, be a decreasing neighborhood base at p. Let $S'_p = S_p$ if $p \in Y$, else let $S'_p = \emptyset$. Then it is easy to check that $S \cap (U_n \setminus S'_p)$, $n \in \omega$, is a decreasing sequence of members of $\mathcal{F}$ converging to ∞ in X. □

There is one case, namely implication 8 in the chart for $n \geq 2$, where we know it is both consistent with and independent of ZFC that the implication reverses. The consistent reversal follows from the following result of Todorcevic [19], which answered a question of Nogura [10].

Theorem 3.3. *(OCA)If $X \times Y$ is Fréchet, then $X \times Y$ is α_4.*

Recall that the Open Coloring Axiom, OCA, is a consequence of PFA.

---

[1] This is also the only one of our CH examples for which $\mathfrak{p} = \mathfrak{c}$ would not suffice (by Theorem 2.2).

Corollary 3.4. *(OCA) If $n \geq 2$, then FU_n and α_4-FU_n are equivalent.*

Proof. Suppose X is FU_n at x, $n \geq 2$. W.l.o.g., every point of X except x is isolated. Then $X^n = X^{n-1} \times X$ is Fréchet by Sipacheva's result mentioned earlier (see also the next section where relationships to products are discussed) , so by Todorcevic's theorem, X^n is α_4. By Theorem 4.4 in the next section, X^n α_4-FU is equivalent to X being α_4-FU_n. ☐

The Fréchet fan is a FU_1, not α_4-FU_1 space in ZFC. Simon [17] constructed under CH countable spaces X and Y with one non-isolated point such that $X \times Y$ is Fréchet but not α_4, which shows that Todorcevic's theorem fails under CH. Here we construct for every $n \geq 2$ a Fréchet FU_n space which is not α_4-FU_n (under CH, but $\mathfrak{p} = \mathfrak{c}$ is enough).

Example 3.5. *(CH) A FU_n, not α_4-FU_n space.*

Proof. Let $n > 1$, and let $Y = n \times \omega \times \omega$. Let $d_{j,k} = n \times \{j\} \times \{k\}$, $D_k = \{d_{j,k} : j \in \omega\}$, and $D = \bigcup_{k\in\omega} D_k$. We are going to make each D_k a π-net (in fact a convergent sequence) of n-element sets, but no selection of one member of each of infinitely many D_k's is going to be convergent. Of course, to make the space FU_n, this means that no subset of D meeting each D_k in a finite set can be a π-net.

Let $\mathcal{P}_\alpha$ and E_α, $\alpha < \omega_1$, list $[[Y]^n \setminus D]]^\omega$ and $\{E \in [D]^\omega : \forall k(|E \cap D_k| < \omega)\}$, respectively. For $S \subset Y$, let $D(S) = \cup\{d \in D : d \cap S \neq \emptyset\}$. We inductively define I_α, $\alpha < \omega_1$, the complements of which will be a subbase for ∞ in X. Let $\mathcal{T}_\alpha$ be the topology on X such that $\{X \setminus \bigcup_{\beta\in F} I_\beta : F \in [\alpha]^{<\omega}\}$ is a base at ∞. We want the following conditions to be satisfied:

(1) $I_\alpha \cap (\cup D_n)$ is finite for all n;
(2) If $\beta \leq \alpha$ and $\mathcal{P}_\beta$ is a π-net at ∞ in $\mathcal{T}_\beta$, then there are $P^\beta_k \in \mathcal{P}_\beta$ such that, if $P^\beta = \bigcup_{k\in\omega} P^\beta_k$, then $|P^\beta \cap I_\gamma| < \omega$ for each $\gamma \leq \alpha$;
(3) $P^\alpha_k = P^\alpha_{k,0} \cup P^\alpha_{k,1}$, and the following holds: Let $P^\alpha_e = \bigcup_{k\in\omega} P^\alpha_{k,e}$. There are $j_\alpha \in \omega$ and a finite $F \subset \alpha$ such that (i) $P^\alpha_0 \subset [\bigcup_{i\leq j_\alpha} \cup D_i] \cup [\bigcup_{\beta\in F} P^\beta]$, (ii) $|P^\alpha_1 \cap P^\beta| < \omega$ for each $\beta < \alpha$, (iii) for each $j \in \omega$, $|\{k : P^\alpha_{k,1} \cap \cup D_j \neq \emptyset\}| \leq 1$, and (iv) $D(P^\alpha_1) \setminus P^\alpha_1 \subset I_\alpha$;
(4) If E_α is a π-net in $\mathcal{T}_\alpha$, then there is some finite $F \subset \alpha + 1$ such that $d \in E_\alpha \Rightarrow d \cap (\bigcup_{\beta\in F} I_\beta) \neq \emptyset$.

First let's see that if everything is constructed according to the above conditions, the resulting space has the desired properties. Clearly (1) gives that each $\cup D_k$ is convergent, and hence D_k is a π-net. Condition (4) shows that no choice of one member of D_k for infinitely many k will be convergent, for otherwise the set of choices would appear as some E_α and would be a π-net, but by (4) E_α gets destroyed as a π-net at step α. So X will not be α_4-FU_n. Finally, we need to check that X is FU_n. Suppose P is a π-net of n-element sets. Then either $P \cap D$ or $P \setminus D$ is too. The latter case is taken care of by condition (2) for some β where $P \setminus D = P_\beta$. In the former case, it follows from (4) that $P \cap D_k$ is infinite for some k, and this would be a convergent sequence.

Suppose we have defined everything satisfying the above conditions up to α. If $\mathcal{P}_\alpha$ is a π-net in the topology $\mathcal{T}_\alpha$, find P^α_k such that $|I_\gamma \cap (\bigcup_{k\in\omega} P^\alpha_k)| < \omega$ for each $\gamma < \alpha$. To get (3), we thin out as follows. If there are $x_k \in P^\alpha_k \cap S$ for infinitely many k, where $S = \cup D_j$ for some j or $S = P^\beta$ for some $\beta < \alpha$, then pass to that infinite subsequence. Then if there are $y_k \in P^\alpha_k \setminus \{x_k\} \cap S$ for infinitely many k and some S as before, do it again. Continue until there is no longer such an infinite selection. Since $|P^\alpha_k| = n$, this will occur in $\leq n$ steps. The set of points selected from the P^α_k's is $P^\alpha_{k,0}$, and $P^\alpha_{k,1} = P^\alpha_k \setminus P^\alpha_{k,0}$. Then it is easy to see that (3)(i) and (3)(ii) are satisfied, and that we may thin out again if necessary to get (3)(iii).

Let's see that making sure (3)(iv) holds does not destroy the convergence of the previous P_β's. Let $I_{\alpha,0} = D(P^\alpha_1) \setminus P^\alpha_1$, and suppose $I_{\alpha,0} \cap P^\beta$ is infinite for some $\beta < \alpha$. Assume β is the least such; then by (3)(i) and (3)(iii), it must be that $I_{\alpha,0} \cap P^\beta_1$ is infinite. But then $D(P^\beta_1) \setminus P^\beta_1 \cap P^\alpha_1$ is an infinite subset of $I_\beta \cap P^\alpha_1$, contradicting P^α convergent in $\mathcal{T}_\alpha$.

Now look at E_α. If there is a finite $F \subset \alpha$ such that $d \in E_\alpha \Rightarrow d \cap [I_{\alpha,0} \cup (\bigcup_{\beta\in F} I_\beta)] \neq \emptyset$, then simply let $I_\alpha = I\alpha, 0$ and all conditions will be satisfied. Otherwise, let $\{J_k\}_{k\in\omega}$ list $\{I_\beta\}_{\beta<\alpha} \cup \{I_{\alpha,0}\}$ and let $\{S_k\}_{k\in\omega}$ list $\{P^\beta : \beta \leq \alpha\} \cup \{\cup D_k : k \in \omega\}$. Then inductively construct $J'_k \subset J_k$ such that $|J_k \setminus J'_k| < \omega$ and $J'_k \cap \bigcup_{i\leq k} S_i = \emptyset$. Then we let $I_{\alpha,1} = I_{\alpha,0} \cup (\bigcup_{k\in\omega} J'_k$, and $I_{\alpha,2} = \cup\{d \in E_\alpha : d \cap I_{\alpha,1} = \emptyset\}$. Finally, let $I_\alpha = I_{\alpha,0} \cup I_{\alpha,1} \cup I_{\alpha,2}$. This satisfies condition (4) with $F = \{\alpha\}$.

Conditions (1) and (3) are clear, so it remains to check condition (2). We need to see that $P^\beta \cap I_\alpha$ is finite for all $\beta \leq \alpha$. We already saw that $I_{\alpha,0}$ does not ruin this condition, and by construction neither does $I_{\alpha,1}$. So we want to show that $I_{\alpha,2} \cap P^\beta$ is finite for any $\beta \leq \alpha$. Suppose by way of contradiction that β is least such that this set is infinite. Then $I_{\alpha,2} \cap P^\beta_1$ is infinite, and hence $I_{\alpha,2} \cap I_\beta$ is infinite. But then $I_{\alpha,2} \cap J'_k$ is infinite for some k, contradicting $I_{\alpha,2} \cap I_{\alpha,1} = \emptyset$. □

Example 3.6. *(CH) A α_4-FU_n-space which is not FU_{n+1} or α_2-FU_n (or even α_2).*

Proof. This construction is with very minor modifications the same as the construction of Example 16 in [6] of a FU_n not FU_{n+1}-space. So here we will only indicate the necessary changes.

In our construction in [6], all potential π-nets of n-sized sets are listed as T_α, $\alpha < \omega_1$, and at stage α, a certain subset S_α of T_α is chosen so that S_α will be convergent if T_α happens to be a π-net. If we instead let the T_α's index all potential *sequences* of π-nets, and choose S_α to be a diagonalizing sequence through infinitely many terms of T_α, the same proof goes through easily. □

Why is it that the same construction as in the previous example is not adaptable to obtain our next example, an α_2-FU_n-space which is not FU_{n+1}? For one thing, it is important in the above proof to be able to thin out at will a preliminary choice S for S_α, which α_4-FU_n allows. But in fact it *cannot* be α_2-FU_n by the neighborhood structure, which is generated by complements of an almost disjoint family. Indeed, combining results of Arhangel'skii and Nogura, it follows easily that, more generally, compact α_3-spaces are boundedly FU_{fin}. Since the direct argument may be more illuminating, we give that as well.

Proposition 3.7. *Every compact α_3-FU space X is boundedly FU_{fin}.*

Proof 1. Arhangel'skii [2] showed that any α_3-FU space times a countably compact Fréchet space is Fréchet, and Nogura [11] showed that the α_3-property (as well as the α_1 and α_2 properties, but not the α_4 property) is countably productive.

Hence an easy induction shows X^n is Fréchet (and α_3) for all n, which implies boundedly FU_{fin}.

Proof 2. Suppose X is not boundedly FU_{fin} at point q. Then there is a least integer k such that there exists a π-net $\mathcal{P}$ consisting of $k+1$ sized sets $P = \{x_i^P : i \le k\}$ with no convergent subsequence. Let P^- denote $\{x_i^P : i < k\}$.

Let Q be the set of all limit points of convergent sequences of the form $\{x_k^{P_n}\}_{n\in\omega}$, where $P_n \in \mathcal{P}$ and $P_n^- \to q$. It follows easily from Fréchetness of X and minimality of k that $q \in \overline{Q}$. Choose $q_m \in Q$ with $q_m \to q$.

For each $m \in \omega$, choose $P_{mn} \in \mathcal{P}$ such that

$$P_{mn}^- \to q \text{ and } x_k^{P_{mn}} \to q_m \text{ as } n \to \infty.$$

By applying the α_3 property k times, it follows that there are an infinite $B \subset \omega$ and infinite $C_m \subset \omega$ for each $m \in B$ such that

$$\cup\{P_{mi}^- : m \in B \text{ and } i \in C_m\}$$

converges to q. (Applying α_3 to the $x_0^{P_{mn}}$'s, we see that we can pass to infinite subsequences of infinitely many of the sequences P_{mn}^-, $n \in \omega$, such that the set of all first terms of these P's converges to q; restrict to these P's and apply α_3 to their second terms x_1^P, etc. After k steps we have our claimed B and C_m's.)

Now, since q is in the closure of $\{x_k^{P_{mi}} : m \in B, i \in C_m\}$, there are $m_0, m_1, \dots$ and $i_0, i_1, \dots$ such that $x_k^{P_{m_j i_j}}$ converges to q as $j \to \infty$. It follows that $P_{m_j i_j} \to q$, contradiction. □

Thus we need a different type of example to show that α_2-FU_n need not imply FU_{n+1}. It turns out that gap spaces work.

Example 3.8. *(CH) A α_2-FU_n space which is not FU_{n+1}.*

Proof. We construct a gap $(a_\alpha, b_\alpha : \alpha < \omega_1)$ on $\omega \times (n+1)$ in such a way that the family

$$\{\{m\} \times (n+1)\} : m \in \omega\}$$

is a π-net but no infinite subset converges in the corresponding gap space $X = [\omega \times (n+1)] \cup \{\infty\}$.

Let $\{x_\alpha : \alpha < \omega_1\}$ be an enumeration of $[[\omega \times n+1]^{\le n}]^\omega$. By recursion we construct $\{a_\alpha : \alpha < \omega_1\}$ and $\{b_\alpha : \alpha < \omega_1\}$ such that

1. $a_\alpha, b_\alpha \subseteq \omega \times n+1$ for all $\alpha < \omega_1$,
2. $a_\alpha \subseteq^* a_\beta \subseteq b_\beta \subseteq^* b_\alpha$ for all $\alpha < \beta$,
3. $|a_{\alpha+1} \setminus a_\alpha| = |b_\alpha \setminus b_{\alpha+1}| = \aleph_0$ for all $\alpha < \omega_1$.
4. $\{m\} \times (n+1) \not\subseteq a_\alpha$ for all $m \in \omega$ and all $\alpha < \omega_1$.
5. For each $\alpha < \omega_1$ there is an infinite $b'_\alpha \subseteq \omega$ such that $b_\alpha = a_\alpha \cup (b'_\alpha \times (n+1))$ and $a_\alpha \cap (b'_\alpha \times (n+1)) = \emptyset$.
6. If $\{s \in x_\alpha : s \subseteq a_\alpha \setminus (m \times n+1)\} = \emptyset$ for some m, then either
   (a) for all m, $\{s \in x_\alpha : s \subseteq a_{\alpha+1} \setminus (m \times n+1)\}$ is infinite, or
   (b) there is an m such that $\{s \in x_\alpha : s \subseteq b_{\alpha+1} \setminus (m \times n+1)\} = \emptyset$.

Hypothesis 6 assures two things: Since all infinite subsets of $\omega \times n+1$ are included in the enumeration of the x_α's, we have that $\{b_\alpha : \alpha < \omega_1\}$ generates the gap filter $\mathcal{F}_g$. Also, if, after the construction, any x_α is a π-net at ∞, then there is an infinite subset that converges. Indeed, if x_α is a π-net, then by construction $\{s \in x_\alpha : s \subseteq a_{\alpha+1} \setminus (m \times n+1)\}$ is infinite for every m. So, we can easily extract a convergent sequence. Hence the gap space X is FU_n. As noted in the previous section, every gap space is α_2, and by Theorem 4.2 in the next section, α_2 and FU_n together is equivalent to α_2-FU_n.

Hypotheses 4 and 5 assure that $\{\{m\} \times n+1 : m \in \omega\}$ is a π-net with no convergent subset. So X is not FU_{n+1}.

It suffices then to show how to carry out the construction. Suppose that $\gamma < \omega_1$ and that $\{a_\beta : \beta < \gamma\}$ and $\{b_\beta : \beta < \gamma\}$ have been fixed so that the inductive hypotheses are satisfied.

Case 1. γ is a successor. Let β be such that $\gamma = \beta + 1$. First suppose that $\{s \in x_\gamma : s \subseteq a_\beta \setminus (m \times n+1)\}$ is infinite for every m. Then we need not worry about hypothesis 6. To define a_γ and b_γ, partition $b'_\beta = B_0 \cup B_1$ into infinite pairwise disjoint sets. Let $a_\gamma = a_\beta \cup B_0 \times \{0\}$ and let $b_\gamma = a_\gamma \cup (B_1 \times n+1)$. It is easy to see that the hypotheses 1-6 hold for $\{a_\beta, b_\beta : \beta \leq \gamma\}$.

In the case that $\{s \in x_\gamma : s \subseteq a_\beta \setminus (m \times n+1)\} = \emptyset$ for some m, the construction is similar: First note that we may assume that $\{s \in x_\gamma : s \subseteq b_\beta \setminus (m \times n+1)\}$ is infinite for all m (if not, the previous construction may be used and hypothesis 6(b) is satisfied).

By our assumption, we may recursively define an increasing sequence of $k_m \in \omega$ and $s_m \in x_\gamma$ such that for each m

$$s_m \subseteq b_\beta \cap ((k_{m+1} \setminus k_m) \times (n+1)) \text{ and } s_m \cap (b'_\beta \times (n+1)) \neq \emptyset.$$

For each m let $s'_m = s_m \cap (b'_\beta \times (n+1))$. If we now let

$$a_\gamma = a_\beta \cup \bigcup \{s'_m : m \text{ even}\} \text{ and } b'_\gamma = \bigcup \{b'_\beta \cap (k_{m+1} \setminus k_m) : m \text{ odd }\}.$$

Then for each even m we have that $s_m \subseteq a_\gamma \setminus (k_m \times n+1)$. And the inductive hypotheses are easily seen to be satisfied. In particular, 6 is satisfied by item 6(a).

Case 2. γ is a limit. In this case we have nothing to do to preserve hypothesis 3 and 6. However, preserving the other hypotheses requires a little work. Choose $\{\gamma_j : j \in \omega\}$ increasing and cofinal in γ. Let b'_γ be a pseudo-intersection of the b'_{γ_j}. Thus $b'_\gamma \times (n+1) \cap a_\beta$ is finite for every $\beta < \gamma$.

Recursively define a_j as follows. Let $a_0 = a_{\gamma_0}$. Having defined $a_j =^* a_{\gamma_j}$, choose k_j large enough so that $a_j \setminus (k_j \times 3) \subseteq a_{\gamma_{j+1}}$. Let

$$a_{j+1} = a_j \cup (a_{\gamma_{j+1}} \setminus (k_j \times (n+1))).$$

Let $a_\gamma = \bigcup a_j$. The main property to note is that $\{m\} \times (n+1) \not\subseteq a_\gamma$ for every $m \in \omega$. To see this, suppose by way of contradiction that $m \times (n+1) \subseteq a_\gamma$. Let j be minimal such that $m \times (n+1) \subseteq a_j$. Then $j \neq 0$. So

$$a_j = a_{j-1} \cup a_{\gamma_j} \setminus (k_{j-1} \times n+1).$$

Thus, $m > k_{j-1}$. It follows that $m \times n+1 \subseteq a_{\gamma_j}$ since $a_{j-1} \setminus (k_{j-1} \times n+1) \subseteq a_{\gamma_j}$. But this contradicts our inductive hypothesis 4 for a_{γ_j}.

By going to a subset of a_γ we preserve hypothesis 4 for a_γ so we may assume that

$$a_\gamma \subseteq^* b_\beta \text{ for every } \beta < \gamma, \text{ and } a_\gamma \cap (b'_\gamma \times n+1) = \emptyset.$$

We let $b_\gamma = a_\gamma \cup (b'_\gamma \times n+1)$. Clearly the rest of the hypotheses are satisfied for all $\alpha \leq \gamma$.

This completes the construction. □

Recall that a_2- and α_4-boundedly FU_{fin} are equivalent. The following example shows that boundedly FU_{fin} does not (consistently) imply α_2-boundedly FU_{fin} (even when it is α_2).

Example 3.9. *(CH) A boundedly FU_{fin} space which is α_2 (and hence α_2-FU_n for each n) but is not α_4-boundedly FU_{fin}.*

Proof. This example is a modification of the example of Theorem 4 of [6]. Let $\mathbb{Q}$ denote the rationals in the unit interval $I = [0,1]$. Our space X will be $\mathbb{Q} \cup \{\infty\}$, where points of $\mathbb{Q}$ are isolated, and the neighborhood filter of ∞ will be generated by complements of finite subsets of $\mathbb{Q}$, together with complements of certain sequences S_x of rationals converging to x, for some points $x \in I$. We will choose at most one S_x for each x; by Proposition 3.2, this will guarantee the space is bisequential, hence boundedly FU_{fin}. We carry out an inductive construction to make sure it is α_2 but not α_2-boundedly FU_{fin}.

Let $\mathcal{H} = \{H_{nm} : n, m \in \omega\}$ be a pairwise disjoint collection of subsets of $\mathbb{Q}$ such that for each n, $|H_{nm}| = n+1$, $\{H_{nm} : m \in \omega\} \to q_n$, and $diam(\bigcup_{m\in\omega} H_{nm}) < 1/2^n$. We will make sure that, for each n, each S_x meets only finitely many members of $\mathcal{H}_n = \{H_{nm} : m \in \omega\}$, and hence that each $\mathcal{H}_n$ is a π-net.

Let $\mathcal{X}_\alpha = \{X_{\alpha i}\}_{i\in\omega}$ and g_α, for $\alpha < \omega_1$, list all sequences of infinite subsets of $\mathbb{Q}$ and all $g : \omega \to \omega$, respectively. Suppose at stage α we have chosen, for each $\beta < \alpha$, subsets S_β and T_β of $\mathbb{Q}$ satisfying:

(1) $S_\beta \to x_\beta$ in $[0,1]$;
(2) $x_\beta \neq x_\gamma$ if $\beta \neq \gamma < \alpha$;
(3) If $\gamma < \alpha$ and $X_{\gamma i} \cap S_\beta$ is finite for all $\beta < \gamma$ and $i \in \omega$, then for all $i \in \omega$, $T_\gamma \cap X_{\gamma i} \neq \emptyset$;
(4) $\forall \beta, \gamma \in \alpha$ ($S_\beta \cap T_\gamma$ is finite);
(5) For each $n \in \omega$ and $\beta < \alpha$, S_β meets only finitely many members of $\mathcal{H}_n$;
(6) For infinitely many $n \in \omega$, $S_\beta \cap H_{ng_\beta(n)} \neq \emptyset$.

First, suppose we can carry out the indicated construction. Then clearly condition (5) ensures that each $\mathcal{H}_n$ is a π-net, and (6) ensures that no selection of a member of each $\mathcal{H}_n$ converges to ∞; hence X is not α_2-boundedly FU_{fin}. On the other hand, conditions (3) and (4) ensure that X is α_2.

So it remains to check that the inductive construction can be done. If $X_{\alpha i} \cap S_\beta$ is infinite for some $\beta < \alpha$, let $T_\alpha = \emptyset$. Otherwise, let S'_n, $n \in \omega$, index $\{S_\beta\}_{\beta<\alpha}$. Since $\mathcal{H}$ is a pairwise-disjoint collection, it is easy to see that we may choose points $t_{\alpha n} \in X_{\alpha n} \setminus \bigcup_{i\leq n} S'_i$ such that no member of $\mathcal{H}$ contains more than one $t_{\alpha n}$. Let $T_\alpha = \{t_{\alpha n}\}_{n\in\omega}$. This gets conditions (3) and (4) with $\gamma = \alpha$. Now we choose S_α so that (1)-(6) hold with $\beta = \alpha$. Let T'_n, $n \in \omega$, index $\{T_\beta : \beta \leq \alpha\}$. At step n, since each T'_n meets each $H \in \mathcal{H}$ in at most one point, and $|H_{ni}| = n+1$ for all i, we can choose $s_{\alpha n} \in H_{ng_\alpha(n)} \setminus \bigcup_{j<n} T'_j$. Note that the $s_{\alpha n}$'s are Euclidean dense in $[0,1]$. Thus there is some $x_\alpha \notin \{x_\beta : \beta < \alpha\}$ and infinite $A_\alpha \subset \omega$ such that $\{s_{\alpha n} : n \in A_\alpha\} \to x_\alpha$. Then setting $S_\alpha = \{s_{\alpha n} : n \in A_\alpha\}$ clearly works. □

Let us recall the following construction due to Nyikos [13] (see also Example 1 of [6]). Let $T = 2^{<\omega}$ be the Cantor tree, and let $A \subset 2^\omega$. Let $X_A = T \cup \{\infty\}$ be the space with T as the set of isolated points, and subbasic neighborhoods of ∞ are complements of "branches" $b_x = \{x|n : n \in \omega\}$ of the tree for $x \in A$. X_A is always bisequential (e.g., by Proposition 3.2), hence boundedly FU_{fin}. Nyikos showed that X_A is α_2-FU if A is a λ'-set (i.e., if $B \subset 2^\omega$ is countable, then B is G_δ in $A \cup B$). The next result strengthens this just a bit, the proof being a mild extension of Nyikos's argument, and shows that there is in ZFC a countable α_2-boundedly FU_{fin} space which is not first-countable.

Example 3.10. *If A is a λ'-set, then the space X_A is α_2-boundedly FU_{fin}.*

Proof. Suppose $\mathcal{P}_0, \mathcal{P}_1, \dots$ is a sequence of π-nets at ∞, where $sup\{|P| : P \in \mathcal{P}_i\} \leq k_i$. Since X_A is boundedly FU_{fin}, we may assume $\mathcal{P}_i = \{P_{ij}\}_{j\in\omega} \to \infty$, and $|P_{ij}| = k_i$ for all i, j.

We are going to use the compact metrizable topology on $T \cup 2^\omega$ generated by the basis $T \cup \{\sigma^* : \sigma \in 2^{<\omega}\}$, where

$$\sigma^* = \{x \in 2^\omega : \sigma \subset x\} \cup \{t \in T : \sigma \subset t\}.$$

Note that the subspace 2^ω inherits its usual product topology.

For each i, j, let $\vec{p}_{ij} = \langle p_{ijm}\rangle_{m<k_i}$, where $P_{ij} = \{p_{ijm}\}_{m<k_i}$. W.l.o.g., we may assume that for each i, $\{\vec{p}_{ij}\}_{j\in\omega}$ converges to a point $\langle x_{im}\}_{m<k_i}\rangle$ in the k_i^{th} power of the above compact metrizable topology, and that $P_{ij} \cap (\cup\{b_{x_{im}} : m < \kappa_i$ and $x_{im} \in A\} = \emptyset$. Let $B = \{x_{im} : i, m \in \omega\}$.

Let U_n, $n \in \omega$, be a decreasing sequence of open sets in 2^ω with $\bigcap_{n\in\omega} U_n \cap (A \cup B) = B$. Let $U_n^* = \cup\{\sigma^* : \sigma \subset U_n\}$, and let $A \cap B = \{y_n : n \in \omega\}$. It is easy to check that for each i we can find a large enough $j_i \in \omega$ so that $P_{ij_i} \subset U_i^*$ and $P_{ij_i} \cap (\cup\{b_{y_k}\}_{k\leq i}) = \emptyset$.

Let $Q = \{P_{ij_i}\}_{i\in\omega}$. We need to show that $Q \to \infty$, i.e., that each b_x, $x \in A$, meets P_{ij_i} for at most finitely many i. For $x \in A \cap B$, this is clear by the construction. Suppose $x \in A \setminus B$. Then for sufficiently large i, $x \notin U_i$. Note that in this case $b_x \cap U_i^* = \emptyset$, and hence $P_{ij_i} \cap b_x = \emptyset$. □

The previous example does not separate in ZFC the α_2-boundedly FU_{fin} property from FU_{fin}. Nyikos showed that X_A is FU_{fin} iff A is a γ-set, and A. Miller [9] showed that it is consistent that every λ'-set is γ. There are many models, however, which have λ'-sets which are not γ-sets (e.g., any model of CH or MA), so it does give consistent examples showing that implication 1 need not reverse.

Example 3.11. *If A is a λ'-set which is not a γ-set, then X_A is α_2-boundedly FU_{fin} space which is not FU_{fin}.*

Finally, Nyikos [13] noted that if $A = 2^\omega$, X_A is not α_2:

Example 3.12. *If $A = 2^\omega$, then X_A is boundedly FU_{fin} space but not α_2.*

4. Relationships to games, products, and α_i spaces

Let X be a space, $x \in X$, and $k \in \omega$. We define the game $G_{O,P}(X, x)$ (respectively, $G^k_{O,P}(X, x)$, $G^{fin}_{O,P}(X, x)$) as follows. The players are O and P. In the n^{th} round, O chooses an open neighborhood U_n of x, and P responds with a singleton (respectively, k-sized set, finite set) $P_n \subset \bigcap_{i\leq n} U_i$. O *wins* if $P_n \to x$.

The game in which the P_n's are singletons was introduced and studied in [3]. Of course, O has a winning strategy in any of these games if X is first-countable at x. It is also not very difficult to see that O has a winning strategy in any one of these games iff O has a winning strategy in all of the games. (This was proven in [3] for $G_{O,P}$ and $G^{fin}_{O,P}$.) The situation for P is different, however. In [6], we showed that X is FU_{fin} at x iff P has no winning strategy in $G^{fin}_{O,P}(X,x)$. What the proof really shows is that the game property is equivalent to α_2-FU_{fin}, and the same proof virtually word for word shows the following result:

Theorem 4.1. *Let $*$ be either k, where $k \in \omega \setminus \{0\}$, or fin. Then P has no winning strategy in $G^*_{O,P}(X,x)$ iff X is α_2-FU_* at x.*

It follows that if X is the gap space of Example 2.1, which is α_2-FU_1 (=Fréchet α_2) but not FU_2, then P has no winning strategy in the singleton game, but P does have a winning strategy in the doubleton game.

We remark that the case $n = 1$ in the above theorem, which boils down to "P has no winning strategy in $G_{O,P}(X,x)$ iff X is Fréchet α_2 at x" was proven by P.L. Sharma [16].

Next we discuss product spaces. Sipacheva [18] showed that X is FU_n at x iff X^n if Fréchet at the diagonal point $\langle x, x, \dots, x\rangle$. Since the Fréchet property is a pointwise property, it follows, e.g., that there is a space X such that X^n is Fréchet but X^{n+1} is not Fréchet iff there is one with only one non-isolated point iff there is an X which is FU_n but not FU_{n+1}. There is a similar relationship to Fréchetness in products for many of the other properties we are considering, summarized by the chart below. In the chart, X is a space with exactly one non-isolated point. This is a critical assumption. For example, Example 15 of [6] gives an example (under CH) of two countable FU_{fin} spaces whose product is not Fréchet, so if X is their topological sum, then X is FU_{fin} yet X^2 is not Fréchet.

$$
\begin{array}{ccccc}
X\ \alpha_2\text{-}\mathrm{FU}_n & \iff & X^n\ \alpha_2\text{-FU} & \iff & X\ \mathrm{FU}_n + \alpha_2 \\
 & & \Downarrow & & \\
 & & X^n\ \alpha_3\text{-FU} & \iff & X\ \mathrm{FU}_n + \alpha_3 \\
 & & \Downarrow & & \\
 & & X^n\ \alpha_4\text{-FU} & \iff & X\ \alpha_4\text{-}\mathrm{FU}_n \\
 & & & & \Downarrow \\
 & & & & X\ \mathrm{FU}_n + \alpha_4 \\
 & & & & \Downarrow \\
 & & & & X\ \mathrm{FU}_n \iff X^n\ \text{Fréchet}
\end{array}
$$

$$
\begin{array}{ccccc}
\forall n(X\ \alpha_2\text{-}\mathrm{FU}_n) & \iff & X^\omega\ \alpha_2\text{-FU} & \iff & \forall n(X^n\ \alpha_2\text{-FU}) \\
 & & \Downarrow & & \\
 & & X^\omega\ \text{Fréchet} & & \\
 & & \Downarrow & & \\
\forall n(X\ \mathrm{FU}_n) & \iff & \forall n(X^n\ \text{Fréchet}(+\alpha_4)) & \iff & X\ \text{boundedly}\ \mathrm{FU}_{fin}
\end{array}
$$

The down arrows in the chart are trivial. The equivalences in the bottom "connected" piece are immediate from those in the top piece. We should also mention Nogura's result [11] that, for $i = 1, 2, 3$, if X^n is Fréchet α_i for all $n \in \omega$ then X^ω Fréchet α_i. For $i = 2$ this is precisely one of the implications in the chart. The next three results establish the equivalences in the top piece.

Theorem 4.2. *The following are equivalent for a space X with one non-isolated point:*

(a) *X is α_2-FU_n;*

(b) *X^n is α_2-FU;*
(c) *X is FU_n and α_2.*

Proof. (b)⇒(c) is immediate from Sipacheva's result that X FU_n iff X^n is Fréchet.

If X satisfies (a), then X^n is Fréchet for the same reason. That X^n is α_2 is easily shown by translating convergent sequences of points in X^n to the corresponding sequences of π-nets of $\leq n$-sized sets consisting of the coordinates of the points, applying α_2-FU_n in X to these π-nets, and translating back again to points in X^n.

Suppose X satisfies (c), and let $\mathcal{P}_0, \mathcal{P}_1, \dots$ be a sequence of π-nets of n-sized sets. By FU_n, we may assume each $\mathcal{P}_k = \{P_{ki}\}_{i\in\omega}$ is convergent. Let $P_{ki} = \{p_{kij} : j < n\}$. Apply α_2 to obtain an infinite $T_k(0)$ of $S_k(0) = \{p_{ki0} : i \in \omega\}$ such that $T(0) = \bigcup_{k\in\omega} T_k(0)$ converges. Then apply α_2 again to obtain an infinite subset $T_k(1)$ of $S_k(1) = \{p_{ki1} : p_{ki0} \in T(0)\}$ such that $T(1) = \bigcup_{k\in\omega} T_k(1)$ converges. Continue in the same manner for each $j < n$. Then the set $T = \{P_{ki} : p_{kij} \in T(j) \text{ for all } j < n\}$ is a convergent selection of infinitely many sets from each $\mathcal{P}_k$. So X satisfies (a). □

By essentially the same method as the proof of (c)⇒(a) above, it is easy to establish the following:

Theorem 4.3. *For a space X with one non-isolated point, X^n is α_3-FU iff X is FU_n and α_3.*

Theorem 4.4. *For a space X with one non-isolated point, X is α_4-FU_n iff X^n is α_4-FU.*

Proof. Translate back and forth between convergent sequences in X^n and π-nets of n-sized sets. □

Note that the analogue of condition (c) in Theorem 4.2 cannot necessarily be added to the list of equivalences in the previous theorem. Of course, it is equivalent for $n = 1$, but Example 3.2 in the previous section shows this need not be the case (i.e., the down arrow third from the top need not reverse) for $n \geq 2$. However, Corollary 3.4 shows that it does reverse (for $n \geq 2$) under OCA. Regarding the non-reversibility of the other down arrows, any of the (known, ZFC) examples differentiating α_3 from α_2, and α_4 from α_3, show that the top two do not reverse (for $n = 1$), and any Fréchet non-α_4 space the fourth one (which *does* reverse for $n \geq 2$, since FU_2 implies α_4). The example in [4] under MA shows that the bottom down arrow need not reverse. The remaining down arrow, and also the top one for $n \geq 2$, is taken care of by the boundedly FU_{fin} non-α_2 space X of Example 3.12 in the previous section. This X is α_3. Then it follows from Theorem 4.3 above that X^n is Fréchet α_3 for all n, so by Nogura's result mentioned earlier, X^ω is α_3-FU. But we do not know the answer to:

Question 2. *Does X^ω Fréchet imply X is α_3?*

We also do not know if there are ZFC examples which show that the bottom down arrow, or the second from the top for $n \geq 2$, do not reverse:

Question 3. *Is there in ZFC a space X which is*
(1) *boundedly FU_{fin} but X^ω is not Fréchet?*
(2) *α_4-FU_n but not α_3 (for $n \geq 2$)?*

If X is FU_{fin}, then X is α_2-FU_n for all n, so it follows from the chart that X^ω is α_2-Fu. In fact, we can show that X FU_{fin} is equivalent to X^ω FU_{fin}, and the same holds for α_2-boundedly FU_{fin}.

Theorem 4.5. *Let X be a space with exactly one non-isolated point. Then:*

(1) *X is FU_{fin} iff X^ω is FU_{fin};*
(2) *X is α_2-boundedly FU_{fin} iff X^ω is α_2-boundedly FU_{fin}.*

Proof. We show (1), with (2) being entirely analogous. To prove the non-trivial direction, suppose X is FU_{fin}. Denote the non-isolated point of X by ∞, and let $\mathcal{P}$ be a π-net at $\langle\infty,\infty,\dots\rangle \in X^\omega$ consisting of finite subsets of X^ω.

For each $P \in \mathcal{P}$ and $n \in \omega$, let $\pi(P,n) = \{x(i) : i \leq n, x \in P\}$, and let $\mathcal{P}(n) = \{\pi(P,n) : P \in \mathcal{P}\}$. It is elementary to check that each $\mathcal{P}(n)$ is a π-net at ∞ of finite subsets of X. Hence we can find $P_n \in \mathcal{P}$ such that $\pi(P_n, n) \to \infty$.

Let us see that $P_n \to \langle\infty,\infty,\dots\rangle$ in X^ω. A basic open set containing $\langle\infty,\infty,\dots\rangle$ has the form $U^k \times X^\omega$ for some $k \in \omega$. For sufficiently large n, we have $\pi(P_n,n) \subset U$. If also $n \geq k$, it follows that $P_n \subset U^k \times X^\omega$. $\square$

It follows from the above results, together with the examples of the previous section, that X^ω can (consistently) have any one of the properties we are considering that imply boundedly FU_{fin} without having any stronger property.

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