WHEN THE COLLECTION OF $\epsilon$-BALLS IS LOCALLY FINITE

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Abstract. Consider the class $\mathcal{N}$ of metrizable spaces which admit a metric $d$ such that, for every $\epsilon > 0$, the collection $\{B(x, \epsilon) : x \in X\}$ of all $\epsilon$-balls is locally finite. We show that $\mathcal{N}$ is precisely the class of strongly metrizable spaces, i.e., $X \in \mathcal{N}$ iff $X$ is homeomorphic to a subspace of $\kappa^\omega \times [0, 1]^\omega$ for some cardinal $\kappa$ (where $\kappa$ carries the discrete topology). In particular, this shows that not every metrizable space admits such a metric, thereby answering a question of Nagata.

1. Introduction

Let $(X, d)$ be a metric space. For $\epsilon > 0$, we let $B_d(x, \epsilon)$ denote the $\epsilon$-ball $\{y \in X : d(x, y) < \epsilon\}$ about $x$, and we let $B_d(\epsilon)$ denote the collection $\{B_d(\epsilon) : x \in X\}$ of all $\epsilon$-balls in $X$. We may delete the subscript $d$ in case the metric is understood.

In [N1], J. Nagata showed that every metrizable space $X$ admits a metric $d$ such that, for every $\epsilon > 0$, $B_d(\epsilon)$ is closure-preserving. Indeed, implicit in his paper is the fact that every separable metric space admits a metric $d$ such that $B_d(\epsilon)$ is finite for every $\epsilon > 0$. For the metric he builds has the property that, for each $\epsilon > 0$, there is a locally finite open cover $G_\epsilon$ of $X$ such that $B_d(\epsilon) = \{st(x, G_\epsilon) : x \in X\}$, where $st(x, G_\epsilon) = \bigcup\{G \in G_\epsilon : x \in G\}$ and is called the “star” of $G_\epsilon$ at $x$. (It is easy to check that the collection of stars of a locally finite collection is closure-preserving.) Since a locally finite collection in a compact space must be finite, it follows then that the Hilbert cube admits a metric $d$ such that, for each $\epsilon > 0$, $B_d(\epsilon)$ is finite. Hence, so does any separable metrizable space, for the restriction of such $d$ to any subspace of the Hilbert cube has the same property.

In [N2], Nagata asks if every metrizable space admits a metric such that each $B(\epsilon)$ is locally finite. (He uses the term “hereditarily closure-preserving” in place of “locally finite”, but these notions are equivalent in the class of first-countable, in particular metrizable, spaces.) In this note, we characterize the class $\mathcal{N}$ of metrizable spaces which admit such a metric as precisely the class of strongly metrizable spaces, where a metrizable space $X$ is strongly metrizable iff $X$ has a base which is the union of countably many star-finite open covers, or equivalently (see [P], Proposition 3.27), $X$ is embeddable in $\kappa^\omega \times I^\omega$ for some cardinal $\kappa$, where $I = [0, 1]$ and $\kappa$ carries the discrete topology. This gives a negative answer to Nagata’s question;

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in particular, any space with a non-separable component, such as a hedgehog with uncountably many spines, is not embeddable in $\kappa^\omega \times I^\omega$ and hence does not admit a metric such that each $B(\epsilon)$ is locally finite.

2. Main results

We first show that it matters not if one changes the question by replacing “locally finite” with “point-finite” or “star-finite”. (Recall that a collection $\mathcal{U}$ of subsets of $X$ is star-finite if each member of $\mathcal{U}$ meets only finitely other members.)

**Lemma 1.1.** Let $(X, d)$ be a metric space, and $\epsilon > 0$. Then the following are equivalent:

- (a) $B_d(\epsilon)$ is locally finite;
- (b) $B_d(\epsilon)$ is point-finite;
- (c) $B_d(\epsilon)$ is star-finite;
- (d) There is a star-finite open cover $\mathcal{U}$ such that $B_d(\epsilon) = \{st(x, \mathcal{U}) : x \in X\}$.

**Proof.** That (c)⇒(a)⇒(b) is trivial, and (d)⇒(b) is easy to check.

We now prove (b)⇒(c), thereby establishing equivalence of (a)-(c). To this end, suppose $B_d(\epsilon)$ is point-finite. For each $B \in B_d(\epsilon)$, let $C(B) = \{x \in B : B = B_d(x, \epsilon)\}$; i.e., $C(B)$ is the set of centers of the ball $B$. Note that the $C(B)$'s partition $X$. That (b)⇒(c) is then immediate from the following two claims.

Claim 1. Each $B \in B_d(\epsilon)$ meets at most finitely many $C(B)$'s. Suppose by way of contradiction that $x_n \in C(B_n) \cap B, n \in \omega$. Let $x \in C(B)$. Then $d(x, x_n) < \epsilon$ for all $n$, so since $x_n \in C(B_n)$ we have $x \in B_n$ for all $n$, contradicting (b).

Claim 2. Each $C(B)$ meets only finitely many $B$'s in $B_d(\epsilon)$. To see this, suppose $x_n \in C(B) \cap B_n$ for all $n$. Let $y \in C(B)$, and pick $z_n \in C(B_n)$. Then $d(x_n, z_n) < \epsilon$, so $z_n \in B_d(x_n, \epsilon) = B$. This implies $d(y, z_n) < \epsilon$ for all $n$, whence $y \in B_n$ for all $n$, again contradicting (b).

It remains to prove that (c)⇒(d). Assume $B_d(\epsilon)$ is star-finite. For each $p \neq q \in X$ with $d(p, q) < \epsilon$, let

$$U(p, q) = \bigcap\{B \in B_d(\epsilon) : \{p, q\} \subset B\}.$$

We first show that $d(x, y) < \epsilon$ for any two points $x, y \in U(p, q)$. Clearly $U(p, q) \subset B_d(p, \epsilon) \cap B_d(q, \epsilon)$. It follows that $p, q \in B_d(x, \epsilon)$, whence $y \in U(p, q) \subset B_d(x, \epsilon)$. So $d(x, y) < \epsilon$ as claimed.

Now let $\mathcal{U}' = \{U(p, q) : p \neq q \text{ and } d(p, q) < \epsilon\}$. Since each member of $\mathcal{U}'$ is a finite intersection of members of a star-finite collection, $\mathcal{U}'$ is also star-finite. It covers all non-isolated points, but possibly not all isolated points, so we let $\mathcal{U} = \mathcal{U}' \cup \{\{x\} : x \in X \setminus \mathcal{U}'\}$, which of course is also star-finite. Consider any $x \in X$. By the previous paragraph, we have $st(x, \mathcal{U}) \subset B_d(x, \epsilon)$. Suppose $y \in B_d(x, \epsilon) \setminus \{x\}$. Then $x, y \in U(x, y) \subset U$, so $y \in st(x, \mathcal{U})$. Thus $st(x, \mathcal{U}) = B_d(x, \epsilon)$, which completes the proof. □

The proof of the next lemma is similar to the proof of the equivalence of 1.1(b) and 1.1(c) and hence is omitted.

**Lemma 1.2.** Let $(X, d)$ be a metric space, and $\epsilon > 0$. Then the following are equivalent:

- (a) $B_d(\epsilon)$ is point-countable;
- (b) $B_d(\epsilon)$ is star-countable;
- (c) $B_d(\epsilon)$ is $\kappa$-point-finite; and
- (d) There is a star-countable open cover $\mathcal{U}$ such that $B_d(\epsilon) = \{st(x, \mathcal{U}) : x \in X\}$.
Let $\mathcal{N}$ be as defined in the introduction. We note the following easy lemma.

**Lemma 1.3.** $\mathcal{N}$ is closed under subspaces and countable products.

*Proof.* If the metric $d$ witnesses that $X \in \mathcal{N}$, clearly the restriction of $d$ to any subspace $X' \in \mathcal{N}$. To prove closure under countable products, suppose $X_0, X_1, \ldots$ are in $\mathcal{N}$, witnessed by $d_0, d_1, \ldots$. Let $d'_n(x, y)$ be the minimum of $1/2^n$ and $d_i(x, y)$. It is easy to check that $d'_n$ also witnesses that $X_i$ is in $\mathcal{N}$, and that $d(x, y) = \max_{i \in \omega} d'_n(x_i, y_i)$ witnesses that $\prod_{i \in \omega} X_i \in \mathcal{N}$. \hfill $\square$

Since discrete spaces are obviously in $\mathcal{N}$, and the Hilbert cube is in $\mathcal{N}$ by Nagata's result mentioned in the Introduction, it follows from Lemma 1.3 that, for any cardinal $\kappa$, any subspace of $\kappa^\omega \times [0, 1]^\omega$ is in $\mathcal{N}$ (where $\kappa$ has the discrete topology). The next result, our main one, shows that this characterizes $\mathcal{N}$.

**Theorem 1.4.** The following are equivalent for a metrizable space $X$:

(i) $X \in \mathcal{N}$, i.e., $X$ admits a metric such that, for all $\epsilon > 0$, $B(\epsilon)$ is locally finite;

(ii) $X$ admits a metric such that, for all $\epsilon > 0$, $B(\epsilon)$ is point-finite;

(iii) $X$ admits a metric such that, for all $\epsilon > 0$, $B(\epsilon)$ is star-finite;

(iv) $X$ admits a metric such that, for all $\epsilon > 0$, $B(\epsilon)$ is point-countable;

(v) $X$ admits a metric such that, for all $\epsilon > 0$, $B(\epsilon)$ is star-countable;

(vi) There is a sequence of star-finite open covers $\mathcal{G}_n$, $n < \omega$, of $X$ such that, for each $n$, $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$ and, for each $x \in X$, $\{st(x, \mathcal{G}_n) : n \in \omega\}$ is a base at $x$.

(vii) There is a sequence of star-countable open covers $\mathcal{G}_n$, $n < \omega$, of $X$ such that, for each $n$, $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$ and, for each $x \in X$, $\{st(x, \mathcal{G}_n) : n \in \omega\}$ is a base at $x$.

(viii) $X$ is homeomorphic to a subspace of $\kappa^\omega \times [0, 1]^\omega$ for some cardinal $\kappa$.

*Proof.* By the previous lemmas, (i)-(iii) are equivalent, as are (iv) and (v). As noted following the proof of Lemma 1.3, we also have (viii)⇒(i). Furthermore, it is clear that (iii) implies (iv),(v), and (vi), and (v) and (vi) both imply (vii). We now prove (vii) implies (viii); the theorem then follows from this and the aforementioned implications.

Let $\{\mathcal{G}_n\}_{n \in \omega}$ satisfy condition (vii). For $U, V \in \mathcal{G}_n$, define $U \sim_n V$ iff there is a finite sequence $U_0, U_1, \ldots, U_k$ of elements of $\mathcal{G}_n$ with $U = U_0$, $V = U_k$, and $U_i \cap U_{i+1} \neq \emptyset$ for all $i < k$. Then each equivalence class $[U]_n$ is countable, and the collection $\mathcal{P}_n = \{\bigcup[U]_n : U \in \mathcal{G}_n\}$ is a clopen partition of $X$. Since $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$, $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$.

Let $\mathcal{P}_0 = \{P_{(\alpha)} : \alpha < \kappa_0\}$ for some cardinal $\kappa_0$. Then for each $\alpha < \kappa_0$, let $\mathcal{P}_0 = \{P_{(\alpha)} : \alpha < \kappa_0\}$, and for each $\beta < \kappa_0$, let $\mathcal{P}_0 = \{P_{(\alpha, \beta)} : \beta < \kappa_0\}$, where $P_{(\alpha, \beta)}$ is a base at $x \in U$. Then each equivalence class $[U]_n$ is countable, and the collection $\mathcal{P}_n = \{\bigcup[U]_n : U \in \mathcal{G}_n\}$ is a clopen partition of $X$. Since $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$, $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$.

Let $\mathcal{P}_\sigma = \bigcup \mathcal{P}_\sigma \in \mathcal{G}_k$, where $k = |\sigma| - 1$, such that $\mathcal{P}_\sigma = \bigcup [G_\sigma]_k$. Let $G_\sigma$ denote the equivalence class $[G_\sigma]_k$; since equivalence classes are countable, we may index $G_\sigma$ by $\{G_{\sigma, j} : j \in \omega\}$.

Let $\kappa$ be the supremum of all the defined $\kappa_\alpha$'s. For any $\sigma \in \kappa^\omega$ for which $P_\sigma$ has not been defined, let $P_\sigma = \emptyset = G_\sigma$. Note that for each $n$,

$$G_n = \bigcup \{G_{\sigma, j} : \sigma \in \kappa^{n+1}\}.$$
Since $X$ is metrizable, hence paracompact, there is a locally finite closed shrinking $\{H(G) : G \in \mathcal{G}_n\}$ of $\mathcal{G}_n$. For $G \in \mathcal{G}_n$, let $f_G : X \to [0, 1/2^n]$ be continuous such that $f_G(H(G)) = \{1/2^n\}$ and $f_G(X \setminus G) = \{0\}$. Let $\mathcal{F}_n = \{f_G : G \in \mathcal{G}_n\}$.

Now we can define our desired embedding $\theta$, which will map $X$ into $\kappa^\omega \times I^\omega \times \omega$. Pick $x \in X$. Then for each $n$, there is a unique $\sigma_n^x \in \kappa^{n+1}$ with $x \in P_{\sigma_n^x}$. Note that $\sigma_{n+1}^x$ extends $\sigma_n^x$, so there is a unique $\tau^x \in \kappa^\omega$ such that $\tau^x \restriction (n+1) = \sigma_n^x$ for all $n$. Then let

$$\theta(x) = \{\tau^x\} \times \{f_{G_{\tau^x}(k+1,j)}(x)(k,j)) \in \omega \times \omega\}.$$ 

Let us check that $\theta$ is one-to-one. Suppose $x \neq y$, yet $\theta(x) = \theta(y)$. Then $\tau^x = \tau^y = \tau \in \kappa^\omega$, i.e., $x$ and $y$ are always in the same member of the partitions $\mathcal{P}_n$. Choose $k$ sufficiently large so that $y \notin \text{st}(x, \mathcal{G}_k)$, and choose $G \in \mathcal{G}_k$ with $x \in H(G)$. Then for some $j$, $G = G_{\tau^x(k+1),j}$, and $f_{G_{\tau^x(k+1),j}}(x) = 1/2^k$ while $f_{G_{\tau^x(k+1),j}}(y) = 0$. Thus $\theta(x)$ and $\theta(y)$ differ on coordinate $(k,j)$, contradiction.

Now we show that $\theta$ is continuous. Let $\theta(x) = (\theta_1(x), \theta_2(x))$, where $\theta_1(x) \in \kappa^\omega$ and $\theta_2(x) \in I^\omega \times \omega$. Suppose a sequence of points $x_n$, $n < \omega$, converges to $x \in X$. Fix $k \in \omega$. Then for sufficiently large $n$, $x_n \in P_{\tau^x(k+1)}$ and hence also $\tau^x_n \restriction (k+1) = \tau^x \restriction (k+1)$. It easily follows from this that as $n$ gets large, $\theta_1(x_n)(k) = \tau^x_n(k)$ converges to (in fact, is equal to) $\theta_1(x)(k) = \tau^x(k)$ and that, if we also fix $j$, $\theta_2(x_n)(k,j)$ converges to $\theta_2(x)(k,j)$. Thus $\theta$ is continuous.

It remains to prove that $\theta$ is a homeomorphism onto its range. To this end, let $O$ be open in $X$; it will suffice to show that $\theta(O)$ is relatively open in $\theta(X)$. Take a point $\theta(x) \in \theta(O)$. We need to find a neighborhood $V$ of $\theta(x)$ in $\kappa^\omega \times I^\omega \times \omega$ such that $V \cap \theta(X) \subset \theta(O)$.

Choose $k$ sufficiently large so that $\text{st}(x, \mathcal{G}_k) \subset O$. Let $J = \{j \in \omega : x \in H(G_{\tau^x(k+1),j})\}$; obviously, $J$ is finite. Let $\epsilon = 1/2^k$, and let $k' = k + 1$. Then the set

$$V = \{(z_1, z_2) \in \kappa^\omega \times I^\omega \times \omega : z_1 \restriction k' = \tau^x \restriction k' & \forall j \in J(|z_2(k,j) - \theta_2(x)(k,j)| < \epsilon)\}$$

is an open neighborhood of $\theta(x)$.

It remains to show $V \cap \theta(X) \subset \theta(O)$. Suppose $\theta(y) \in V \cap \theta(X) \setminus \theta(O)$. Then $y \notin O$. Since $\theta(y) \in V$, $\tau^y \restriction (k+1) = \tau^x \restriction (k+1)$. Let $G \in \mathcal{G}_k$ with $x \in H(G)$. Then $G \in \{G_{\tau^x(k+1),j}\}$ and so $G = G_{\tau^y(k+1),j}$ for some $j$; of course, $j \in J$. Since $y \notin O$, $y \notin G$. Thus $\theta_2(y)(k,j) = f_{G_{\tau^y(k+1),j}}(y) = f_{G_{\tau^x(k+1),j}}(y) = f_G(y) = 0$, while $\theta_2(x)(k,j) = f_G(x) = 1/2^k$. But this contradicts $\theta(y) \in V$, and thus completes the proof. \(\square\)

**Corollary 1.5.** $\mathcal{N}$ is precisely the class of strongly metrizable spaces.

**Remark.** Recall that a space is strongly paracompact if every open cover has a star-finite open refinement. It is well-known and easy to see that every strongly paracompact metrizable space is strongly metrizable and hence, by our result, is in $\mathcal{N}$. But not every member of $\mathcal{N}$ is strongly paracompact; e.g., it is known $[N_3]$ that $\omega_1^\omega \times (0,1)$, which of course embeds in $\omega_1^\omega \times I^\omega$ and hence is strongly metrizable, is not strongly paracompact.

We also remark that Y. Hattori [H] obtained another characterization of strongly metrizable spaces in terms of a metric.
3. An Example

By Lemma 1.1, the only way the collection of $\epsilon$-balls ($\epsilon$ fixed) can be locally finite is for this collection to be precisely the stars of some star-finite open cover. Recall that Nagata showed that every metrizable space admits a metric such that the collection of $\epsilon$-balls is closure-preserving by constructing a metric such that the collection of $\epsilon$-balls is precisely the collection of stars of some locally finite open cover. So it is natural to ask if this is the only way the collection of $\epsilon$-balls can be closure-preserving. The following example shows that the answer is “no”.

**Example.** There is a metric space $(X, d)$ such that, for every $\epsilon > 0$, $\mathcal{B}_d(\epsilon)$ is closure-preserving but there is no locally finite open cover $\mathcal{G}$ with $\mathcal{B}_d(\epsilon) = \{st(x, \mathcal{G}(\epsilon)) : x \in X\}$.

**Proof.** Let the set $X$ be $\omega \times \mathbb{R}$, viewed as a subset of the plane. For $x, y \in X$, denote the usual Euclidean distance between $x$ and $y$ by $|x - y|$. Then for $x = (n_x, r_x)$ and $y = (n_y, r_y)$ in $X$, define $d(x, y)$ to be $|x - y|$ if $x = y$, or if $n_x \neq n_y$, or if $n_x = n_y = n$ and $|x - y| > 1/2^n$. Let $d(x, y) = 1/2^n$ otherwise. It is easy to check that $d$ is a metric on the set $X$, and that $d$ generates the discrete topology on $X$. So any collection of subsets of $X$ is closure-preserving, in particular $\mathcal{B}_d(\epsilon)$.

Fix $\epsilon > 0$. We aim to show that $\mathcal{B}_d(\epsilon)$ cannot be precisely the collection of stars of some locally finite open cover. To this end, choose $n$ such that $1/2^n < \epsilon$, and note that the trace of $\mathcal{B}_d(\epsilon)$ on $\{n\} \times \mathbb{R}$ contains $\{(n) \times (x - \epsilon, x + \epsilon) : x \in \mathbb{R}\}$, i.e., it contains all open intervals on $\{n\} \times \mathbb{R}$ of length $2\epsilon$. Thus establishing the following claim will complete the proof.

**Claim.** There is no point-finite cover $\mathcal{G}$ of $\mathbb{R}$ such that every open interval of length $2\epsilon$ is the union of some finite subcollection of $\mathcal{G}$.

**Proof of Claim.** Suppose $\mathcal{G}$ is such a point-finite cover of $\mathbb{R}$. Let $z \in \mathbb{R}$. There is a finite subcollection $\mathcal{G}_z$ of $\mathcal{G}$ such that $\cup \mathcal{G}_z = (z - 2\epsilon, z)$. Since $\mathcal{G}$ is point-finite, if we let

$$z' = \sup(\cup \{G \in \mathcal{G}_z : \sup(G) < z\})$$

then $z' < z$. Choose $q_z \in \mathbb{Q}$ between $z'$ and $z$. Pick $G_z \in \mathcal{G}_z$ with $q_z \in G_z$. Note that $\sup(G_z) = z$, hence $z \neq z'$ implies $G_z \neq G_{z'}$. But there must be $q$ such that $q_z = q$ for uncountably many $z$, contradicting point-finiteness of $\mathcal{G}$ at $q$. □

**References**


