WHEN THE COLLECTION OF ϵ-BALLS IS LOCALLY FINITE

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Abstract. Consider the class $\mathcal{N}$ of metrizable spaces which admit a metric d such that, for every $\epsilon > 0$, the collection $\{B(x, \epsilon) : x \in X\}$ of all ϵ-balls is locally finite. We show that $\mathcal{N}$ is precisely the class of strongly metrizable spaces, i.e., $X \in \mathcal{N}$ iff X is homeomorphic to a subspace of $\kappa^\omega \times [0, 1]^\omega$ for some cardinal κ (where κ carries the discrete topology). In particular, this shows that not every metrizable space admits such a metric, thereby answering a question of Nagata.

1. Introduction

Let (X, d) be a metric space. For $\epsilon > 0$, we let $B_d(x, \epsilon)$ denote the ϵ-ball $\{y \in X : d(x, y) < \epsilon\}$ about x, and we let $\mathcal{B}_d(\epsilon)$ denote the collection $\{B_d(\epsilon) : x \in X\}$ of all ϵ-balls in X. We may delete the subscript d in case the metric is understood.

In [N_1], J. Nagata showed that every metrizable space X admits a metric d such that, for every $\epsilon > 0$, $\mathcal{B}_d(\epsilon)$ is closure-preserving. Indeed, implicit in his paper is the fact that every separable metric space admits a metric d such that $\mathcal{B}_d(\epsilon)$ is finite for every $\epsilon > 0$. For the metric he builds has the property that, for each $\epsilon > 0$, there is a locally finite open cover $\mathcal{G}_\epsilon$ of X such that $\mathcal{B}_d(\epsilon) = \{st(x, \mathcal{G}_\epsilon) : x \in X\}$, where $st(x, \mathcal{G}_\epsilon) = \cup\{G \in \mathcal{G}_\epsilon : x \in G\}$ and is called the "star" of $\mathcal{G}_\epsilon$ at x. (It is easy to check that the collection of stars of a locally finite collection is closure-preserving.) Since a locally finite collection in a compact space must be finite, it follows then that the Hilbert cube admits a metric d such that, for each $\epsilon > 0$, $\mathcal{B}_d(\epsilon)$ is finite. Hence, so does any separable metrizable space, for the restriction of such d to any subspace of the Hilbert cube has the same property.

In [N_2], Nagata asks if every metrizable space admits a metric such that each $\mathcal{B}(\epsilon)$ is locally finite. (He uses the term "hereditarily closure-preserving" in place of "locally finite", but these notions are equivalent in the class of first-countable, in particular metrizable, spaces.) In this note, we characterize the class $\mathcal{N}$ of metrizable spaces which admit such a metric as precisely the class of strongly metrizable spaces, where a metrizable space X is *strongly metrizable* iff X has a base which is the union of countably many star-finite open covers, or equivalently (see [P], Proposition 3.27), X is embeddable in $\kappa^\omega \times I^\omega$ for some cardinal κ, where $I = [0, 1]$ and κ carries the discrete topology. This gives a negative answer to Nagata's question;

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in particular, any space with a non-separable component, such as a hedgehog with uncountably many spines, is not embeddable in $\kappa^\omega \times I^\omega$ and hence does not admit a metric such that each $\mathcal{B}(\epsilon)$ is locally finite.

2. Main results

We first show that it matters not if one changes the question by replacing "locally finite" with "point-finite" or "star-finite". (Recall that a collection $\mathcal{U}$ of subsets of X is *star-finite* if each member of $\mathcal{U}$ meets only finitely other members.)

Lemma 1.1. *Let (X,d) be a metric space, and $\epsilon > 0$. Then the following are equivalent:*

(a) *$\mathcal{B}_d(\epsilon)$ is locally finite;*
(b) *$\mathcal{B}_d(\epsilon)$ is point-finite;*
(c) *$\mathcal{B}_d(\epsilon)$ is star-finite;*
(d) *There is a star-finite open cover $\mathcal{U}$ such that $\mathcal{B}_d(\epsilon) = \{st(x,\mathcal{U}) : x \in X\}$.*

Proof. That (c)⇒(a)⇒(b) is trivial, and (d)⇒(b) is easy to check.

We now prove (b)⇒(c), thereby establishing equivalence of (a)-(c). To this end, suppose $\mathcal{B}_d(\epsilon)$ is point-finite. For each $B \in \mathcal{B}_d(\epsilon)$, let $C(B) = \{x \in B : B = B_d(x,\epsilon)\}$; i.e., $C(B)$ is the set of centers of the ball B. Note that the $C(B)$'s partition X. That (b)⇒(c) is then immediate from the following two claims.

Claim 1. Each $B \in \mathcal{B}_d(\epsilon)$ meets at most finitely many $C(B)$'s. Suppose by way of contradiction that $x_n \in C(B_n) \cap B$, $n \in \omega$. Let $x \in C(B)$. Then $d(x,x_n) < \epsilon$ for all n, so since $x_n \in C(B_n)$ we have $x \in B_n$ for all n, contradicting (b).

Claim 2. Each $C(B)$ meets only finitely many B's in $\mathcal{B}_d(\epsilon)$. To see this, suppose $x_n \in C(B) \cap B_n$ for all n. Let $y \in C(B)$, and pick $z_n \in C(B_n)$. Then $d(x_n, z_n) < \epsilon$, so $z_n \in B_d(x_n,\epsilon) = B$. This implies $d(y,z_n) < \epsilon$ for all n, whence $y \in B_n$ for all n, again contradicting (b).

It remains to prove that (c)⇒(d). Assume $\mathcal{B}_d(\epsilon)$ is star-finite. For each $p \neq q \in X$ with $d(p,q) < \epsilon$, let

$$U(p,q) = \bigcap\{B \in \mathcal{B}_d(\epsilon) : \{p,q\} \subset B\}.$$

We first show that $d(x,y) < \epsilon$ for any two points $x,y \in U(p,q)$. Clearly $U(p,q) \subset B_d(p,\epsilon) \cap B_d(q,\epsilon)$. It follows that $p,q \in B_d(x,\epsilon)$, whence $y \in U(p,q) \subset B_d(x,\epsilon)$. So $d(x,y) < \epsilon$ as claimed.

Now let $\mathcal{U}' = \{U(p,q) : p \neq q \text{ and } d(p,q) < \epsilon\}$. Since each member of $\mathcal{U}'$ is a finite intersection of members of a star-finite collection, $\mathcal{U}'$ is also star-finite. It covers all non-isolated points, but possibly not all isolated points, so we let $\mathcal{U} = \mathcal{U}' \cup (\{\{x\} : x \in X \setminus \cup\mathcal{U}'\})$, which of course is also star-finite. Consider any $x \in X$. By the previous paragraph, we have $st(x,\mathcal{U}) \subset B_d(x,\epsilon)$. Suppose $y \in B_d(x,\epsilon) \setminus \{x\}$. Then $x,y \in U(x,y) \in \mathcal{U}$, so $y \in st(x,\mathcal{U})$. Thus $st(x,\mathcal{U}) = B_d(x,\epsilon)$, which completes the proof. □

The proof of the next lemma is similar to the proof of the equivalence of 1.1(b) and 1.1(c) and hence is omitted.

Lemma 1.2. *Let (X,d) be a metric space, and $\epsilon > 0$. Then the following are equivalent:*

(a) *$\mathcal{B}_d(\epsilon)$ is point-countable;*
(b) *$\mathcal{B}_d(\epsilon)$ is star-countable.*

Let $\mathcal{N}$ be as defined in the introduction. We note the following easy lemma.

Lemma 1.3. *$\mathcal{N}$ is closed under subspaces and countable products.*

Proof. If the metric d witnesses that $X \in \mathcal{N}$, clearly the restriction of d to any subspace X' witnesses that $X' \in \mathcal{N}$. To prove closure under countable products, suppose $X_0, X_1, \dots$ are in $\mathcal{N}$, witnessed by $d_0, d_1, \dots$. Let $d_i'(x,y)$ be the minimum of $1/2^i$ and $d_i(x,y)$. It is easy to check that d_i' also witnesses that X_i is in $\mathcal{N}$, and that $d(\vec{x},\vec{y}) = \max_{i\in\omega} d_i'(x_i,y_i)$ witnesses that $\Pi_{i\in\omega}X_i \in \mathcal{N}$. $\square$

Since discrete spaces are obviously in $\mathcal{N}$, and the Hilbert cube is in $\mathcal{N}$ by Nagata's result mentioned in the Introduction , it follows from Lemma 1.3 that, for any cardinal κ, any subspace of $\kappa^\omega \times [0,1]^\omega$ is in $\mathcal{N}$ (where κ has the discrete topology). The next result, our main one, shows that this characterizes $\mathcal{N}$.

Theorem 1.4. *The following are equivalent for a metrizable space X:*

(i) *$X \in \mathcal{N}$, i.e., X admits a metric such that, for all $\epsilon > 0$, $\mathcal{B}(\epsilon)$ is locally finite;*
(ii) *X admits a metric such that, for all $\epsilon > 0$, $\mathcal{B}(\epsilon)$ is point-finite;*
(iii) *X admits a metric such that, for all $\epsilon > 0$, $\mathcal{B}(\epsilon)$ is star-finite;*
(iv) *X admits a metric such that, for all $\epsilon > 0$, $\mathcal{B}(\epsilon)$ is point-countable;*
(v) *X admits a metric such that, for all $\epsilon > 0$, $\mathcal{B}(\epsilon)$ is star-countable;*
(vi) *There is a sequence of star-finite open covers $\mathcal{G}_n$, $n < \omega$, of X such that, for each n, $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$ and, for each $x \in X$, $\{st(x,\mathcal{G}_n) : n \in \omega\}$ is a base at x.*
(vii) *There is a sequence of star-countable open covers $\mathcal{G}_n$, $n < \omega$, of X such that, for each n, $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$ and, for each $x \in X$, $\{st(x,\mathcal{G}_n) : n \in \omega\}$ is a base at x.*
(viii) *X is homeomorphic to a subspace of $\kappa^\omega \times [0,1]^\omega$ for some cardinal κ.*

Proof. By the previous lemmas, (i)-(iii) are equivalent, as are (iv) and (v). As noted following the proof of Lemma 1.3, we also have (viii)$\Rightarrow$(i). Furthermore, it is clear that (iii) implies (iv),(v), and (vi), and (v) and (vi) both imply (vii). We now prove (vii) implies (viii); the theorem then follows from this and the aforementioned implications.

Let $\{\mathcal{G}_n\}_{n\in\omega}$ satisfy condition (vii). For $U, V \in \mathcal{G}_n$, define $U \sim_n V$ iff there is a finite sequence $U_0, U_1, \dots, U_k$ of elements of $\mathcal{G}_n$ with $U = U_0$, $V = U_k$, and $U_i \cap U_{i+1} \neq \emptyset$ for all $i < k$. Then each equivalence class $[U]_n$ is countable, and the collection $\mathcal{P}_n = \{\cup[U]_n : U \in \mathcal{G}_n\}$ is a clopen partition of X. Since $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$, $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$.

Let $\mathcal{P}_\emptyset = \mathcal{P}_0 = \{P_{\langle\alpha\rangle} : \alpha < \kappa_\emptyset\}$ for some cardinal $\kappa_\emptyset$. Then for each $\alpha < \kappa_\emptyset$, let $\mathcal{P}_{\langle\alpha\rangle} = \{P \in \mathcal{P}_1 : P \subset P_{\langle\alpha\rangle}\} = \{P_{\langle\alpha,\beta\rangle} : \beta < \kappa_{\langle\alpha\rangle}\}$ for some cardinal $\kappa_{\langle\alpha\rangle}$. If $\mathcal{P}_{\langle\alpha_0,\alpha_1,\dots,\alpha_{n-1}\rangle} = \{P_{\langle\alpha_0,\alpha_1,\dots,\alpha_{n-1},\alpha_n\rangle} : \alpha_n < \kappa_{\langle\alpha_0,\alpha_1,\dots,\alpha_{n-1}\rangle}\}$ is defined, then let $\mathcal{P}_{\langle\alpha_0,\alpha_1,\dots,\alpha_n\rangle} = \{P \in \mathcal{P}_{n+1} : P \subset P_{\langle\alpha_0,\alpha_1,\dots,\alpha_n\rangle}\} = \{P_{\langle\alpha_0,\alpha_1,\dots,\alpha_n,\alpha_{n+1}\rangle} : \alpha_{n+1} < \kappa_{\langle\alpha_0,\alpha_1,\dots,\alpha_n\rangle}\}$. For each finite sequence σ for which P_σ has been defined, select $G_\sigma \in \mathcal{G}_k$, where $k = |\sigma| - 1$, such that $P_\sigma = \cup[G_\sigma]_k$. Let $\mathcal{G}_\sigma$ denote the equivalence class $[G_\sigma]_k$; since equivalence classes are countable, we may index $\mathcal{G}_\sigma$ by $\{G_{\sigma,j} : j \in \omega\}$.

Let κ be the supremum of all the defined κ_s's. For any $\sigma \in \kappa^{<\omega}$ for which P_σ has not been defined, let $P_\sigma = \emptyset = \mathcal{G}_\sigma$. Note that for each n,

$$\mathcal{G}_n = \bigcup\{\mathcal{G}_\sigma : \sigma \in \kappa^{n+1}\}$$

.

Since X is metrizable, hence paracompact, there is a locally finite closed shrinking $\{H(G) : G \in \mathcal{G}_n\}$ of $\mathcal{G}_n$. For $G \in \mathcal{G}_n$, let $f_G : X \to [0, 1/2^n]$ be continuous such that $f_G(H(G)) = \{1/2^n\}$ and $f_G(X \setminus G) = \{0\}$. Let $\mathcal{F}_n = \{f_G : G \in \mathcal{G}_n\}$.

Now we can define our desired embedding θ, which will map X into $\kappa^\omega \times I^{\omega\times\omega}$. Pick $x \in X$. Then for each n, there is a unique $\sigma_n^x \in \kappa^{n+1}$ with $x \in P_{\sigma_n^x}$. Note that σ_{n+1}^x extends σ_n^x, so there is a unique $\tau^x \in \kappa^\omega$ such that $\tau^x \restriction (n+1) = \sigma_n^x$ for all n. Then let

$$\theta(x) = \{\tau^x\} \times \{\langle f_{G_{\tau^x\restriction(k+1),j}}(x)\rangle_{(k,j)\in\omega\times\omega}\}.$$

Let us check that θ is one-to-one. Suppose $x \neq y$, yet $\theta(x) = \theta(y)$. Then $\tau^x = \tau^y = \tau \in \kappa^\omega$, i.e., x and y are always in the same member of the partitions $\mathcal{P}_n$. Choose k sufficiently large so that $y \notin st(x, \mathcal{G}_k)$, and choose $G \in \mathcal{G}_k$ with $x \in H(G)$. Then for some j, $G = G_{\tau\restriction(k+1),j}$, and $f_{G_{\tau\restriction(k+1),j}}(x) = 1/2^k$ while $f_{G_{\tau\restriction(k+1),j}}(y) = 0$. Thus $\theta(x)$ and $\theta(y)$ differ on coordinate (k, j), contradiction.

Now we show that θ is continuous. Let $\theta(x) = (\theta_1(x), \theta_2(x))$, where $\theta_1(x) \in \kappa^\omega$ and $\theta_2(x) \in I^{\omega\times\omega}$. Suppose a sequence of points x_n, $n < \omega$, converges to $x \in X$. Fix $k \in \omega$. Then for sufficiently large n, $x_n \in P_{\tau^x\restriction(k+1)}$ and hence also $\tau^{x_n} \restriction (k+1) = \tau^x \restriction (k+1)$. It easily follows from this that as n gets large, $\theta_1(x_n)(k) = \tau^{x_n}(k)$ converges to (in fact, is equal to) $\theta_1(x)(k) = \tau^x(k)$ and that, if we also fix j, $\theta_2(x_n)(k, j)$ converges to $\theta_2(x)(k, j)$. Thus θ is continuous.

It remains to prove that θ is a homeomorphism onto its range. To this end, let O be open in X; it will suffice to show that $\theta(O)$ is relatively open in $\theta(X)$. Take a point $\theta(x) \in \theta(O)$. We need to find a neighborhood V of $\theta(x)$ in $\kappa^\omega \times I^{\omega\times\omega}$ such that $V \cap \theta(X) \subset \theta(O)$.

Choose k sufficiently large so that $st(x, \mathcal{G}_k) \subset O$. Let $J = \{j \in \omega : x \in H(G_{\tau^x\restriction(k+1),j})\}$; obviously, J is finite. Let $\epsilon = 1/2^k$, and let $k' = k + 1$. Then the set

$$V = \{(z_1, z_2) \in \kappa^\omega \times I^{\omega\times\omega} : z_1 \restriction k' = \tau^x \restriction k' \ \&\ \forall j \in J(|z_2(k, j) - \theta_2(x)(k, j)| < \epsilon)\}$$

is an open neighborhood of $\theta(x)$.

It remains to show $V \cap \theta(X) \subset \theta(O)$. Suppose $\theta(y) \in V \cap \theta(X) \setminus \theta(O)$. Then $y \notin O$. Since $\theta(y) \in V$, $\tau^y \restriction (k+1) = \tau^x \restriction (k+1)$. Let $G \in \mathcal{G}_k$ with $x \in H(G)$. Then $G \in [G_{\tau^x\restriction(k+1)}]_k$ and so $G = G_{\tau^x\restriction(k+1),j}$ for some j; of course, $j \in J$. Since $y \notin O$, $y \notin G$. Thus $\theta_2(y)(k, j) = f_{G_{\tau^y\restriction(k+1),j}}(y) = f_{G_{\tau^x\restriction(k+1),j}}(y) = f_G(y) = 0$, while $\theta_2(x)(k, j) = f_G(x) = 1/2^k$. But this contradicts $\theta(y) \in V$, and thus completes the proof. □

Corollary 1.5. *$\mathcal{N}$ is precisely the class of strongly metrizable spaces.*

Remark. Recall that a space is *strongly paracompact* if every open cover has a star-finite open refinement. It is well-known and easy to see that every strongly paracompact metrizable space is strongly metrizable and hence, by our result, is in $\mathcal{N}$. But not every member of $\mathcal{N}$ is strongly paracompact; e.g., it is known [N3] that $\omega_1^\omega \times (0, 1)$, which of course embeds in $\omega_1^\omega \times I^\omega$ and hence is strongly metrizable, is not strongly paracompact.

We also remark that Y. Hattori [H] obtained another characterization of strongly metrizable spaces in terms of a metric.

3. An Example

By Lemma 1.1, the only way the collection of ϵ-balls (ϵ fixed) can be locally finite is for this collection to be precisely the stars of some star-finite open cover. Recall that Nagata showed that every metrizable space admits a metric such that the collection of ϵ-balls is closure-preserving by constructing a metric such that the collection of ϵ-balls is precisely the collection of stars of some locally finite open cover. So it is natural to ask if this is the only way the collection of ϵ-balls can be closure-preserving. The following example shows that the answer is “no”.

Example. *There is a metric space (X, d) such that, for every $\epsilon > 0$, $\mathcal{B}_d(\epsilon)$ is closure-preserving but there is no locally finite open cover $\mathcal{G}(\epsilon)$ with $\mathcal{B}_d(\epsilon) = \{st(x, \mathcal{G}(\epsilon)) : x \in X\}$.*

Proof. Let the set X be $\omega \times \mathbb{R}$, viewed as a subset of the plane. For $x, y \in X$, denote the usual Euclidean distance between x and y by $|x - y|$. Then for $x = (n_x, r_x)$ and $y = (n_y, r_y)$ in X, define $d(x, y)$ to be $|x - y|$ if $x = y$, or if $n_x \neq n_y$, or if $n_x = n_y = n$ and $|x - y| > 1/2^n$. Let $d(x, y) = 1/2^n$ otherwise. It is easy to check that d is a metric on the set X, and that d generates the discrete topology on X. So any collection of subsets of X is closure-preserving, in particular $\mathcal{B}_d(\epsilon)$.

Fix $\epsilon > 0$. We aim to show that $\mathcal{B}_d(\epsilon)$ cannot be precisely the collection of stars of some locally finite open cover. To this end, choose n such that $1/2^n < \epsilon$, and note that the trace of $\mathcal{B}_d(\epsilon)$ on $\{n\} \times \mathbb{R}$ contains $\{\{n\} \times (x - \epsilon, x + \epsilon) : x \in \mathbb{R}\}$, i.e., it contains all open intervals on $\{n\} \times \mathbb{R}$ of length 2ϵ. Thus establishing the following claim will complete the proof.

Claim. There is no point-finite cover $\mathcal{G}$ of $\mathbb{R}$ such that every open interval of length 2ϵ is the union of some finite subcollection of $\mathcal{G}$.

Proof of Claim. Suppose $\mathcal{G}$ is such a point-finite cover of $\mathbb{R}$. Let $z \in \mathbb{R}$. There is a finite subcollection $\mathcal{G}_z$ of $\mathcal{G}$ such that $\cup \mathcal{G}_z = (z - 2\epsilon, z)$. Since $\mathcal{G}$ is point-finite, if we let

$$z' = sup(\cup\{G \in \mathcal{G}_z : sup(G) < z\})$$

then $z' < z$. Choose $q_z \in \mathbb{Q}$ between z' and z. Pick $G_z \in \mathcal{G}_z$ with $q_z \in G_z$. Note that $sup(G_z) = z$, hence $z \neq z'$ implies $G_z \neq G_{z'}$. But there must be q such that $q_z = q$ for uncountably many z, contradicting point-finiteness of $\mathcal{G}$ at q. $\square$

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