Covering compacta by discrete and other separated sets

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1 Introduction

I. Juhasz and J. van Mill [4] asked the following question, which was unsolved even for first countable spaces :

Question 1.1. *If X is a compact Hausdorff crowded space, is it true that X cannot be covered by fewer than $\mathfrak{c}$-many discrete subspaces?*

Juhasz and van Mill denote by $dis(X)$ the least cardinal of a cover of X by discrete subspaces. They show that $dis(X) \geq \mathfrak{c}$ (and hence the answer to Question 1.1 is positive) for any compact crowded hereditarily normal X. In fact, this follows from their stronger result that for such X, $rs(X) + ls(X) \geq \mathfrak{c}$, where $rs(X)$ (resp., $ls(X)$) is the least cardinal of a cover of X by right(resp., left)-separated subspaces.

Here we prove that the answer to Question 1.1 is positive, without any further assumptions. Indeed, this is a corollary to our more general result that the property of being the union of $\leq \kappa$-many discrete subspaces (i.e., $dis(X) \leq \kappa$) is preserved by perfect mappings, a result proven earlier for the case $\kappa = \omega$ by D. Burke and R. Hansell [1].

It is still not known if either $ls(X) \geq \mathfrak{c}$ or $rs(X) \geq \mathfrak{c}$ holds for any compact crowded X. In [4], it is noted that $rs(X)$ is at least $\mathfrak{m}$, tbe least cardinal of a cover of the real line by meager sets, and Juhasz and Szentmiklossy [5] showed that $ls(X) \geq \mathfrak{m}$ also. We obtain the partial result that both $rs(X) \geq \mathfrak{c}$ and $ls(X) \geq \mathfrak{c}$ hold for first countable crowded compacta, provided $\mathfrak{c}$ is a regular cardinal.

In [4], it is noted that any counterexample to Question 1.1 contains a separable counterexample which must have cardinality $\mathfrak{c}$. Their argument clearly works for the right and left-separated questions too.

2 Discrete subspaces

The purpose of this section is to prove:

Theorem 2.1. *No crowded compact Hausdorff space is the union of fewer than $\mathfrak{c}$-many discrete subspaces.*

Any crowded compact space admits a perfect mapping onto the unit interval, and the unit interval is not the union of less than $\mathfrak{c}$-many discrete subspaces. Hence this result follows easily from:

Theorem 2.2. *Let κ be an infinite cardinal. If X is the union of κ-many discrete subspaces, then so is any perfect image of X.*

The main lemma is:

Lemma 2.3. *Suppose $X = \cup\mathcal{D}$, where $\mathcal{D}$ is a collection of κ-many discrete subspaces, and let $K \subset X$ be compact. For each $D \in \mathcal{D}$, let $D' = \overline{D} \setminus D$. Then there is some finite $\mathcal{E} \subset \mathcal{D}$ and $E \in \mathcal{D} \setminus \mathcal{E}$ such that $\emptyset \neq K \cap \bigcap_{D\in\mathcal{E}} D' \cap \overline{E} \subset E$.*

Proof. Choose $D_0 \in \mathcal{D}$ such that $D_0 \cap K \neq \emptyset$. If $K \cap \overline{D_0} \subset D_0$, we are done (with $\mathcal{E} = \emptyset$ and $E = D_0$). If $K \cap \overline{D_0} \not\subset D_0$, this means K meets both D_0 and D'_0, and we continue an inductive construction as follows.

Suppose $D_\beta \in \mathcal{D}$ have been chosen for each $\beta < \alpha$ such that:

$$K \cap \bigcap_{\gamma<\beta} D'_\gamma \text{ meets both } D_\beta \text{ and } D'_\beta.$$

Then $K \cap \bigcap_{\gamma<\alpha} D'_\gamma \neq \emptyset$. Choose $D_\alpha \in \mathcal{D}$ such that

$$D_\alpha \cap K \cap \bigcap_{\gamma<\alpha} D'_\gamma \neq \emptyset.$$

If $D'_\alpha \cap K \cap \bigcap_{\gamma<\alpha} D'_\gamma \neq \emptyset$, we continue the induction, otherwise we stop.

Since $D \cap D' = \emptyset$ for each $D \in \mathcal{D}$, it follows that the D_β's are distinct and that $K \cap \bigcap_{D\in\mathcal{D}} D' = \emptyset$. Hence we must arrive at some stage δ in the induction such that the induction stops, i.e., we have

$$D_\delta \cap K \cap \bigcap_{\gamma<\delta} D'_\gamma \neq \emptyset$$

but

$$D'_\delta \cap K \cap \bigcap_{\gamma<\delta} D'_\gamma = \emptyset.$$

It follows from compactness of K that there is some finite $F \subset \delta$ such that

$$D'_\delta \cap K \cap \bigcap_{\gamma\in F} D'_\gamma = \emptyset.$$

But then the conclusion of the lemma holds with $\mathcal{E} = \{D_\gamma : \gamma \in F\}$ and $E = D_\delta$. □

Proof of Theorem 2.2. Suppose $X = \cup\mathcal{D}$, where each $D \in \mathcal{D}$ is discrete, and $|\mathcal{D}| = \kappa \geq \omega$. Let $f : X \to Y$ be perfect and onto. For each $y \in Y$, by the lemma there are finite $\mathcal{E}_y \subset \mathcal{D}$ and $E_y \in \mathcal{D} \setminus \mathcal{E}$ such that

$$\emptyset \neq f^{-1}(y) \cap \bigcap_{D \in \mathcal{E}_y} D' \cap \overline{E_y} \subset E_y.$$

It follows that if $f_y = f \restriction \bigcap_{D \in \mathcal{E}_y} D' \cap \overline{E_y}$, then $f_y^{-1}(y) \subset E_y$.

Now for each finite $\mathcal{E} \subset \mathcal{D}$ and $E \in \mathcal{D} \setminus \mathcal{E}$, let

$$Y(\mathcal{E}, E) = \{y \in Y : \mathcal{E}_y = \mathcal{E} \text{ and } E_y = E\}.$$

Then

$$Y = \bigcup\{Y(\mathcal{E}, E) : \mathcal{E} \in [\mathcal{D}]^{<\omega} \text{ and } E \in \mathcal{D} \setminus \mathcal{E}\}$$

so it suffices to show that each $Y(\mathcal{E}, E)$ is discrete.

Let $f' = f \restriction (\bigcap_{D \in \mathcal{E}} D' \cap \overline{E})$. For each $y \in Y(\mathcal{E}, E)$, we have $f_y = f'$ and $f'^{-1}(y) \subset E$. Since f' is the restriction of a perfect mapping to a closed subset, it is also perfect, and so is $f' \restriction f'^{-1}(Y(\mathcal{E}, E))$. So $Y(\mathcal{E}, E)$ is the perfect image of a discrete space, hence is discrete. □

Remark. Juhasz and Szentmiklossy[5] have recently used our Lemma 2.3 to obtain a result more general than Theorem 2.1, namely that if X is a compact Hausdorff space and $\chi(x, X) \geq \kappa$ for each $x \in X$, then $dis(X) \geq 2^\kappa$. They also ask the following interesting question, a positive answer to which would be even more general: Is $dis(X) \geq \Delta(X)$ true for any compact X? ($\Delta(X)$ is the least cardinal of a nonempty open set in X.)

3 Right and left separated subspaces

The purpose of this section is to prove that for any first countable crowded compact Hausdorff space X, we have $rs(X) \geq \mathfrak{c}$ and $ls(X) \geq \mathfrak{c}$, provided $\mathfrak{c}$ is regular. It is unsolved whether or not either first countability or the restriction on $\mathfrak{c}$ is a necessary assumption.

Theorem 3.1. *If the continuum $\mathfrak{c}$ is regular, then no crowded first-countable compact Hausdorff space can be covered by fewer than $\mathfrak{c}$-many right-separated subspaces.*

Proof. Let X be a first-countable crowded compactum, let $\kappa < \mathfrak{c}$, and suppose $X = \bigcup_{\alpha<\kappa} D_\alpha$, where each D_α is right-separated. Then $|X| = \mathfrak{c}$, so $w(X) \leq \mathfrak{c}$. We assume $\mathfrak{c}$ is regular. Then $|D_\alpha| = \mathfrak{c}$ for some α, so it follows that $w(X) = \mathfrak{c}$. Since κ must be uncountable by ???, we have that $\mathfrak{c} > \omega_1$.

We may view X as a subspace of $I^\mathfrak{c}$, where $I = [0, 1]$. For $\alpha < \mathfrak{c}$, let π_α be the projection on I^α, and for $x, y \in X$, define $x \sim_\alpha y$ iff $\pi_\alpha(x) = \pi_\alpha(y)$. Let $[x]_\alpha$ denote the $\sim_\alpha$ equivalence class of x. Let

$$T = \{[x]_\alpha : \alpha < \mathfrak{c}, x \in X\}.$$

Then T is a tree under reverse inclusion consisting of closed subsets of X. Let $L_\alpha = \{[x]_\alpha : x \in X\}$. No L_α contains only singleton elements, for otherwise π_α would be one-to-one, which would imply $w(X) \le |\alpha|$.

Let $\mathcal{B}$ be a countable base for I. For each finite subset F of $\mathfrak{c}$ and $\sigma : F \to \mathcal{B}$, let $V(F,\sigma)$ denote the basic open set in X determined by F and σ, i.e.,

$$V(F,\sigma) = \{x \in X : \forall\gamma \in F(x(\gamma) \in \sigma(\gamma))\}.$$

The following easily verified fact will be used: If $t \in L_\alpha$, and U is open in X and contains t, then there are a finite subset F of α and a map $\sigma : F \to \mathcal{B}$ such that $t \subset V(F,\sigma) \subset U$.

Claim 1. There is no stationary $S \subset \mathfrak{c}$ and $t_\alpha \in L_\alpha$, $\alpha \in S$, such that each t_α is countable, and $\{t_\alpha : \alpha \in S\}$ is pairwise-disjoint.

Proof of Claim 1. Suppose otherwise. Since $\mathfrak{c} > \omega_1$, we may assume each t_α has the same (countable) Cantor-Bendixson height and the same (finite) number of top points (points of maximal height), and that this height and number of top points is minimal for all counterexamples to Claim 1 for all counterexamples to the theorem embedded in $I^\mathfrak{c}$.

We need the following probably well-known (though we don't have a reference) set-theoretic result:

Lemma 3.2. *Suppose S is a stationary subset of a regular uncountable cardinal κ, and ∇ is a well-order of S. Then S contains a stationary T such that ∇ agrees with the natural order on T.*

Proof. We first claim that there is a stationary $S' \subset S$ such that, for each $\alpha \in S'$, the set of ∇ predecessors of α is non-stationary. To see this, first observe we can take $S' = S$ if no $\alpha \in S$ has stationary many ∇ predecessors. Otherwise, let $\delta \in S$ be ∇-least with stationary many predecessors, and take $S' = \{\alpha \in S : \alpha \nabla \delta\}$.

Now, for each $\alpha \in S'$, there is a c.u.b. C_α which misses all ∇ predecessors of α. Let C be the diagonal intersection of the C_α's. It is easy to check that taking $T = C \cap S'$ works. □

For $\alpha \in S$, let $x_0(\alpha) \in t_\alpha$ be a top point of t_α. It follows from the lemma that there is a stationary subset S' of S such that $\{x_0(\alpha) : \alpha \in S'\}$ is right-separated by the ordering of the indices. Let $U_0(\alpha)$ be a regular open in X neighborhood of $x_0(\alpha)$ such that $x_0(\alpha') \notin U_0(\alpha)$ for any $\alpha' \ge \alpha$, $\alpha' \in S'$.

We note that the set $N = \{\alpha \in S' : U_0(\alpha) \supset t_\alpha\}$ is non-stationary. Suppose otherwise. Then for each $\alpha \in N$, there is a finite subset F_α of α and a map $\sigma_\alpha : F_\alpha \to \mathcal{B}$, such that $t_\alpha \subset V(F_\alpha, \sigma_\alpha) \subset U_0(\alpha)$. Applying the pressing down lemma, there are F and σ such that $\{\alpha \in N : F_\alpha = F \text{ and } \sigma_\alpha = \sigma\}$ is a stationary subset N' of N. But then $U_0(\alpha) \supset V(F,\sigma) \supset t_\beta$ for any $\alpha, \beta \in N'$, a contradiction to the $U_0(\alpha)$'s witnessing right-separation of the $x_0(\alpha)$'s.

So we can pass to a stationary subset S'' of S' such that $t_\alpha \setminus U_0(\alpha) \ne \emptyset$ for each $\alpha \in S''$, and moreover, all $t_\alpha \setminus U_0(\alpha)$'s have the same Cantor-Bendixson height and the same number of top points. Note that one or the other is less than it was for t_α. Let $x_1(\alpha)$ be a top point of $t_\alpha \setminus U_0(\alpha)$, and pass to a stationary subset S''' of S'' such that the corresponding $x_1(\alpha)$'s are right-separated by their indices, witnessed by the regular open neighborhood $U_1(\alpha)$ of $x_1(\alpha)$.

Similar to the above, we claim that $N = \{\alpha \in S''' : U_0(\alpha) \cup U_1(\alpha) \supset t_\alpha\}$ is non-stationary. Suppose otherwise. For each $\alpha \in N$, let $U_0'(\alpha)$ and $U_1'(\alpha)$ be regular open sets which cover t_α and whose closures are contained in $U_0(\alpha)$ and $U_1(\alpha)$, respectively.

By a similar pressing down argument as before, there are a finite $F \subset \mathfrak{c}$ and $\sigma : F \to \mathcal{B}$, and a stationary $N' \subset N$, such that $U_0'(\alpha) \cup U_1'(\alpha) \supset V(F,\sigma) \supset t_\beta$ for any $\alpha, \beta \in N'$. Fix $\alpha \in N'$. Then for any $\beta \in N'$, $\beta > \alpha$, we have $t_\beta \subset U_0'(\alpha) \cup U_1'(\alpha)$, but t_β is not contained in either $U_0(\alpha)$ or $U_1(\alpha)$. Note that $t_\beta \cap \overline{U_0'(\alpha)}$ and $t_\beta \setminus U_0(\alpha)$ are nonempty, and at least one of them has either lower height than t_β or fewer top points. Also, $\overline{U_0'(\alpha)}$ and $X \setminus U_0(\alpha)$ are regular closed, hence also crowded, subsets of X. Thus one of these two contradicts the minimality of the height and number of top points for any example in $I^{\mathfrak{c}}$ (with respect to the stationary set $N' \setminus \{\alpha\}$).

Thus we can pass to a stationary subset $S^{(4)}$ of S''' such that $t_\alpha \setminus (U_0(\alpha) \cup U_1(\alpha)) \neq \emptyset$ for each $\alpha \in S^{(4)}$, and continue as in the previous step, choosing a top point $x_2(\alpha) \in t_\alpha \setminus (U_0(\alpha) \cup U_1(\alpha))$, etc. In this way we produce a sequence $x_0(\alpha_0), x_1(\alpha_1), \dots$ of points of nonincreasing height, such that every constant subsequence of their heights is finite. This is a contradiction which finishes the proof of the Claim 1.

Let

$$C_\alpha = \{[x]_\alpha : 2 \leq |[x]_\alpha| < \mathfrak{c} \text{ or } |[x]_\alpha| = 1 \text{ and } \alpha \text{ is least such that } |[x]_\alpha| = 1\}.$$

Claim 2. There is a club $D \subset \mathfrak{c}$ such that $C_\alpha = \emptyset$ for every $\alpha \in D$.

Proof of Claim 2. If not, then there is a stationary $S \subset \mathfrak{c}$ such that, for each $\alpha \in S$, there is $t_\alpha \in C_\alpha$. Note that for any $\alpha \in S$, since t_α is countable, there is $\gamma_\alpha < \mathfrak{c}$ such that $t_\beta \not\subset t_\alpha$ (and so $t_\beta \cap t_\alpha = \emptyset$) for any $\beta > \gamma_\alpha$. It follows that there is some stationary subset of S on which that t_α's are disjoint. But this contradicts Claim 1.

Having established the claims, we finish the proof of the theorem. Let $D \subset \mathfrak{c}$ be a club as in Claim 2. Let $\{d_\alpha : \alpha < \mathfrak{c}\}$ be a continuous enumeration of D. Choose any nonsingleton $t_0 \in L_{d_0}$. Then $|t_0| = \mathfrak{c}$, so t_0 contains a counterexample, and hence has a nonsingleton successor t_1 in L_{d_1}. Note that $|t_1| = \mathfrak{c}$ since $d_1 \in D$. Suppose nonsingleton $t_\beta \in L_{d_\beta}$ have been defined for all $\beta < \alpha$, where $\alpha < \mathfrak{c}$. If α is the successor of α', choose a nonsingleton successor $t_\alpha \in L_{d_\alpha}$ of $t_{\alpha'}$. If α is a limit, then $t_\alpha = \bigcap_{\beta<\alpha} t_\beta$ is in L_{d_α}, hence again has cardinality $\mathfrak{c}$ (if t_α were countable, it would be in C_{d_α}, contradiction). In this way we construct a decreasing $\mathfrak{c}$-chain of nonempty $\mathfrak{c}$-sized closed subsets of X whose intersection is a single point, contradicting first-countability. □

We now show that with a bit more work the above theorem holds for left-separated subspaces as well.

Theorem 3.3. *If the continuum $\mathfrak{c}$ is regular, then no crowded first-countable compact Hausdorff space can be covered by fewer than $\mathfrak{c}$-many left-separated subspaces.*

Proof. Suppose X is a counterexample. Clearly it suffices to prove the analogue of Claim 1 of the proof of the previous theorem. Instead of using Cantor Bendixson height, we will use the fact that any first countable compact scattered space is homeomorphic to some countable successor ordinal. So, suppose S is a stationary subset of $\mathfrak{c}$ and t_α, $\alpha \in S$, satisfy the conditions of Claim 1. W.l.o.g., there is a countable ordinal δ^0 such that $t_\alpha \cong \delta^0+1$ for every $\alpha \in S$. For $\delta \leq \delta^0$, we will denote by δ_α the copy of the point (under some fixed homeomorphism) of δ in t_α.

The set $\{\delta^0_\alpha : \alpha \in S\}$ is the union of fewer than $\mathfrak{c}$-many left-separated subsets, and $\mathfrak{c}$ is assumed to be regular, so there is some stationary subset S_0 of S that left-separates, that is:

$$\forall \alpha \in S_0(\delta^0_\alpha \notin \overline{\{\delta^0_\beta : \beta \in \alpha \cap S_0\}}).$$

Choose an open nbhd $U_0(\alpha)$ of δ^0_α witnessing the left-separation. W.l.o.g, by passing to a stationary subset if necessary, $U_0(\alpha) \cap \delta^0_\alpha + 1 = (\delta^1_\alpha, \delta^0_\alpha]$ for some $\delta^1 < \delta^0$.

Let us denote by "$\forall^* \alpha \in S$..." the statement "$\exists$ a club C such that $\forall \alpha \in C \cap S$...".

Next pass to a stationary $S_1 \subset S_0$ such that either

(1) $U_0(\alpha) \cap \{\delta^1_\beta : \beta \in \alpha \cap S_1\} = \emptyset$; or

(2) $\forall$ stationary $T \subset S_1$, $\forall^* \alpha \in T(\{\beta \in \alpha \cap T : \delta^1_\beta \in U_0(\alpha)\}$ is cofinal in α.

Let us see that such S_1 can be chosen. Suppose no stationary $S_1 \subset S_0$ satisfies (1). We claim that taking $S_1 = S_0$ works to satisfy (2). Suppose otherwise. Then there is a stationary $T \subset S_0$ such that for stationarily many $\alpha \in T$, the set $\{\beta \in \alpha \cap T : \delta^1_\beta \in U_0(\alpha)\}$ is bounded below α. Then by a pressing down argument, some stationary subset T' of T satisfies (1), so we have a contradiction.

Observe that for every stationary subset of S_1, either (1) or (2) is satisfied too, depending on whether S_1 satisfies (1) or (2).

Next pass to a stationary $S_2 \subset S_1$ such that either

(3) $\forall \alpha \in S_2(\delta^1_\alpha \notin \overline{\{\delta^0_\beta : \beta \in \alpha \cap S_2\}})$; or

(4) $\forall$ stationary $T \subset S_2$, $\forall^* \alpha \in T(\delta^1_\alpha \in \overline{\{\delta^0_\beta : \beta \in \alpha \cap T\}})$.

Now take a stationary $S_3 \subset S_2$ that left-separates $\{\delta^1_\alpha : \alpha \in S_3\}$, and choose an open nbhd $U_1(\alpha)$ of δ^1_α witnessing this left-separation, so that if (3) above holds, then $U_1(\alpha) \cap \{\delta^0_\beta : \beta \in \alpha \cap S_3\} = \emptyset$. Similar to the choice of $U_0(\alpha)$, we can assume there is some $\delta^2 < \delta^1$ such that $U_1(\alpha) \cap t_\alpha = (\delta^2_\alpha, \delta^1_\alpha]$.

Now pass to a stationary $S_4 \subset S_3$ such that the analogue of condition (1) or (2) holds with respect to $\delta^2, U_1(\alpha)$ in place of $\delta^1, U_0(\alpha)$, and condition (3) or (4) holds with δ^2, δ^1 in place of δ^1, δ^0; then pass to a stationary $S_5 \subset S_4$ such that the analogues of (1) or (2), and (3) or (4), hold with respect to δ^2, δ^0. Continue in like manner, defining $\delta^0 > \delta^1 > \ldots$ and $U_0(\alpha), U_1(\alpha), \ldots$ until a stage n is reached such that $t_\alpha \subset \bigcup_{i \le n} U_i(\alpha)$.

At this stage we also have defined a stationary S^* such that for every $i \le n$, $U_i(\alpha)$, $\alpha \in S^*$, witness left-separation of $\{\delta^i_\alpha : \alpha \in S^*\}$, and for every $i \ne j \le n$, if $i > j$ then the analogue of (1) or (2) holds with δ^1 replaced by δ^i and $U_0(\alpha)$ replaced by $U_j(\alpha)$ and S_1 replaced by S^*, while if $i < j$ then the analogue of (3) or (4) holds with similar replacements. Next define a relation $\to$ on $n+1$ as follows: put $i \to j$ if $i > j$ and (2) holds, or if $i < j$ and (4) holds. We note that $\to$ satisfies the following for all $i, j, k \le n$:

(i) Either $i \to j$ or $j \to i$ is false;

(ii) If $i \to j$, $j \to k$, and $i < j$, then $i \to k$.

Let us check (i). Suppose $i < j$ and both $i \to j$ and $j \to i$ hold. Then the analogue of (4) holds with δ^1, δ^0 replaced by δ^j, δ^i; hence the set $T = \{\alpha \in S^* : \delta^j_\alpha \in \overline{\{\delta^i_\beta : \beta \in \alpha \cap S^*\}}\}$ is stationary. Then, since the analogue of (2) holds with δ^1 replaced by δ^j and $U_0(\alpha)$ replaced by $U_i(\alpha)$, there are $\beta < \alpha \in T$ with $\delta^j_\beta \in U_i(\alpha)$. But δ^j_β is in the closure of prior δ^i_γ's, contradicting that $U_i(\alpha)$ witnesses left-separation of the δ^i's.

To check (ii), first note that $k \ne i$, else (i) would be violated. There are several other cases, each taken care of in a similar manner to the checking of (i). We are assuming $i < j$, so δ^j's are limit points of prior δ^i's (we use this phrasing as a shorthand for saying that a certain analogue of (4) holds). If $j < k$, then the δ^k's are limit points of prior δ^j's; so we put these facts together to see that the δ^k's are limits of prior δ^i's, whence $i \to k$. On the other hand, if $k < j$, then the $U_k(\alpha)$'s contain prior δ^j's, hence contain prior δ^i's too. If $k < i$, this clearly implies $i \to k$. If $k > i$, then the $U_k(\alpha)$'s would not be chosen to contain any prior δ^i's unless it had to, i.e., unless δ^k_α was a limit of them. But then again we have $i \to k$.

Now we use (i) and (ii) to establish:

(iii) There is $k \le n$ such that $k \to j$ is false for every $j \le n$.

Suppose otherwise. Let $k_0 = 0$, and let k_1 be least such that $k_0 \to k_1$. Let k_2 be least such that $k_1 \to k_2$. From (i) and (ii) and minimality of k_1, it easily follows that $k_2 > k_1$. Then let k_3 be least such that $k_2 \to k_3$ and note again that $k_3 > k_2$. This can't go on indefinitely, so we obtain a contradiction which proves (iii).

Finally we are set up to finish the proof of the theorem (by finishing the proof of the analogue of Claim 1 of the previous theorem). Let $k \le n$ satisfy (iii). It follows that for each $\beta < \alpha \in S^*$ and each $j \le n$, we have that $\delta^k_\beta \notin U_j(\alpha)$. For if $k > j$, this is true since $j \to k$ fails so the appropriate analogue of condition

(1) must hold; if $k < j$ this is true since an analogue of (3) must hold and therefore we chose $U_j(\alpha)$ to miss prior δ^k's; and if $j = k$ this holds because the $U_j(\alpha)$'s witness left-separation of the δ^j's.

Let $V_\alpha = \bigcup_{i \le n} U_i(\alpha)$. Then $t_\alpha \subset V_\alpha$, but for each $\beta < \alpha \in S^*$, $\delta^k_\beta \notin V_\alpha$ and so $t_\beta \not\subset V_\alpha$. However, a pressing down argument as done in the proof of the previous result shows that there is an open set V and a stationary $T \subset S^*$ such that $t_\alpha \subset V \subset V_\alpha$ for every $\alpha \in T$, from which we easily obtain a contradiction. □

The same argument gets the following:

Theorem 3.4. *Suppose $\mathfrak{c}$ is regular, and X is a compact Hausdorff crowded space. If there is a cardinal κ such that $\kappa^+ < \mathfrak{c}$, and every closed subset of X either has cardinality $\le \kappa$ or $\ge \mathfrak{c}$, then X is not the union of fewer than $\mathfrak{c}$-many left-separated or right-separated subspaces.*

Proof. Modify the proof of Claim 1 of the previous two results as follows.

New Claim 1. There is no stationary $S \subset \mathfrak{c}$ and $t_\alpha \in L_\alpha$, $\alpha \in S$, and $\delta < \mathfrak{c}$, such that each t_α is scattered with Cantor-Bendixson height δ and $\{t_\alpha : \alpha \in S\}$ is pairwise-disjoint.

The proof in the situation of Theorem 1 is the same, while for Theorem 2 one needs to induct on the height and number of top points, as was done in Theorem 1.

For Claim 2, redefine C_α as follows:

$$C_\alpha = \{[x]_\alpha : |[x]_\alpha| < \mathfrak{c} \text{ and } \alpha \text{ is least such that } |[x]_\alpha| < \mathfrak{c}\}.$$

It follows that if $t_\alpha \in C_\alpha$ and $t_\beta \in C_\beta$ with $\alpha \neq \beta$, then $t_\alpha \cap t_\beta = \emptyset$. So we avoid having to pass to a stationary subset to get $t_\alpha \in C_\alpha$, $\alpha \in S$, pairwise-disjoint. Each $t_\alpha \in C_\alpha$ is scattered, and by condition (i), $ht(t_\alpha) < \kappa^+$. Since $\kappa^+ < \mathfrak{c}$, we can pass to a stationary set in which the t_α's all have the same height.

Now the completion of the proof of Theorem 1 obtains the contradiction by constructing a decreasing $\mathfrak{c}$-sequence of sets of size $\mathfrak{c}$ whose intersection is a singleton p. In the same way, we can construct, also by choosing only t's in levels of the club D, a complete binary tree of height $\mathfrak{c}$ contained in T. By the remark at the end of the introduction, this gives a contradiction since we may assume $|X| = \mathfrak{c}$. □

Corollary 3.5. *Suppose $\mathfrak{c}$ is regular, and X is a compact Hausdorff crowded space with $\chi(X) \le \kappa$. If $\mathfrak{c} = o_1$ or $\kappa^+ < \mathfrak{c}$, then X is not the union of fewer than $\mathfrak{c}$-many left or right-separated subspaces.*

References

[1] D. Burke and R. Hansell, *Perfect maps and relatively discrete collections*, Papers on General topology and applications (Amsterdam, 1994), 54–56, Ann. New York Acad. Sci., 788, New York.

[2] J. Gerlits, I. Juhász, and Z. Szentmiklóssy, *Two improvements on Tkačenko's addition theorem*, preprint.

[3] G. Gruenhage, *Metrizability number and perfect maps*, Questions Answers Gen. Topology 20 (2002), no. 2, 95–100.

[4] I. Juhász and J. van Mill, *Covering compacta by discrete subspaces*, Topology and its Applications, to appear

[5] I. Juhász and Z. Szentmiklóssy, *A strengthening of the Čech-Pospišil theorem*, preprint.