PRODUCTS OF COZERO COMPLEMENTED SPACES

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Abstract. We answer several questions of Levy and Shapiro, and Henriksen and Woods, on products of cozero complemented spaces.

0. Introduction

We assume all spaces are Tychonoff. Recall that a subset H of a space X is a *zero set* if there is a continuous real-valued $f : X \to \mathbb{R}$ with $H = f^{-1}(0)$, and $U \subset X$ is a *cozero set* if $X \setminus U$ is a zero set. A space X is *cozero complemented* if, given any cozero set U, there is a disjoint cozero set V such that $U \cup V$ is dense in X. This property is of interest due to connections with the algebraic properties of the ring $C(X)$ of continuous real-valued functions on X. See [HW] for a concise survey of this connection, which has long been known. Here we only mention that X is cozero complemented iff the space of minimal prime ideals of $C(X)$ is compact iff the ring of fractions of $C(X)$ is von Neumann regular.

In a recent preprint, R. Levy and J. Shapiro[LS] study cozero complemented spaces (under the name *z-good*) and ask a number of questions about them, including:

(1) If X and Y are cozero-complemented, must $X \times Y$ be too?
(2) If X is cozero complemented, and M is metrizable, must $X \times M$ be cozero complemented?
(3) If $X \times Y$ is cozero complemented, must both X and Y be too?

M. Henriksen and G. Woods[HW] answer the first question in the negative by showing that for an uncountable discrete space D, the Stone-Čech compactification βD of D is cozero complemented but $\beta D \times \beta D$ is not. They ask a version of (2) above in case X is βD:

(4) If D is discrete and M metrizable, must $\beta D \times M$ be cozero complemented?

They show that the answer is positive when M is locally separable.

They then ask about the case of *ccc* spaces. Recall that X satisfies the *countable chain condition*, abreviated *ccc*, if every pairwise-disjoint collection of open subsets of X is countable. Any *ccc* space is easily seen to be cozero complemented [HW], so it is consistent with ZFC (e.g., it is implied by Martin's Axiom together with the negation of the Continuum Hypothesis; see, e.g., [K]) that any product of *ccc* spaces is cozero complemented. Henriksen and Woods ask:

(5) Is it true in ZFC that any product of *ccc* spaces is cozero complemented?

2000 *Mathematics Subject Classification.* 54B10, 54C40.

The particular case of Souslin lines is mentioned in connection with this question. They also ask:

(6) If X is cozero complemented, and Y is separable, must $X \times Y$ be cozero complemented?
(7) If X is cozero complemented, and Y is a P-space, must $X \times Y$ be cozero complemented?

(Recall that a *P-space* is a space in which G_δ-sets are open. It follows that cozero sets are clopen, and hence all P-spaces are cozero complemented[HW].)

In this note, we show the following:

(I) If X is cozero complemented, and M is a separable metrizable space, then $X \times M$ is cozero complemented;
(II) If X is the one-point Lindelöfization of a discrete space of cardinality ω_2, and M is the Baire metric space ω_1^ω, then $X \times M$ is not cozero complemented;
(III) For any space X, there is a metrizable space M such that $X \times M$ is cozero complemented.
(IV) If D is discrete and M metrizable, then $\beta D \times M$ is cozero complemented.
(V) There is a cozero complemented space X and a separable (even countable) space Y such that $X \times Y$ is not cozero complemented;
(VI) The product of two locally compact Souslin lines is always cozero complemented;
(VII) It is consistent that there are compact *ccc* spaces X and Y such that $X \times Y$ is not cozero complemented.

We should mention that in (V), X may be taken to be βD, where D is discrete of cardinality ω_1. Recall that connected implies locally compact for ordered spaces, so (VI) holds for Souslin lines which are connected (which is sometimes included in the definition of Souslin line–our definition is "ordered *ccc* non-separable space"). Models in which spaces as in (VII) exist include any model of CH, or any model obtained by adding Cohen reals.

Regarding (IV), we show more generally that the class of spaces called "fraction dense" by Hager and Martinez[HM], and what we will call "cozero approximated", is closed under products with a metrizable space. Cozero approximated implies cozero complemented, and any βD is cozero approximated.

Results (I)-(VII) completely answer questions (2)-(7) above. Note that (I) and (II) together show that the answer to question (2) is positive for separable metric spaces, but negative otherwise. (II) also gives a negative answer to (7), even when the other factor is metrizable. (III) implies a negative answer to question (3), (IV) a positive answer to (4), and (V) and (VII) a negative answer to (6) and (5), respectively. The following questions remain unsolved:

(8) Does X^2 cozero complemented imply X is cozero complemented? (This question is essentially in [HW].)
(9) If $X \times Y$ is cozero complemented, must at least one of X and Y be too?
(10) Can there be two Souslin lines whose product is not cozero complemented?

Used througout the paper without reference is the basic fact that any countable union of cozero sets is cozero. The following notation is also used: if A is any set, and κ a cardinal, $[A]^\kappa$ (resp., $[A]^{<\kappa}$) is the set of all subsets of A having cardinality κ (resp., $< \kappa$).

1. Products with metrizable

In this section, we prove statements (I)-(IV) of the Introduction.

Theorem 1.1. *If X is cozero complemented, and M is a separable metrizable space, then $X \times M$ is cozero complemented.*

Proof. Let $\mathcal{B}$ a countable base for M, and D a countable dense subset. Suppose U is a cozero subset of $X \times M$. For each $d \in D$, let $U_d \subset X$ such that $U_d \times \{d\} = U \cap (X \times \{d\}$. It is easy to check that U_d is cozero.

For each $B \in \mathcal{B}$, let $U_B = \bigcup_{d \in D} U_d$; then U_B is cozero too. Let V_B be its cozero complement in X, and let $V = \bigcup_{B \in \mathcal{B}} V_B \times B$. Then V is cozero in $X \times M$.

Claim 1. $U \cap V = \emptyset$. Suppose otherwise. Since D is dense, there is $(x, d) \in V \cap U$ for some $x \in X$ and $d \in D$. Also, there is $B \in \mathcal{B}$ with $(x, d) \in V_B \times B$. But $(x, d) \in U$ and $d \in B$ implies $x \in U_d \subset U_B$, so $x \notin V_B$, contradiction.

The next claim completes the proof of the theorem.

Claim 2. $U \cup V$ is dense in $X \times M$. If not, then for some O open in X and $B \in \mathcal{B}$, $(O \times B) \cap (U \cup V) = \emptyset$. Since $O \times B$ misses U, we have $O \cap U_B = \emptyset$. Then $O \cap V_B \neq \emptyset$, whence $\emptyset \neq (O \times B) \cap (V_B \times B) \subset (O \times B) \cap V$, contradiction. □

The following example shows that there is a metrizable space of weight ω_1 whose product with a cozero complemented space, in fact a P-space, is not cozero complemented. Recall that the "one-point Lindelöfization of a discrete space D" is the space $D \cup \{\infty\}$, where neighborhoods of ∞ are the complements of countable subsets of D.

Example 1.2. *If $L(\omega_2)$ is the one-point Lindelöfization of a discrete space of cardinality ω_2, and ω_1^ω the countable power of discrete ω_1, then $L(\omega_2) \times \omega_1^\omega$ is not cozero complemented.*

Proof. Let θ be a bijection from $[\omega_1]^{<\omega}$ to ω_1. Let

$$U = \cup\{\{\theta(\sigma)\} \times [\sigma] : \sigma \in \omega_1^{<\omega}\},$$

where $[\sigma] = \{x \in \omega_1^\omega : x \supset \sigma\}$. Note that $\{[\sigma] : \sigma \in \omega_1^{<\omega}\}$ is a clopen base for ω_1^ω, and for each n, $\{[\sigma] : \sigma \in \omega_1^n\}$ is a discrete collection. It is easy to check that $U_n = \cup\{\{\theta(\sigma)\} \times [\sigma] : \sigma \in \omega_1^n\}$ is clopen, and so $U = \bigcup_{n \in \omega} U_n$ is cozero.

We claim that U has no cozero complement V. Suppose otherwise. Let $V = \bigcup_{n \in \omega} V_n$, where V_n is open and $\overline{V}_n \subset V$. For each $\alpha \geq \omega_1$, since $\{\alpha\} \times \omega_1^\omega$ is a clopen subset of the complement of U, there is some $n_\alpha \in \omega$ and $\sigma_\alpha \in \omega_1^{<\omega}$ with $\{\alpha\} \times [\sigma_\alpha] \subset V_{n_\alpha}$. Then there is $n \in \omega$, $\sigma \in \omega_1^{<\omega}$, and an ω_2-sized subset W of $\omega_2 \setminus \omega_1$ such that $n_\alpha = n$ and $\sigma_\alpha = \sigma$ for every $\alpha \in W$. Let ∞ be the non-isolated point of $L(\omega_2)$ and let $y \in [\sigma]$. Then $(\infty, y) \in \overline{\cup\{\{\alpha\} \times [\sigma] : \alpha \in W\}} \subset \overline{V}_n \subset V$, so there is a countable subset A of ω_2 and $\tau \in \omega_1^{<\omega}$ with $y \in [\tau] \subset [\sigma]$ such that $(L(\omega_2) \setminus A) \times [\tau] \subset V$. Since θ is one-to-one and $|A| \leq \omega$, there is an extension τ' of τ with $\theta(\tau') \notin A$. But then $\{\theta(\tau')\} \times [\tau'] \subset U \cap V$, contradiction. □

There are however many situations in which a product with a metrizable space will be cozero complemented. Let us call a space X *cozero approximated* if, given any open set U, there is a cozero set V such that $\overline{U} = \overline{V}$; we call such a V the *cozero approximation* of U. "Cozero approximated" was called "fraction dense" by Hager and Martinez[HM], since it characterizes a certain property of $C(X)$ concerning its ring of quotients. (As they show, another characterization is that the space of minimal prime ideals of $C(X)$ is compact and extremally disconnected.) We use the term cozero approximated here since it is more descriptive of the topological concept.

The following is a collection of easily verified claims about cozero approximated spaces. Recall that a collection $\mathcal{B}$ of non-empty open subsets of X is a *π-base* for X if every non-empty open set in X contains some member of $\mathcal{B}$.

Proposition 1.3.

(a) *Cozero approximated spaces are cozero complemented;*
(b) *Perfectly normal spaces, ccc spaces, extremally disconnected spaces, and spaces having a σ-discrete π-base are cozero approximated.*

The space of countable ordinals is an example of a cozero complemented space that is not cozero approximated, the latter witnessed by any unbounded, co-unbounded set of isolated points. It is noted in [HW] that any ordinal space is cozero complemented.

By Proposition 1.3, βD for discrete D is always cozero approximated. Thus the following result gives a positive answer to question (4) in the introduction.

Theorem 1.4. *Let X be cozero approximated. Then for any metrizable space M, $X \times M$ is cozero approximated .*

Proof. Let $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \dots$ be a base for the metrizable space M, where each $\mathcal{B}_n$ is is a discrete collection of open subsets of M. Let U be open in $X \times M$.

For each $B \in \mathcal{B}$, let

$$U(B) = \cup\{W : W \times B \subset U, W \text{ open in } X\}.$$

That is, $U(B)$ is the maximal open subset of X such that $U(B) \times B$ is contained in U. Let $V(B)$ be the cozero approximation of $U(B)$.

Now let $V_n = \cup\{V(B) \times B : B \in \mathcal{B}_n\}$. Then V_n is the union of a discrete collection of cozero sets, hence is cozero. Finally, let V be the union of the V_n's. Then V is cozero, and an elementary argument shows that $\overline{V} = \overline{U}$. Thus $X \times M$ is cozero approximated. □

The next result yields a negative answer to question (3) in the introduction. Recall that the π-weight $\pi w(X)$ is the least cardinal $(+\omega)$ of a π-base for the space X.

Theorem 1.5. *Let κ be a cardinal, $\kappa \geq \pi w(X)$. Then $X \times \kappa^\omega$, where κ carries the discrete topology, has a σ-discrete π-base and hence is cozero approximated.*

Proof. For each $\sigma \in \kappa^{<\omega}$, let $[\sigma] = \{x \in \kappa^\omega : \sigma \subset x\}$. Let $\Sigma_n = \{[\sigma] : \sigma \in \kappa^n\}$, and $\Sigma = \bigcup_{n\in\omega} \Sigma_n$. Then the Σ a base for κ^ω, and each Σ_n is a discrete collection of clopen sets.

Let $\mathcal{B}$ be a π-base for X of cardinality not greater than κ. Given $\sigma \in \kappa^{<\omega}$, and $\alpha \in \kappa$, let $\sigma^\frown\langle\alpha\rangle$ denote the sequence σ followed by α. Since $\kappa \geq |\mathcal{B}|$, it is easy to construct a function $\theta : \kappa^{<\omega} \to \mathcal{B}$ satisfying:

$$\forall\sigma \in \kappa^{<\omega}(\mathcal{B} = \{\theta(\sigma^\frown\langle\alpha\rangle) : \alpha \in \kappa\}).$$

Now let

$$\mathcal{U}_n = \{\theta(\sigma) \times [\sigma] : \sigma \in \kappa^n\}.$$

Then $\mathcal{U}_n$ is a discrete collection of open sets, and it is not difficult to check that $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1 \cup \ldots$ is a σ-discrete π-base for $X \times \kappa^\omega$. □

As noted in both [LS] and [HW], the one-point compactification of an uncountable discrete space is never cozero complemented. Thus if ω_1 carries the discrete topology, and X is its one-point compactification, $X \times \omega_1^\omega$ is a cozero complemented product with a non-cozero complemented factor, which gives a negative answer to question (3) in the introduction. We don't know, however, if there is a cozero complemented product $X \times Y$ with neither factor cozero complemented.

2. Products with separable

The following gives a negative answer to question (6) in the introduction.

Theorem 2.1. *Let D be a discrete space of cardinality ω_1. Then there is a countable regular space Y such that $\beta D \times Y$ is not cozero complemented.*

To prove this, we first establish a series of lemmas.

Lemma 2.2. *Let $\mathcal{A}$ be a collection of subsets of ω, and let $|D| = |\mathcal{A}|$. Then there are subsets D_n, $n \in \omega$, of D such that, for any $x \in \mathcal{A}$, there is some $d_x \in D$ with $x = \{n \in \omega : d_x \in D_n\}$.*

Proof. W.l.o.g., set $D = \mathcal{A}$. For each $n \in \omega$, let $D_n = \{x \in D : n \in x\}$. It is elementary to check that this works. □

The next lemma shows a property that a countable regular space Y must have for $\beta D \times Y$ to be cozero complemented. It says that, given any countable collection $\mathcal{P}$ of open subsets of Y, and $|D|$-many non-empty sets of the form $Int(Y \setminus \cup\mathcal{P}')$, where $\mathcal{P}' \subset \mathcal{P}$, there has to be what one might call a countable π-base for this collection.

Lemma 2.3. *Let D be a set of cardinality not greater than the continuum, let Y be a countable regular space, and $\mathcal{P}$ a countable collection of open subsets of Y. Let $\mathcal{R}$ be a collection of $|D|$-many non-empty sets of the form $Int(Y \setminus \cup\mathcal{P}')$, where $\mathcal{P}' \subset \mathcal{P}$. If $\beta D \times Y$ is cozero complemented, then there is a countable collection Q of non-empty open subsets of Y satisfying:*

$$\forall R \in \mathcal{R}[\exists Q \in \mathcal{Q}(Q \subset R)].$$

Proof. Let $\mathcal{P} = \{P_n\}_{n\in\omega}$. For each $R \in \mathcal{R}$, let $A_R \subset \omega$ such that $R = Int(Y \setminus \bigcup_{n\in A_R} P_n)$. Then let $\mathcal{A} = \{A_R : R \in \mathcal{R}\}$, and let $\{D_n\}_{n\in\omega}$ be as in Lemma 2.2. Now let $U = \bigcup_{n\in\omega} \overline{D}_n \times P_n$, where the closure $\overline{D}_n$ of D_n is taken in βD. Then U is cozero in $\beta D \times Y$.

Suppose V is a cozero complement of U. Then V is σ-compact, so there are $E_n \subset D$ and non-empty open $Q_n \subset Y$ such that $V = \bigcup_{n\in\omega} \overline{E}_n \times Q_n$. Note that, for each $d \in D$,

$$U \cap (\{d\} \times Y) = \{d\} \times (\cup\{P_n : d \in D_n\}),$$

with the analogous formula holding for $V \cap (\{d\} \times Y)$. Since U and V are disjoint and the union of their traces on $\{d\} \times Y$ must be dense in the clopen set $\{d\} \times Y$, it follows that if $\cup\{P_n : d \in D_n\}$ is not dense in Y, then there is some Q_m contained in (the interior of) its complement. By our choice of the D_n's, for any $R \in \mathcal{R}$, there is some $d \in D$ such that $R = Int(Y \setminus \cup\{P_n : d \in D_n\})$. Hence R contains Q_m for some $m \in \omega$, and the result follows. □

Our task now is to construct a countable regular Y, a countable collection $\mathcal{P}$ of open subsets of Y, and ω_1-many sets of the form $Int(Y \setminus \cup\mathcal{P}')$, where $\mathcal{P}' \subset \mathcal{P}$, such that there is no countable $\mathcal{Q}$ satisfying the conditions of Lemma 2.3. The next lemma will be helpful.

Lemma 2.4. *Let $(Y, \mathcal{T})$ be a countable dense-in-itself metrizable space, and let $\mathcal{C}$ be a countable collection of clopen sets in Y which forms a base for a weaker Hausdorff topology $\mathcal{T}'$ on Y. Then $(Y, \mathcal{T})$ contains a nowhere-dense, dense-in-itself, subset K which is $\mathcal{T}'$-closed.*

Proof. Let $\{B_n\}_{n\in\omega}$ index a countable clopen base $\mathcal{B}$ for $\mathcal{T}$. We are going to choose, for each $\sigma \in 2^{<\omega}$, a point $r_\sigma \in Y$, and K is going to be $\{r_\sigma : \sigma \in 2^{<\omega}\}$.

Let $Y = \{y_n\}_{n\in\omega}$. Let $r_\emptyset = y_0$, and let $B_\emptyset \in \mathcal{B}$ such that either $B_\emptyset \subset B_0$ or $B_\emptyset \cap B_0 = \emptyset$. Let $C_0 \in \mathcal{C}$ be arbitrary.

Suppose $r_\sigma \in Y$ and $B_\sigma \in \mathcal{B}$ have been defined for all $\sigma \in 2^{\leq n}$, and $C_m \in \mathcal{C}$ defined for all $m \leq n$, such that the following hold:

(i) $r_\sigma \in B_\sigma$ and for all $m \leq n$, $\{B_\sigma\}_{\sigma\in 2^m}$ is pairwise-disjoint;
(ii) $\sigma \in 2^{\leq n}$ extends τ implies $B_\sigma \subset B_\tau$;
(iii) $\sigma \in 2^{<n}$ implies $B_\sigma \setminus (B_{\sigma\frown\langle 0\rangle} \cup B_{\sigma\frown\langle 1\rangle}) \neq \emptyset$;
(iv) $\sigma \in 2^m$, $m \leq n$, implies $B_\sigma \subset B_m$ or $B_\sigma \cap B_m = \emptyset$;
(v) $k \leq m \leq n$ implies $\{r_\sigma\}_{\sigma\in 2^m} \supset \{r_\tau\}_{\tau\in 2^k}$;
(vi) For $m \leq n$, if $y_m \notin \{r_\sigma\}_{\sigma\in 2^{\leq m}}$, then $y_m \in C_m$ and $C_m \cap (\bigcup_{\sigma\in 2^m} B_\sigma) = \emptyset$.

To make the definitions at the next stage, first look at y_{n+1}. If $y_{n+1} \notin \{r_\sigma\}_{\sigma\in 2^{\leq n}}$, choose $C_{n+1} \in \mathcal{C}$ such that $C_{n+1} \cap \{r_\sigma\}_{\sigma\in 2^{\leq n}} = \emptyset$; otherwise, let $C_{n+1} \in \mathcal{C}$ be arbitrary. For each $\sigma \in 2^n$, let $r_{\sigma\frown\langle 0\rangle} = r_\sigma$, and let $r_{\sigma\frown\langle 1\rangle}$ be any other point in B_σ. Now it is easy to define the $B_{\sigma\frown\langle e\rangle}$'s, $\sigma \in 2^n$ and $e < 2$, so that conditions (i)-(vi) hold.

Let $K_n = \{r_\sigma : \sigma \in 2^n\}$, and let $K = \bigcup_{n\in\omega} K_n$. Let $U_n = \bigcup_{\sigma\in 2^n} B_\sigma$. Conditions (i) and (ii) imply that $U_n \supset K_n$ and the U_n's are decreasing. Condition (v) implies that the K_n's are increasing. It follows that $U_n \supset K$ for all n. Condition (vi) then implies that $\bigcap_{n\in\omega} U_n = K$ and that K is $\mathcal{T}'$-closed. If $r \in K$, then conditions (ii) and (iv) imply that $\{B_\sigma : r \in B_\sigma\}$ is a local base at r (in $\mathcal{T}$). Since no $B_\sigma \cap K$ is a singleton, it follows that K is dense-in-itself.

To see that K is nowhere-dense, suppose otherwise. Then $r \in B_n \subset K$ for some n. So if $\sigma \in 2^n$ and $r \in B_\sigma$, then by (iv), $B_\sigma \subset B_n$. Then condition (iii) implies U_{n+1}, and hence K, does not contain B_n, contradiction. □

Lemma 2.5. *Let $\mathbb{Q}$ be the rationals and $\mathcal{T}'$ the usual Euclidean topology on $\mathbb{Q}$. Then there is a regular topology $\mathcal{T}$ on $\mathbb{Q}$ and subsets K_α, $\alpha < \omega_1$, of $\mathbb{Q}$, satisfying:*

(i) *Each K_α is $\mathcal{T}'$-closed and $\mathcal{T}$-open;*
(ii) *For any uncountable $W \subset \omega_1$, $\bigcap_{\alpha\in W} K_\alpha$ is nowhere-dense in $(\mathbb{Q}, \mathcal{T})$.*

Proof. Let $\mathcal{C}$ be a countable clopen base for $\mathcal{T}'$. We define $\mathcal{B}_\alpha$, $\alpha < \omega_1$, satisfying:

(a) Each $\mathcal{B}_\alpha$ is a countable clopen base for a Hausdorff (hence regular) dense-in-itself topology on $\mathbb{Q}$;

(b) For each $\alpha < \beta < \omega_1$, $\mathcal{C} \subset \mathcal{B}_\alpha \subset \mathcal{B}_\beta$;
(c) $K_\alpha \in \mathcal{B}_\alpha$;
(d) K_α is $\mathcal{T}'$-closed;
(e) K_α is nowhere-dense in the topology $\mathcal{T}_\alpha$ generated by $\mathcal{C} \cup \bigcup_{\beta<\alpha} \mathcal{B}_\beta$.

Suppose $\mathcal{B}_\beta$ has been constructed for all $\beta < \alpha$, and let $\mathcal{T}_\alpha$ be the topology generated by $\mathcal{C} \cup \bigcup_{\beta<\alpha} \mathcal{B}_\beta$. Apply Lemma 2.4 to $(\mathbb{Q}, \mathcal{T}_\alpha)$ and $\mathcal{C}$ to obtain a $\mathcal{T}'$-closed K_α which is nowhere-dense and dense-in-itself in $\mathcal{T}_\alpha$. Let $\mathcal{B}_\alpha$ be the set of all finite intersections of elements of $\{K_\alpha, \mathbb{Q} \setminus K_\alpha\} \cup \mathcal{C} \cup \bigcup_{\beta<\alpha} \mathcal{B}_\beta$. It is easy to check that (a)-(e) are satisfied.

Let $\mathcal{T}$ be the topology on $\mathbb{Q}$ generated by $\bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha$. Condition (i) of the lemma clearly holds for the K_α, so it remains to check condition (ii). Suppose that $W \subset \omega_1$ is uncountable, but the closed set $\bigcap_{\alpha \in W} K_\alpha$ fails to be nowhere-dense. Then there is some $\gamma < \omega_1$ and $B \in \mathcal{B}_\gamma$ with $B \subset \bigcap_{\alpha \in W} K_\alpha$. But this contradicts, for $\alpha > \gamma$, K_α being nowhere-dense in the topology generated by $\bigcup_{\beta<\alpha} \mathcal{B}_\beta$. □

Proof of Theorem 2.1. Let $|D| = \omega_1$, and let Y be the countable regular space $(\mathbb{Q}, \mathcal{T})$ given by Lemma 2.5. Let $\mathcal{T}'$ and K_α, $\alpha < \omega_1$, satisfy the conditions of Lemma 2.5 (in particular, $\mathcal{T}'$ is the Euclidean topology on $\mathbb{Q}$), and let $\mathcal{P}$ be a countable clopen base for $\mathcal{T}'$. Note that for each $\alpha < \omega_1$, since K_α is Euclidean closed, and open in Y, there is a subset $\mathcal{P}_\alpha$ of $\mathcal{P}$ with $K_\alpha = Y \setminus \cup \mathcal{P}_\alpha = Int(Y \setminus \cup \mathcal{P}_\alpha)$. Let $\mathcal{R} = \{K_\alpha : \alpha < \omega_1\}$. We claim that there is no countable collection $\mathcal{Q}$ of non-empty open subsets of Y satisfying the conclusion of Lemma 2.3 with respect to this collection $\mathcal{R}$, and hence that $\beta D \times Y$ is not cozero complemented.

Suppose there were such a collection $\mathcal{Q}$. Then for each $\alpha < \omega_1$, there must be $Q_\alpha \in \mathcal{Q}$ with $Q_\alpha \subset K_\alpha$. But some Q_α must be the same element of $\mathcal{Q}$ uncountably often; this contradicts condition (ii) of Lemma 2.5. □

3. Products of Souslin lines

The purpose of this section is to prove that the product of two locally compact Souslin lines is cozero complemented. Recall that connected implies locally compact for ordered spaces, so the result holds for connected Souslin lines too (which is sometimes part of the definition of Souslin line). We don't know if the result holds for arbitrary Souslin lines.

Theorem 3.1. *If S_0 and S_1 are locally compact Souslin lines, then $S_0 \times S_1$ is cozero complemented.*

Proof. If S_0 and S_1 are locally compact Souslin lines, so is their topological sum, so it suffices to prove that the square of any locally compact Souslin line S is cozero complemented. Now, given S, define two points a, b in S to be equivalent if the closed interval with endpoints a, b is compact. By local compactness, the equivalence classes are clopen, so it suffices to consider S with only one equivalence class.

Given an S with only one such equivalence class, now define two points to be equivalent if the interval between them is finite. These equivalence classes are countable, closed, and convex, and collapsing these to points yields a connected Souslin line.

We will give our proof for a connected Souslin line S; this avoids some awkwardness in having to deal with isolated points, or points with immediate successors;

but these are not essential difficulties, and it should be clear that the proof can be adapted to the more general case. When we come to it, we will point out in the proof below where local compactness is crucial.

Let T be the associated tree obtained from S by applying the standard construction of a Souslin tree from a Souslin line (see, e.g, [K]). Let us recall the following facts about T:

(a) T consists of open intervals of S and the order on T that makes it a Souslin tree is reverse inclusion;
(b) The α^{th} level $Lev_\alpha T$ is a pairwise-disjoint collection of intervals of S and its union is dense in S;
(c) If $s \in Lev_\alpha T$, $t \in Lev_\beta T$, $\beta < \alpha$, and $s \cap t \neq \emptyset$, then $\overline{s} \subset t$;
(d) A chain $\mathcal{C}$ in T has a upper bound in T iff $\cap\mathcal{C}$ contains a nonempty interval.

Since S is connected and therefore densely ordered, we may also assume:

(e) Every $t \in T$ has at least two successors in T, and the endpoints of any successor of t are in t.

By the connectedness, it follows that the intersection of every branch (=maximal chain) of T must be a single point of S, since it is a well-ordered, by reverse inclusion, sequence of intervals, each one containing the closure of its successor, whose intersection has empty interior. Let S' be the subset of S consisting of all points that are the intersection of a branch. For each $x \in S'$, let B_x denote the corresponding branch of T with $\{x\} = \cap B_x$, and note that the intervals in the branch B_x form a local base at the point x.

Claim 1. S' is dense in S.

Proof. Consider any interval (a, b) in S. Since there are no uncountable chains in $\mathcal{T}$, there must be a level $\alpha < \omega_1$ such that neither a nor b is in $\cup Lev_\alpha T$. Since $\cup Lev_\alpha T$ is dense, there is $t \in Lev_\alpha T$ with $t \subset (a, b)$. Now consider all chains in $\mathcal{T}$ containing t and going up to the next limit level. There are uncountably many such chains (since every node in $\mathcal{T}$ has at least two successors), and by *ccc* only countably many of them have an intersection consisting of more than one point. So most of them have an intersection consisting of a single point lying in t and hence in (a, b). □

Now let U be an open subset of S^2. Let $\mathcal{T}(U) = \{(s,t) \in T^2 : s \times t \subset U\}$, and let $\mathcal{M}(U)$ be set of minimal members of $\mathcal{T}(U)$ with respect to the following partial ordering: $(s,t) \leq (s',t')$ iff $s \leq s'$ and $t \leq t'$ (in the tree order). Note that both $\{s \times t : (s,t) \in \mathcal{T}(U)\}$ and $\{s \times t : (s,t) \in \mathcal{M}(U)\}$ cover the open set U.

Claim 2. $\{s \times t : (s,t) \in \mathcal{M}(U)\}$ is star-countable.

Proof. Fix $(s,t) \in \mathcal{M}(U)$, and suppose $s \times t$ meets uncountably many $s' \times t'$'s for $(s',t') \in \mathcal{M}(U)$. If $s \times t$ meets $s' \times t' \in \mathcal{M}(U)$, then by minimality, either $s' \leq s$ and $t' \geq t$, or $s' \geq s$ and $t' \leq t$. W.l.o.g., assume the former occurs uncountably often. Then uncountably often, s' is the same predecessor of s in $\mathcal{T}$. But by *ccc*, two of the corresponding t''s are related, contradicting that they are both minimal elements of $\mathcal{T}(U)$. □

For $(s,t), (s',t') \in \mathcal{M}(U)$, define $(s,t) \sim (s',t')$ iff there are $(s_i,t_i) \in \mathcal{M}(U)$, $i \leq n$, where $(s_0,t_0) = (s,t)$, $(s_n,t_n) = (s',t')$, and $s_i \times t_i \cap s_{i+1} \times t_{i+1} \neq \emptyset$ for each $i < n$. Then $\sim$ is an equivalence relation, and by Claim 1, each equivalence class is countable. Let $\mathcal{M}(U)/\sim$ denote the collection of equivalence classes. For each equivalence class $\mathcal{E}$ of $\mathcal{M}(U)$, let $U_{\mathcal{E}} = \cup\{s \times t : (s,t) \in \mathcal{E}\}$, and let $\mathcal{P}(U) = \{U_{\mathcal{E}} : \mathcal{E} \in \mathcal{M}(U)/\sim\}$.

Claim 3. $\mathcal{P}(U)$ is pairwise-disjoint collection of cozero subsets of U whose union covers $U \cap (S' \times S')$ and hence is dense in U.

Proof. Pairwise-disjointness follows easily from the definition of $\sim$, and cozeroness from the fact that each equivalence class is countable. Now suppose $(x,y) \in U \cap (S' \times S')$. There are $(s,t) \in B_x \times B_y$ with $s \times t \subset U$. Then $(s,t) \in \mathcal{T}(U)$, so there is $(s',t') \in \mathcal{M}(U)$ with $s' \leq s$ and $t' \leq t$. If $\mathcal{E}$ is the equivalence class of (s',t'), then $(x,y) \in U_{\mathcal{E}} \subset \cup\mathcal{P}(U)$. $\square$

Claim 5 below is a key reason why S^2 is cozero-complemented. To prove Claim 5, we first need:

Claim 4. Suppose (s_α, t_α), $\alpha < \omega_1$, are in $\mathcal{T}^2$, and that $lev(s_\alpha)$ and $lev(t_\alpha)$ are both $\geq \alpha$. Then there exists $(x,y) \in (S')^2$ such that every neighborhood of (x,y) contains $s_\alpha \times t_\alpha$ for some α.

Proof. For $(s,t) \in T^2$, let $\phi(s,t) = \{\alpha : s_\alpha \geq s \text{ and } t_\alpha \geq t\}$. Note that for every $\beta < \omega_1$, there is $(s,t) \in (Lev_\beta T)^2$ such that $|\phi(s,t)| = \omega_1$.

Choose $(s_\emptyset, t_\emptyset) \in (Lev_{\beta(\emptyset)} T)^2$ with $|\phi(s_\emptyset, t_\emptyset)| = \omega_1$. Let $\chi(\emptyset) = \{(s,t) \in T^2 : s \geq s_\emptyset, t \geq t_\emptyset, \text{ and } |\phi(s,t)| = \omega_1\}$. Note that $\chi(\emptyset)$ includes (s,t)'s in $(Lev_\beta T)^2$ for any $\beta > \beta(\emptyset)$. Since the poset of finite antichains of any Aronszajn tree has the *ccc* (see, e.g., Lemma 9.2 of [T]), there are $(s_{\langle 0\rangle}, t_{\langle 0\rangle}), (s_{\langle 1\rangle}, t_{\langle 1\rangle}) \in \chi(\emptyset)$ such that $\{s_{\langle 0\rangle}, t_{\langle 0\rangle}, s_{\langle 1\rangle}, t_{\langle 1\rangle}\}$ is an antichain.

Now let

$$\chi(\langle e\rangle) = \{(s,t) : s \geq s_{\langle e\rangle}, t \geq t_{\langle e\rangle}, \text{ and } |\phi(s,t)| = \omega_1\}.$$

There are $(s_{\langle e,0\rangle}, t_{\langle e,0\rangle}), (s_{\langle e,1\rangle}, t_{\langle e,1\rangle}) \in \chi(\langle e\rangle)$ such that $\{s_{\langle e,0\rangle}, t_{\langle e,0\rangle}, s_{\langle e,1\rangle}, t_{\langle e,1\rangle}\}$ is an antichain.

Continue in like manner to define (s_σ, t_σ) for all $\sigma \in 2^{<\omega}$. By *ccc*, there are only countably many $f \in 2^\omega$ for which $\bigcap_{n\in\omega} s_{f\restriction n}$ or $\bigcap_{n\in\omega} t_{f\restriction n}$ is more than one point. So pick $g \in 2^\omega$ for which $\bigcap_{n\in\omega} s_{g\restriction n} = \{x\}$ and $\bigcap_{n\in\omega} t_{g\restriction n} = \{y\}$. Then $(x,y) \in (S')^2$ satisfies the conditions of Claim 4. $\square$

Remark. This last part of the argument of Claim 4 fails for arbitrary subspaces of a Souslin line. We can see that it is still true that for most $g \in 2^\omega$, the intersections of the corresponding intervals do not contain an interval, but we need these intersections also to be non-empty. This is where a local compactness assumption is crucial to the argument.

Claim 5. Suppose U and V are open subsets of X^2 with $\overline{V} \subset U$. Then $V \cap U_{\mathcal{E}} \neq \emptyset$ for only countably many $\mathcal{E} \in \mathcal{M}(U)/\sim$.

Proof. Suppose not. Then there exist distinct $U_\alpha \in \mathcal{P}(U)$, $\alpha < \omega_1$. Since $V \subset U$, it is easy to see that $\mathcal{P}(V)$ refines $\mathcal{P}(U)$; so let $V_\alpha \in \mathcal{P}(V)$ be such that $V_\alpha \subset U_\alpha$.

Pick $(s_\alpha, t_\alpha) \in \mathcal{M}(V)$ with $s_\alpha \times t_\alpha \subset V_\alpha$. By *ccc* and minimality of the members of $\mathcal{M}(V)$, the indexings $\{s_\alpha\}_{\alpha<\omega_1}$ and $\{t_\alpha\}_{\alpha<\omega_1}$ do not repeat the same element of T uncountably often. Hence by re-indexing if necessary, we may assume $\alpha \leq min\{lev(s_\alpha), lev(t_\alpha)\}$. So there exists $(x,y) \in (S')^2$ satisfying the conclusion of Claim 4. Clearly, every neighborhood of (x,y) meets infinitely many V_α, hence infinitely many U_α. Sine $(x,y) \in \overline{V}$, we have $(x,y) \in U$. But then (x,y) is in some member of $\mathcal{P}(U)$, an open set which misses all other members of $\mathcal{P}(U)$, contradiction. $\square$

Claim 6. If $U \subset S^2$ is open F_σ, then $\mathcal{P}(U)$, and hence $\mathcal{M}(U)$, is countable.

Proof. Let $U = \bigcup_{n\in\omega} U_n$, where U_n is open and $\overline{U}_n \subset U$. If $\mathcal{P}(U)$ were uncountable, then some U_n would meet uncountably many members of $\mathcal{P}(U)$, contradicting Claim 4. $\square$

Claim 7. If $U \subset S^2$ is open F_σ, then $\mathcal{P}(S^2 \setminus \overline{U})$ is countable.

Proof. Suppose not. Then there are distinct $(s_\alpha, t_\alpha) \in \mathcal{M}(S^2 \setminus \bar{(}U))$, $\alpha < \omega_1$. As in the proof of Claim 5, we may assume $a \leq min\{lev(s_\alpha), lev(t_\alpha)\}$. By Claim 6, there exists $\delta < \omega_1$ such that $lev(s), lev(t) < \delta$ for every $(s,t) \in \mathcal{M}(U)$. Take $\alpha > \delta$, and let s'_α,t'_α be the predecessors of s_α and t_α at level δ. By minimality of (s_α, t_α), $s'_\alpha \times t'_\alpha$ is not a subset of $S^2 \setminus \overline{U}$. So $s'_\alpha \times t'_\alpha$ meets some member of $\mathcal{P}(U)$, hence $(s'_\alpha \times t'_\alpha) \cap (s \times t) \neq \emptyset$ for some $(s,t) \in \mathcal{M}(U)$. But then $s'_\alpha \times t'_\alpha \subset s \times t \subset U$, contradiction. $\square$

Now we can complete the proof of the theorem.

Claim 8. S^2 is cozero-complemented.

Proof. Let $U \subset S^2$ be cozero. By Claim 7, $\mathcal{P}(S^2 \setminus \overline{U})$ is countable. Now it follows from Claim 3 that $\cup\mathcal{P}(S^2 \setminus \overline{U})$ cozero and dense in $S^2 \setminus \overline{U})$. $\square$

4. Products of *ccc* spaces

The purpose of this section is to give a negative answer to question (5) in the introduction; namely, it is consistent that there are *ccc* spaces X and Y whose product is not cozero complemented. Our X and Y will be compact.

Let G be a graph on an uncountable vertex set V. Of course the edge set E is a subset of $[V]^2$. Let $X(G)$ be the subspace of 2^V consisting of characteristic functions χ_H of vertex sets H of complete subgraphs of G. Such spaces were considered by Murray Bell[B] among others. Note that $X(G)$ is a closed, hence compact, subspace of 2^V. Now let G' be the graph on V whose edge set is $E' = [V]^2 \setminus E$. It is well-known that $X(G) \times X(G')$ is never *ccc*. To see this, for each $v \in V$, let

$$O_v = \{(x,y) \in X(G) \times X(G') : x(v) = y(v) = 1\}.$$

Then the point $(\chi_{\{v\}}, \chi_{\{v\}}))$ is in O_v, thus each O_v is non-empty, but clearly any two distinct O_v's have empty intersection. Now, CH implies that there is an uncountable G such that both $X(G)$ and $X(G')$ are *ccc*. (This is due to Laver; see [G], or Chapter VIII, Exercises C5 and C8, in [K]). It is also known that such a graph exists in any model obtained by adding one Cohen real (due to Roitman; see Exercise C7 in [K]). Also, $X(G)$ and $X(G')$ are known to be *ccc* for any Cohen generic graph on uncountably many vertices (Fleissner; see Exercise C6 in [K]).

We aim to find a graph G such that $X(G)$ and $X(G')$ are *ccc*, yet their product is not cozero complemented. It turns out that a little tweaking of any uncountable G for which the the corresponding spaces are *ccc* will work. The following lemma helps us see what we need to do.

Lemma 4.1. *Suppose $\mathcal{U}$ is an uncountable pairwise-disjoint collection of non-empty cozero sets in a compact space Y. If there is a countable subcollection $\mathcal{V}$ of $\mathcal{U}$ such that $\overline{\cup\mathcal{V}} \supset Y \setminus \cup\mathcal{U}$, then Y is not cozero complemented.*[1]

Proof. Let $V = \cup\mathcal{V}$, and note that V is cozero. Suppose W is cozero and disjoint from V. Write $W = \bigcup_{n\in\omega} W_n$, where each W_n is closed, and note that W_n cannot meet $\overline{V}$. Hence the compact set W_n is covered by $\mathcal{U}$. Thus some finite subcollection of $\mathcal{U}$ covers W_n, and so W is covered by a countable subcollection of $\mathcal{U}$. As $\mathcal{U}$ is uncountable, for any cozero set disjoint W disjoint from V, $V \cup W$ cannot be dense in Y. □

The idea is to make the collection of O_v's in $X(G) \times X(G')$ satisfy Lemma 4.1. A given G may not work as is, but we will add a countable set of vertices so that it does. Fortunately, adding a countable set of vertices never ruins the *ccc* property.

Lemma 4.2. *Suppose $G = (V, E)$ is a graph such that $X(G)$ and $X(G')$ are both ccc. Let $\hat{G} = (\hat{V}, \hat{E})$ be a graph such that $\hat{V} \supset V$ and $\hat{E} \cap [V]^2 = E$. If $\hat{V} \setminus V$ is countable, then $X(\hat{G})$ and $X(\hat{G}')$ are ccc.*

Proof. Following [K], for any set A let $Fn(A, 2)$ be the collection of finite partial functions from A into 2. For $\sigma \in Fn(\hat{V}, 2)$, let $[\sigma] = \{x \in X(\hat{G}) : x \supset \sigma\}$. Of course, the $[\sigma]$'s form a base for $X(\hat{G})$. Note that $[\sigma]$ is non-empty iff $\sigma^{-1}(1)$ is a complete subgraph of $\hat{G}$, and that $[\sigma] \cap [\tau] \neq \emptyset$ iff σ and τ are compatible as functions and $\sigma^{-1}(1) \cup \tau^{-1}(1)$ is a complete subgraph of $\hat{G}$.

Now suppose $X(G)$ is *ccc* and $[\sigma_\alpha]$, $\alpha < \omega_1$, is an uncountable collection of non-empty basic open sets in $X(\hat{G})$. W.l.o.g., we may assume that $\sigma_\alpha \restriction (\hat{V} \setminus V) = \rho$ for some $\rho \in Fn(\hat{V} \setminus V, 2)$ and every $\alpha < \omega_1$. It follows from the *ccc* property of $X(G)$ that there are $\alpha \neq \beta$ such that $\sigma_\alpha \restriction V$ and $\sigma_\beta \restriction V$ are compatible as functions and that $(\sigma_\alpha \restriction V)^{-1}(1) \cup (\sigma_\beta \restriction V)^{-1}(1)$ is a complete subgraph of G. It is easy to check from this that σ_α and σ_β are compatible as functions and that $\sigma_\alpha^{-1}(1) \cup \sigma_\beta^{-1}(1)$ is a complete subgraph of $\hat{G}$, and hence that $[\sigma_\alpha] \cap [\sigma_\beta] \neq \emptyset$. Thus $X(G)$ is *ccc*. The proof that $X(G')$ is *ccc* is entirely analogous. □

The next lemma tells us what we need about G to obtain a non-cozero complemented product.

Lemma 4.3. *Suppose $G = (V, E)$ is an uncountable graph, $G' = (V, E')$ where $E' = [V]^2 \setminus E$, and there is a countable subset C of V satisfying:*

- $(*)$ *whenever H and K are the vertex sets of finite complete subgraphs of G and G', respectively, and $H \cap K = \emptyset$, there are infinitely many $v \in C$ such that $H \cup \{v\}$ and $K \cup \{v\}$ are the vertex sets of finite complete subgraphs of G and G', respectively.*

Then $X(G) \times X(G')$ is not cozero complemented.

Proof. Let $\mathcal{U} = \{O_v : v \in V\}$, where the O_v's are defined as in the beginning of this section. Let $\mathcal{V} = \{O_v : v \in C\}$, and let $V = \cup\mathcal{V}$. By Lemma 4.1, we are done if we can show that $\overline{\cup\mathcal{V}} \supset X(G) \times X(G') \setminus \cup\mathcal{U}$. To this end, take $(x, y) \in X(G) \times X(G') \setminus \cup\mathcal{U}$, and a basic open set $[\sigma] \times [\tau]$ containing (x, y). Note that (x, y)

[1]The version of this lemma in which $\mathcal{U}$ is a set of isolated points was essentially noted in both [HW] and [LS].

not in $\cup\mathcal{U}$ implies that $x^{-1}(1) \cap y^{-1}(1) = \emptyset$, and hence that $\sigma^{-1}(1) \cap \tau^{-1}(1) = 0$. By the property of C, there is $v \in C \setminus (dom(\sigma) \cup dom(\tau))$ such that $\sigma^{-1}(1) \cup \{v\}$ and $\tau^{-1}(1) \cup \{v\}$ are the vertex sets of finite complete subgraphs of G and G', respectively. It follows that $[\sigma] \times [\tau]$ meets O_v, and hence $(x, y) \in \overline{\cup\mathcal{V}}$. □

Next we show that any uncountable G of cardinality $\leq \mathfrak{c}$ has a countable extension satisfying the conditions of Lemma 4.3.

Lemma 4.4. *Suppose $G = (V, E)$ is a graph with $\omega_1 \leq |V| \leq \mathfrak{c}$. Then there is a graph $\hat{G} = (\hat{V}, \hat{E})$ such that $\hat{V} \supset V$, $\hat{E} \cap [V]^2 = E$, and $C = \hat{V} \setminus V$ is countable and satisfies the conditions of Lemma 4.3 with respect to the graph $\hat{G}$.*

Proof[2]. Let $(V, E) = (V_0, E_0)$. We first claim that there is a countable set C_0 disjoint from V_0, and a set $E_1 \subset [V_1]^2$, where $V_1 = V_0 \cup C_0$, such that:

(i) $E_1 \cap [V_0]^2 = E_0$;
(ii) For each pair H, K of disjoint finite subsets of V_0, there are infinitely many $c \in C_0$ such that $\{\{x, c\} : x \in H\} \subset E_1$ and $\{\{x, c\} : x \in K\} \cap E_1 = \emptyset$.

To see this, note that, since $\omega_1 \leq |V| \leq \mathfrak{c}$, the set V admits a second countable Hausdorff topology with a countable base $\mathcal{B}$ which is closed under finite unions. Let $\{V_B : B \in \mathcal{B}\}$ be a collection of countably infinite sets disjoint from V and from each other. Let $C_0 = \bigcup_{B\in\mathcal{B}} V_B$, and let

$$E_1 = \bigcup_{B\in\mathcal{B}} \{\{x, y\} : x \in B, y \in V_B\}.$$

It is easy to check that this works.

Now a countable set C_1 of vertices can be added to (V_1, E_1) in a similar way, obtaining (V_2, E_2), and so on. Let $G = (\hat{V}, \hat{E}) = (\bigcup_{n\in\omega} V_n, \bigcup_{n\in\omega} E_n)$, and let $C = \bigcup_{n\in\omega} C_n$. To see that condition $(*)$ of Lemma 4.3 holds, suppose H and K are finite disjoint complete subgraphs of G and its complementary graph G', respectively. Choose n such that $H \cup K \subset V_n$. By condition (ii) applied to (V_n, E_n), there are infinitely many $c \in C_n$ that get connected by an edge in E_{n+1} to every member of H but no member of K. Since no edges within any V_j get added or subtracted at later stages, we have that for any such c, $H \cup \{c\}$ and $K \cup \{c\}$ are complete subgraphs of G and G', respectively. □

Now our main result of the section is immediate from the previous lemmas.

Theorem 4.5. *If there is an uncountable graph G such that the spaces $X(G)$ and $X(G')$ are ccc, then there is such a G with $X(G) \times X(G')$ not cozero complemented.*

As mentioned above, such G exist under CH, or in any model obtained by adding Cohen reals.

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[2]We thank Stuart Baldwin for a comment which led to a simplification of our original argument for Lemma 4.4.

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