November 20, 2005

PERFECT COMPACTA AND BASIS PROBLEMS IN TOPOLOGY

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An interesting example of a compact Hausdorff space that is often presented in beginning courses in topology is the unit square $[0, 1] \times [0, 1]$ with the lexicographic order topology. The closed subspace consisting of the top and bottom edges is perfectly normal. This subspace is often called the Alexandroff double arrow space. It is also sometimes called the "split interval", since it can be obtained by splitting each point x of the unit interval into two points x_0, x_1, and defining an order by declaring $x_0 < x_1$ and using the induced order of the interval otherwise. The top edge minus the last point is homeomorphic to the Sorgenfrey line, as is the bottom edge minus the first point. Hence it has no countable base, so being compact, is non-metrizable. There is an obvious two-to-one continuous map onto the interval.

There are many other examples of non-metrizable perfectly normal, if extra set-theoretic hypotheses are assumed. The most well-known is the Suslin line (compactified by adding a first and last point). Filippov[5] showed that the space obtained by "resolving" each point of a Luzin subset of the sphere S^2 into a circle by a certain mapping is a perfectly normal locally connected non-metrizable compactum (see also Example 3.3.5 in [30]). Moreover a number of authors have obtained interesting examples under CH (or sometimes something stronger); see, e.g., Filippov and Lifanov[6], Fedorchuk[4], and Burke and Davis[3].

At some point, researchers began to wonder if there is a sense in which minor variants of the double arrow space are the only ZFC examples of perfectly normal non-metrizable compacta. A first guess was made by David Fremlin, who asked if it is consistent that every perfectly normal compact space is the continuous image of the product of the double arrow space with the unit interval. But this was too strong: Watson and Weiss[31] constructed a counterexample (which looked like the double arrow space with a countable set of isolate points added in a certain way). Finally, the following question, also by Fremlin, became the central one:

Question 1. [8] *Is it consistent that every perfect compactum admits a continuous and at most two-to-one map onto a metric space?*

We call a space which does admit an at most two-to-one continuous map onto a metric space *premetric of order* 2.

Gruenhage noticed a close connection with what is now being called the "basis problem" for uncountable first countable spaces:

Question 2. *Is it consistent that every uncountable first countable regular space contains either an uncountable discrete subspace, or an uncountable subspace of the real line or of the Sorgenfrey line?*

In other words, might there be a three-element basis for uncountable first countable regular spaces? (One might be tempted to remove the requirement of first countability in this question, but this is not possible by Moore's ZFC L-space [15].)

The first author acknowledges support from NSF grant DMS-0405216 and the second author from NSF grant DMS-0401893.

It's clear that if there is any three element basis, it must be the three mentioned in Question 2. The connection to Fremlin's problem is this: a positive answer to the basis problem for first countable spaces implies a positive answer to Fremlin's conjecture, and Fremlin's conjecture is equivalent, under PFA, to the basis conjecture for subspaces of perfectly normal compacta[11].

We will need an appropriate axiom system for ruling out the types of counterexamples which can be constructed assuming the Continuum Hypothesis or Jensen's $\diamondsuit$. Woodin's axiom $(*)$ [32] is provably an optimal hypothesis in giving a positive answer to the problems mentioned above as well as most of the questions which we will consider in this article, in the sense that if the statement can be shown to hold in any sort of "reasonable" model, then $(*)$ implies the statement. Some discussion and quantification of this will be made at the end of this note. In practice, $(*)$ can be rather difficult to apply directly. It is easier to show that the conclusion follows from a forcing axiom such as $\mathsf{MA}_{\aleph_1}$, the Proper Forcing Axiom (PFA), or Martin's Maximum (MM) and then observe that the conclusion follows from $(*)$.[1]

That these axioms have a chance of providing positive solutions are indicated in some previous partial results. Fremlin[7] showed that under Martin's Maximum, any perfectly normal compactum admits a map to a metric space M whose fibers have cardinality two or less on a co-meager subset of M. Gruenhage[10] showed that even without first-countability, PFA implies a positive answer to the basis problem in the class of cometrizable spaces (later, Todorcevic [26] proved that this follows from OCA, a consequence of PFA).

1. Perfect compacta

Predating Fremlin's problem are two other basic questions about perfectly normal compacta:

Question 3. *$(*)$ If $X \times Y$ is perfect and compact, then is either X or Y is metrizable?*

Question 4. *$(*)$ Is every locally connected perfect compactum metrizable?*

The first question is due to Przymusinski[18] and the variant of the second which asks if $MA_{\aleph_1}$ gives a positive answer has been attributed to Rudin (see [17]). If $(*)$ implies a positive answer to either the basis problem or to Fremlin's problem, then both of these questions also have positive answers[9][11].

Concerning Przymusinski's question, suppose that there are disjoint uncountable $A_0, A_1 \subseteq [0,1]$ such that there is no monotonic injection of an uncountable subset of A_0 into A_1. Abraham and Shelah have shown in [1] that such pairs of subsets of $[0,1]$ can exist in a model of $\mathsf{MA}_{\aleph_1}$. On the other hand, Todorcevic proved in [27] that if X_0 and X_1 obtained as in the split interval construction, but with only the points of A_0 and A_1 split, then $X_0 \times X_1$ is perfectly normal. Hence $\mathsf{MA}_{\aleph_1}$ is not sufficient for a positive answer to Przymusinski's question.

Since no uncountable subspace of the Sorgenfrey line is embeddable in a locally connected perfect compactum[9], a positive answer to the following would give a positive answer to Question 4:

[1] It is also worth noting that our choice of the language "Does $(*)$ imply..." is typically not historically accurate in stating these problems.

Question 5. *(∗) Does every non-metrizable perfect compactum contains a copy of an uncountable subspace of the Sorgenfrey line?*

The difference between maps with metric fibers and with ≤ 2-point fibers in this context is unclear:

Question 6. *(∗) Does every perfect compactum admit a map into a metric space with metric fibers?*

A weaker form of this question can be stated as follows. Suppose that $K \subseteq [0,1]^{\omega_1}$ is a perfect compactum and define

$$T(K) = \{f \restriction \alpha : f \in K \text{ and } \exists g \in K(\alpha < \Delta(f,g) < \infty)\}.$$

Question 7. *(∗) If K is a non-metrizable perfect compactum, must $T(K)$ contain an uncountable level?*

A compact Suslin line K is a perfectly normal compactum which does not admit such a map [20], and satisfies that $T(K)$ is Suslin.

Question 8. *(∗) If K is a perfect compactum which maps into a metric space with metric fibers, must K admit an at most two-to-one map into a metric space?*

Filippov's CH example mentioned in the introduction admits an obvious map onto a compact metric space with metric fibers, but is not premetric of order two.

Question 9. *(∗) If X is a perfect compactum and $Y \subseteq X^2$ is scattered, must Y have rank less than ω_1? What if Y is assumed to be locally compact?*

Assuming CH, Gruenhage has constructed an example of a perfect compactum X whose square is a hereditarily normal, hereditarily separable space [12]. In fact, X is premetric of order 2 and X^2 contains a locally compact, locally countable S space. It is possible to show, however, that Question 9 has a positive answer for compacta which are premetric of order 2 ((∗) is required for this deduction).

It is also not known if Fremlin's problem can be reduced to the 0-dimensional case, which motivates the following two questions, the latter suggested by Todorcevic.

Question 10. *Is it consistent[2] that every perfect compactum is the continuous image of a 0-dimensional perfect compactum?*

Question 11. *(∗) Does every non-metrizable perfect compactum contains a closed subspace with $\aleph_1$-many clopen sets?*

2. Uncountable spaces

Call a space X *functionally countable* if every continuous real-valued function defined on X has countable range.

Question 12. *(∗) Is every first countable hereditarily functionally countable space countable?*

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[2]Unlike the other questions mentioned in this article, (∗) may not necessarily be an optimal hypothesis for giving a positive solution to this problem. In short the reason is that we can not assume without loss of generality that the space has weight $\aleph_1$. It still seems likely, however, that a forcing axiom is an appropriate hypothesis to yield a positive solution.

Question 13. *($*$) Does every uncountable functionally countable subspace of a countably tight compact space have an uncountable discrete subspace?*

Obviously any uncountable hereditarily functionally countable first countable space is a counterexample to the basis conjecture. Any uncountable left-separated subspace of a Suslin line is a consistent example of such a space. Currently the only known ZFC example of an uncountable functionally countable space with no uncountable discrete subspace is Moore's L-space, which is hereditarily functionally countable. Assuming $\mathsf{MA}_{\aleph_1}$, it is known that there are no first countable L-spaces and that any compactification of an L-space maps continuously onto $[0,1]^{\omega_1}$. Under ($*$), any functionally countable first countable space of countable spread must be both hereditarily Lindelöf and hereditarily separable, and any uncountable one would also be a counterexample to the basis conjecture.

Question 14. *Is it consistent that every uncountable first countable space of countable spread either contains an uncountable subspace of the Sorgenfrey line or has a countable network?*

If a positive answer to this question is consistent with $\mathsf{MA}_{\aleph_1}$, then this would also give a positive answer to the basis question, since $\mathsf{MA}_{\aleph_1}$ implies that any uncountable space with a countable network contains a uncountable separable metrizable subspace [10]. As with the basis conjecture, under PFA[10] (or even OCA[26]), this question has a positive in the class of cometrizable spaces, even without the first countable assumption.

Question 14 is related to some other questions concerning when spaces have a countable network. Recall that a subset Y of a space X is *weakly separated* if one can assign to each $y \in Y$ a neighborhood U_y of y such that $y \neq z$ implies $y \notin U_z$ or $z \notin U_y$. Note that if X has a countable network, then X does not contain an uncountable weakly separated subspace. The converse of this was asked by Tkachenko [19]:

Question 15. *Is it consistent that a space with no uncountable weakly separated must have a countable network?*

Unlike Question 14, this is open even in the non-first countable case. Todorcevic discusses this question in [26] and states that under PFA, if no finite power of a space X has an uncountable weakly separated subspace, then X has a countable network. Note that it follows from this result that under PFA Question 14 and Question 2 are equivalent.

The following also remain unsolved:

Question 16. *(a) Is it consistent that X has a countable network if X^2 has no uncountable discrete subspace? (b) What if X^ω is hereditarily separable and hereditarily Lindelöf?*

Question 16(b) is an old question of Arhangel'skii[2]. The square of Moore's L-space has uncountable spread, so is not a counterexample to Question 16(a). These questions are also open in the the first countable case, and in that case, a positive answer to Question 14 with PFA implies a positive answer to these as well.

3. Approaches, axiomatics, further reading

It should be emphasized that analysis of these problems would benefit greatly from a combinatorial reformulation or approximation, particularly one which is

Ramsey theoretic in nature. If there are positive solutions, Todorcevic's method of building forcings with models as side conditions will likely provide the basic framework. The standard source is [26]; further reading can also be found in [21] and [24]. The methods of [14] can be considered as a continuation of this theme.

In [23], Todorcevic has given positive answers to Fremlin's question and the basis problem in the rather broad class of spaces that can be represented as relatively compact subsets of the class $\mathcal{B}_1(X)$ of all Baire class 1 functions on some Polish space X endowed with the topology of pointwise convergence. Compact subsets of such $\mathcal{B}_1(X)$ are sometimes called 'Rosenthal compacta' since one interpretation of the famous Rosenthal ℓ_1-theorem says that the double dual ball of a separable Banach space containing no ℓ_1 equipped with the weak* topology is one example of such a compactum. The class also contains the split interval, the one point compactification of a discrete set of size at most $2^{\aleph_0}$, and is closed under the operations of taking countable products and closed subspaces. Todorcevic proves that if K is a Rosenthal compactum with no uncountable discrete subspaces, then K is perfect and premetric of order at most 2; moreover, if K is not metrizable, then it contains a full copy of the split interval.

Unlike the broader class of regular spaces, questions about Rosenthal compacta can typically be settled in the framework of ZFC. The analysis in [23], however, has a strong set theoretic theme and a number of the arguments presented there may give some insight into how to approach some of the problems in this article. The reader may also find [28] and [29] informative in a similar manner.

While a complete understanding of Woodin's axiom $(*)$ is probably not necessary for an analysis of these problems, it is worth making a few more remarks about it. Axiom $(*)$ is the assertion that $L(H(\aleph_1^+))$, where $H(\aleph_1^+)$ is the collection of all sets of hereditary cardinality at most $\aleph_1$, is a generic extension of $L(\mathbb{R})$ by the $\mathbb{P}_{\max}$ forcing. Many questions in this article can be cast in the language of $H(\aleph_1^+)$ since it is often possible to assume without loss of generality that the weight and possibly the cardinality of the space is at most $\aleph_1$. Furthermore, the assertions in the questions typically are Π_2 in their complexity — they have a pair $\forall X \exists Y$ of unbounded quantifiers followed by bounded quantification.[3] The $\mathbb{P}_{\max}$ forcing has the effect of making $H(\aleph_1^+)$ satisfy all Π_2 sentences which are Ω-*consistent*. Being Ω-consistent is a natural strengthening of "has a well founded model" — a precise definition can be found in [32]. For our purposes it is sufficient to say that if a statement can always be forced over any ground model with sufficient large cardinals, then it is Ω-consistent. All the forcing axioms and nearly all consistency results in set theoretic topology fit this description. Large cardinals are needed for the analysis of $\mathbb{P}_{\max}$ but these can often be avoided in applications if one wishes to obtain consistency results instead.

Another interesting property of the $\mathbb{P}_{\max}$ extension is its minimality. If G is $\mathbb{P}_{\max}$-generic over $L(\mathbb{R})$ and X is any new element of $H(\aleph_1^+)$, then $L(\mathbb{R})[X] = L(\mathbb{R})[G]$. Since a C-sequence on ω_1 can never be in $L(\mathbb{R})$ under appropriate large cardinal hypotheses, the $\mathbb{P}_{\max}$ extension is always of the form $L(\mathbb{R})[C]$ where C is some C-sequence on ω_1. In this context, $L(\mathbb{R})$ is a model in which the Axiom of Choice fails and which satisfies strong Ramsey theoretic statements (Ramsey's theorem

[3] X usually takes the form of a space, Y usually takes the form of either a substructure (e.g. an uncountable discrete subspace) or a connecting map (e.g. an embedding from an canonical space into X). The bounded quantification is usually made over the base and/or set of points in X.

holds for ω_1, for instance). This gives *a posteriori* explanation as to the role of Todorcevic's method of minimal walks [25] in building counterexamples such as Moore's L space [15]. This method involves an analysis of a number of two place functions which are recursively defined on C-sequences. It is likely that this method will be useful in constructing counterexamples related to the above questions. The reader is referred to [22] for further information.

It also seems plausible that a hypothesis such as the following may be useful in constructing an informative counterexample to some of these questions:

℧: There are continuous $f_\alpha : \alpha \to \omega$ ($\alpha < \omega_1$) such that if $E \subseteq \omega_1$ is closed and unbounded, then there is a δ in E such that f_δ takes all values on $E \cap \delta$.

A similar axiom was postulated long ago by Kunen in order to build a robust example related to the L-space problem. The object postulated by this axiom can naturally be used to strengthen the combinatorial objects constructed using the method of minimal walks. Since quantification is only over the closed unbounded filter, this axiom cannot be negated by c.c.c. forcing and hence is consistent with $\mathsf{MA}_{\aleph_1}$. It is even immune to Axiom A forcings and to the standard forcings built using models as side conditions (see,e.g., [26]). It cannot be used to construct, e.g., an S space. It has been used to construct a counterexample to Shelah's basis conjecture for the uncountable linear orders [16]. Whether ℧ can be used to construct a counterexample can, in general, be used as a litmus test for whether the more involved methods presented in [14] are needed to eliminate counterexamples (as opposed to the more user-friendly techniques of [26]). This axiom was also useful in constructing an L space which later was the prototype for a ZFC construction.

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