ELSEVIER

Topology and its Applications 120 (2002) 365–384

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

Weaker connected and weaker nowhere locally compact topologies for metrizable and similar spaces

G. Gruenhage [a,1,2], V.V. Tkachuk [b,*,1], R.G. Wilson [b,1]

[a] *Department of Mathematics, Auburn University, Auburn, AL 36849, USA*

[b] *Departamento de Matemáticas, Universidad Autónoma Metropolitana, Av. Michoacan y La Purísima, Iztapalapa A.P. 55-532, C.P. 09340, México, DF, Mexico*

Received 30 March 2000; received in revised form 5 January 2001

Abstract

We prove that any metrizable non-compact space has a weaker metrizable nowhere locally compact topology. As a consequence, any metrizable non-compact space has a weaker Hausdorff connected topology. The same is established for any Hausdorff space X with a σ-locally finite base whose weight $w(X)$ is a successor cardinal. © 2002 Elsevier Science B.V. All rights reserved.

AMS classification: Primary 54A10; 54D05; 54E35, Secondary 54C05; 54E40; 54E45

Keywords: Metrizable space; Nowhere locally compact space; Connected space; Condensation

0. Introduction

It was proved in [7] that the free union of countably many copies of the Cantor set cannot be condensed onto a regular connected space. This shows that not every non-compact separable metrizable space has a weaker connected regular topology. However, it is another result of [7], that every non-compact second countable regular space has a weaker Hausdorff connected topology. We extend this theorem to the class of all metric non-compact spaces. To do this, we first prove an auxiliary result which seems to be interesting in itself, namely that every metrizable non-compact space has a weaker metrizable nowhere locally compact topology. Metrizability is essential because there are

* Corresponding author.

E-mail addresses: gruengf@mail.auburn.edu (G. Gruenhage), vova@xanum.uam.mx (V.V. Tkachuk), rgw@xanum.uam.mx (R.G. Wilson).

[1] Research supported by Consejo Nacional de Ciencia y Tecnología (CONACYT) de México, grant 400200-5-3012P-E.

[2] Research of the first author also partially supported by National Science Foundation grant DMS 970-4849.

PII: S0166-8641(01)00082-7

simple examples showing that there exist locally compact non-compact spaces all of whose continuous Tychonoff images are locally compact. Clearly, such spaces can not have a weaker Tychonoff nowhere locally compact topology. The last group of results deals with the class of Hausdorff non-regular spaces with a σ-locally finite base. We prove that a space which belongs to this class has a weaker connected topology in case its weight is a successor cardinal.

1. Notation and terminology

All spaces under consideration are assumed to be Hausdorff. We use the term *condensation* to denote a continuous bijection. Given a set X we denote by $\exp(X)$ the family of all subsets of X. The ordinals are identified with the set of their predecessors. In particular, $n = \{0, \ldots, n-1\}$ for any $n \in \omega$. If (X, τ) is a space and $Y \subset X$ then $\tau|Y = \{U \cap Y\colon U \in \tau\}$. The expression $X \simeq Y$ means that X is homeomorphic to Y. Given a metric space (X, ρ) and an $A \subset X$ we denote by $\operatorname{diam}_\rho(A)$ the diameter of the set A with respect to ρ, i.e., $\operatorname{diam}_\rho(A) = \sup\{\rho(x, y)\colon x, y \in A\}$. The boundary $\overline{U}\backslash U$ is sometimes denoted by ∂U.

The interval $(0, 1] \subset \mathbb{R}$ is denoted by J. Given a cardinal κ, let $H(k) = \{v\} \cup J \times \kappa$, where $v = 0 \in \mathbb{R}$. Given an $\alpha < \kappa$, we will write J_α instead of $J \times \{\alpha\}$. Define a metric d on $H(\kappa)$ in the following way: $d(v, v) = 0$, $d(v, t) = t'$ if $t = (t', \alpha) \in J_\alpha$; given $t, s \in J_\alpha$, $t = (t', \alpha)$ and $s = (s', \alpha)$ let $d(t, s) = |t' - s'|$. If $t = (t', \alpha)$, $s = (s', \beta)$, where $\alpha \neq \beta$, then $d(t, s) = t' + s'$. The space $(H(\kappa), d)$ is called *the hedgehog with κ spines*. All other notions are standard and can be found in [3].

2. Condensations of metrizable spaces

Our purpose is to establish that every non-compact metric space condenses onto a nowhere locally compact metrizable space. When the extent is achievable, i.e., when the given metric space X has a closed discrete subspace of cardinality $w(X)$, the main idea is to make such a closed discrete set dense and nowhere locally compact in the new topology. However, not all metric spaces have an achievable extent, so we have two different cases to consider.

Lemma 2.1. *For any infinite cardinal κ there exists a metrizable locally convex linear topological space M_κ with the following properties*:

(1) $w(M_\kappa) = \kappa$;
(2) *$H(\kappa)^\omega$ embeds into M_κ, where $H(\kappa)$ is the hedgehog with κ spines*;
(3) *M_κ^ω is homeomorphic to M_κ.*

Proof. Take $M = \sigma(\mathbb{R}^\kappa) = \{x \in \mathbb{R}^\kappa\colon$ the set $x^{-1}(\mathbb{R} \setminus \{0\})$ is finite$\}$. The metric ρ on M is defined as follows: $\rho(x, y) = \sup\{|x(\alpha) - y(\alpha)|\colon \alpha < \kappa\}$. Since there are only finitely many non-zero differences, the metric ρ is well-defined. It is easy to see that $w((M, \rho)) = \kappa$.

For each $\alpha < \kappa$ let $I_\alpha = \{t \cdot \chi_{\{\alpha\}}\colon t \in [0,1]\}$. Here $\chi_{\{\alpha\}}:\kappa \to \{0,1\}$ is the characteristic function of the set $\{\alpha\}$, i.e., $\chi_{\{\alpha\}}(\beta) = 0$ if $\beta \neq \alpha$ and $\chi_{\{\alpha\}}(\alpha) = 1$. It is easy to see that the subspace $\bigcup\{I_\alpha\colon \alpha < \kappa\}$ of the space M is homeomorphic to $H(\kappa)$. Thus, $M_\kappa = M^\omega$ has the required properties. □

Let us say that a metric space X has *achievable extent* if $w(X) = \kappa$ and there is a closed discrete $D \subset X$ with $|D| = \kappa$. Note that no infinite metric space with an achievable extent can be compact.

Lemma 2.2. *Let X be an infinite metrizable space with achievable extent. Suppose that D is a closed discrete subspace of X with $|D| = \kappa = w(X)$ and F is a closed subset of X such that $F \cap D = \emptyset$. Then there is a condensation $\varphi : X \to Y$ of X onto a metrizable space Y with the following properties*:

(1) *each open subset of Y has cardinality at least κ*;
(2) *Y is nowhere locally compact*;
(3) *the set $\varphi(D)$ is dense in Y*;
(4) *the set $\varphi(F)$ is closed in Y and $\varphi|_F$ is a homeomorphism.*

Proof. The idea is to use a theorem of Dugundji [2] stating that for any metric space X and for any closed $G \subset X$, if we have a continuous function $f : G \to L$, where L is a locally convex linear topological space, then f extends continuously to a function from X to L.

Now, take a faithful enumeration $\{x_\alpha\colon \alpha < \kappa\}$ of the set D. Since $M_\kappa^\omega \simeq M_\kappa$, there is a closed nowhere dense $N \subset M_\kappa$ homeomorphic to M_κ. Every open set of M_κ has cardinality at least κ. Since the weight of M_κ is equal to κ, there exists a family $\mathcal{P} = \{P_\alpha\colon \alpha < \kappa\}$ such that:

(∗) for each $\alpha < \kappa$ we have $|P_\alpha| = \kappa$ and $P_\alpha \subset (\mathbb{R}\backslash\{0\}) \times (M_\kappa \setminus N)^\omega$ is dense in $(\mathbb{R}\backslash\{0\}) \times (M_\kappa \setminus N)^\omega$ and hence in $\mathbb{R} \times M_\kappa^\omega$;
(∗∗) $P_\alpha \cap P_\beta = \emptyset$ if $\alpha \neq \beta$.

Let $\{p_\alpha\colon \alpha < \kappa\}$ be a faithful enumeration of the set $P = \bigcup\{P_\alpha\colon \alpha \in \kappa\}$. Take any discrete family $\mathcal{U} = \{U_\alpha\colon \alpha < \kappa\}$ of open subsets of X such that $(\bigcup\mathcal{U}) \cap F = \emptyset$ and $x_\alpha \in U_\alpha$ for all $\alpha < \kappa$. For every $\alpha < \kappa$ choose open sets $U_\alpha^n \subset U_\alpha$ so that $\{x_\alpha\} = \bigcap\{U_\alpha^n\colon n \in \omega\}$ and $U_\alpha^{n+1} \subset U_\alpha^n$ for each $n \in \omega$.

Now, fix some embedding $e : X \to N$ (recall that N is homeomorphic to M_κ and hence contains a copy of $H(\kappa)^\omega$ and the latter in turn contains a copy of X). Let $\phi : X \to \mathbb{R}$ be an arbitrary continuous map such that $F = \phi^{-1}(0)$. For each natural number n define a map $\phi_n : D \cup (X \setminus \bigcup\{U_\alpha^n\colon \alpha < \kappa\}) \to M_\kappa$ in the following way: $\phi_n(x_\alpha) = p_\alpha(n)$ for each α and $\phi_n(x) = e(x)$ for any $x \in X \setminus (\bigcup\{U_\alpha^n\colon \alpha < \kappa\})$. Apply the theorem of Dugundji [2] to find a continuous map $\varphi_n : X \to M_\kappa$ such that the restriction of φ_n to $D \cup (X \setminus \bigcup\{U_\alpha^n\colon \alpha < \kappa\})$ is equal to ϕ_n. Then the mapping $\varphi = \phi\Delta(\Delta\{\varphi_n\colon n \in \omega\}) : X \to Y = \varphi(X) \subset \mathbb{R} \times M_\kappa^\omega$ has the required properties.

Note that $P \subset Y$ and therefore Y is dense in $\mathbb{R} \times M_\kappa^\omega$. This implies that Y is nowhere locally compact, i.e., (2) holds. Any open subset of Y has to intersect the set P_α for every $\alpha < \kappa$. This proves (1). Property (3) holds because $\varphi(D) = P$. The set $\varphi(F)$ is closed in Y due to the fact that $\pi_{\mathbb{R}}^{-1}(0) \cap Y = \varphi(F)$, where $\pi_{\mathbb{R}} : \mathbb{R} \times M_\kappa^\omega \to \mathbb{R}$ is the natural projection. Now $\varphi|_F$ is a homeomorphism because it is a diagonal product of ϕ and the homeomorphism $\Delta\{\varphi_n|_F : n \in \omega\}$.

Finally, to see that φ is a condensation, note that the points outside of D are separated by φ_n if they are not in $\bigcup\{U_\alpha^n : \alpha < \kappa\}$. The points of D are separated automatically by construction because of the faithfulness of our enumeration. Finally, the points of D are separated from those of $X \setminus D$ because the images of the points of $X \setminus D$ are eventually in N and the nth projection of P does not meet the set N. □

Corollary 2.3. *Let (X, τ) be a metrizable non-compact space with achievable extent. If $H \subset X$ is nowhere dense and closed, then there is a weaker metrizable nowhere locally compact topology τ' on X such that H is τ'-closed and $\tau'|H = \tau|H$.*

Proof. If $\kappa = w(X)$ then there exists a closed discrete $D \subset X$ such that $|D| = \kappa$ and $D \cap H = \emptyset$. Now apply Lemma 2.2. □

We now proceed to the case of metric spaces in which the extent is not achievable.

Lemma 2.4. *Let X be a metrizable space of uncountable weight in which the extent is not achievable. Denote by K the set of all points of X any neighbourhood of which has weight $w(X)$. Then K is compact and non-empty. As a consequence, we have $w(A) < w(X)$ for every $A \subset X$ with $\overline{A} \cap K = \emptyset$.*

Proof. Lemma 1 in [4] says that K is compact and non-empty. To prove the second part, observe that, without loss of generality, we can assume A to be closed. Now, if $w(A) = w(X) = \kappa$ then A is a closed subspace of X of weight κ such that any $a \in A$ has a neighbourhood in A of weight $< \kappa$. Applying again Lemma 1 from [4], we can conclude that the extent of A is achievable and hence A has a closed discrete subset D of cardinality κ. It is clear that D is also closed and discrete in X, so the extent of X is achieved which is a contradiction. □

Lemma 2.5. *Let (X, τ) be a metrizable space. Given a non-empty $U \in \tau$, suppose that τ_U is a weaker metrizable topology on $\mathrm{cl}_\tau(U)$ such that $\mathrm{cl}_\tau(U) \backslash U$ is τ_U-closed and $\tau_U|(\mathrm{cl}_\tau(U) \backslash U) = \tau|(\mathrm{cl}_\tau(U) \backslash U)$. Then there is a weaker metrizable topology τ' on X such that*:

(a) *the set $\mathrm{cl}_\tau(U)$ is τ'-closed and $\tau'|\mathrm{cl}_\tau(U) = \tau_U$;*
(b) *$\tau'|(X \backslash U) = \tau|(X \backslash U)$;*
(c) *for any $V \in \tau$, the set $U \cup V$ is τ'-open and $\mathrm{cl}_{\tau'}(U \cup V) = \mathrm{cl}_\tau(U \cup V)$.*

Proof. Fix some metric ρ which generates the topology τ. In this proof we denote by $\overline{A}$ the τ-closure of any $A \subset X$. Let $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \omega\}$ be a σ-discrete base for the space $(\overline{U}, \tau_U)$ such that each $\mathcal{B}_n$ is repeated infinitely often in the indexing. Fix a family $\{O_n : n \in \omega\}$ of τ-open sets containing $\overline{U}$ such that $\overline{U} = \bigcap\{O_n : n \in \omega\} = \bigcap\{\overline{O}_n : n \in \omega\}$.

Since (X,τ) is hereditarily collectionwise normal, it is not difficult to construct inductively (on $n\in\omega$) τ-discrete families of τ-open sets $\mathcal{B}_n^*=\{B^*\colon B\in\mathcal{B}_n\}$ so that:

(i) $B^*\cap\overline{U}=B$ and $\overline{B^*}\cap\overline{U}=\overline{B}$ for every $B\in\mathcal{B}_n$;

(ii) $B\subset U$ implies $B^*=B$ for any $B\in\mathcal{B}_n$;

(iii) if $B\in\mathcal{B}_i$ and $C\in\mathcal{B}_j$, where $i<j$ and $\overline{C}\subset B$, then $\overline{C^*}\subset B^*$;

(iv) if $B\in\mathcal{B}_i$ and $C\in\mathcal{B}_j$, where $i<j$ and $\overline{C}\cap\overline{B}=\emptyset$, then $\overline{C^*}\cap\overline{B^*}=\emptyset$;

(v) if $B\in\mathcal{B}_n$ and $\operatorname{diam}_\rho(B\cap\partial U)<1/m$ for some $m\leqslant n$, then $\operatorname{diam}_\rho(B^*\backslash U)<1/m$;

(vi) if $B\in\mathcal{B}_n$ then $B^*\subset O_n$.

Let τ' be the topology generated by the family $\mu\cup\mathcal{B}^*$, where $\mathcal{B}^*=\bigcup\{\mathcal{B}_n^*\colon n\in\omega\}$ and $\mu=\tau|(X\backslash\overline{U})$. Note that the condition (iii) implies that $\mu\cup\mathcal{B}^*$ is a base for τ'. Since $\bigcup\mu=X\backslash\overline{U}\in\tau'$, the set $\operatorname{cl}_\tau(U)$ is τ'-closed. Clearly, $\tau'\subset\tau$ and it is an immediate consequence of (i) that $\tau'|\overline{U}=\tau_U$ which settles (a). We will prove the properties (b) and (c) and the metrizability of τ'.

To establish that $\tau'|(X\backslash U)=\tau|(X\backslash U)$ take any $W\in\tau|(X\backslash U)$. We need to show that $W\in\tau'|(X\backslash U)$. Let $x\in W$. If $x\notin\overline{U}$ then $N_x=W\cap(X\backslash\overline{U})$ is a τ-open neighbourhood of the point x. Clearly, $N_x\in\tau'$ and $x\in N_x\subset W$. Suppose now that $x\in\overline{U}$. There exists an $\varepsilon>0$ such that $B(x,\varepsilon)\cap(X\backslash U)\subset W$, where $B(x,\varepsilon)=\{y\in X\colon \rho(x,y)<\varepsilon\}$ is the ε-ball centered at x. Since $\tau_U|\partial U=\tau|\partial U$, there exists a $B\in\mathcal{B}$ such that $x\in B$ and $B\cap\partial U\subset B(x,1/3m)$, where $m\in\mathbb{N}$ is chosen so that $1/m<\varepsilon$. Each $\mathcal{B}_n$ is repeated infinitely often, so $B\in\mathcal{B}_n$ for some $n\geqslant m$. Now that $\operatorname{diam}_\rho(B\cap\partial U)<1/m$, we can apply (v) to conclude that $\operatorname{diam}(B^*\backslash U)<1/m<\varepsilon$, which implies $B^*\backslash U\subset B(x,\varepsilon)\cap(X\backslash U)\subset W$. Thus, $W\in\tau'|(X\backslash U)$ and (b) is settled.

To prove (c), consider the topologies $\nu=\tau|(X\backslash U)$ and $\nu'=\tau'|(X\backslash U)$; property (b) says that $\nu=\nu'$. Fix any $V\in\tau$; to prove that $U\cup V$ is τ'-open consider the set $F=X\backslash(U\cup V)\subset X\backslash U$. The set $X\backslash U$ is τ-closed and τ'-closed as well, which implies $\operatorname{cl}_\tau(F)=\operatorname{cl}_\nu(F)$ and $\operatorname{cl}_{\tau'}(F)=\operatorname{cl}_{\nu'}(F)$. Now, since $\nu=\nu'$, we have $\operatorname{cl}_\nu(F)=\operatorname{cl}_{\nu'}(F)$ and therefore

$$F=\operatorname{cl}_\tau(F)=\operatorname{cl}_\nu(F)=\operatorname{cl}_{\nu'}(F)=\operatorname{cl}_{\tau'}(F),$$

i.e., the set F is τ'-closed. To show that $\operatorname{cl}_\tau(U\cup V)=\operatorname{cl}_{\tau'}(U\cup V)$, take any $x\in\operatorname{cl}_{\tau'}(U\cup V)$. If $x\in\overline{U}=\operatorname{cl}_{\tau'}(U)$, we are clearly done. If not, then

$$\begin{aligned}x\in\operatorname{cl}_{\tau'}\big(V\backslash\overline{U}\big)&\subset\operatorname{cl}_{\tau'}(V\backslash U)=\operatorname{cl}_{\nu'}(V\backslash U)\\&=\operatorname{cl}_\nu(V\backslash U)=\operatorname{cl}_\tau(V\backslash U)\subset\overline{V}\subset\overline{U\cup V},\end{aligned}$$

so (c) holds.

The topology τ' is regular. The regularity at any $x\in X\backslash\overline{U}$ follows easily from $\tau|(X\backslash U)=\tau'|(X\backslash U)$. So suppose that $x\in\overline{U}$ and $x\in V\in\tau'$. For some natural numbers $i<j$ we have $x\in C\subset\operatorname{cl}_{\tau_U}(C)\subset B$, where $B\in\mathcal{B}_i$, $C\in\mathcal{B}_j$ and $B^*\subset V$. Pick an arbitrary $y\in\operatorname{cl}_{\tau'}(C^*)$. There are two possible cases: $y\in\operatorname{cl}_{\tau'}(C^*\cap\overline{U})$ or $y\in\operatorname{cl}_{\tau'}(C^*\backslash\overline{U})$. In the first case

$$y\in\operatorname{cl}_{\tau'}\big(C^*\cap\overline{U}\big)=\operatorname{cl}_{\tau_U}\big(C^*\cap\overline{U}\big)=\operatorname{cl}_{\tau_U}(C)\subset B\subset B^*.$$

If the second case holds, we have

$$y \in \mathrm{cl}_{\tau'}\big(C^*\backslash\overline{U}\big) \subset \mathrm{cl}_{\tau'}\big(C^*\backslash U\big) = \mathrm{cl}_{\tau}\big(C^*\backslash U\big) \subset \overline{C^*} \subset B^*,$$

where the last inclusion follows from (iii). As y is an arbitrary point of $\mathrm{cl}_{\tau'}(C^*)$, we have shown that $\mathrm{cl}_{\tau'}(C^*) \subset B^*$, so the regularity at x is proved.

The topology τ' is metrizable. There is a base $\mathcal{C} = \bigcup\{\mathcal{C}_n\colon n \in \omega\}$ for $X\backslash\overline{U}$ such that $\mathcal{C}_n$ is a τ-discrete family for every $n \in \omega$ and $(\bigcup\mathcal{C}_n) \cap O_m = \emptyset$ for some $m \in \omega$. Then $\mathcal{C}$ is also a σ-discrete base in τ' for $X\backslash\overline{U}$. Since the family $\mathcal{B}^* = \bigcup\{\mathcal{B}_n^*\colon n \in \omega\}$ is a base for $\overline{U}$, we will be finished if we show that $\mathcal{B}_n^*$ is τ'-discrete for any $n \in \omega$.

If $x \notin \overline{U}$ then there is a $V \in \tau$ such that $x \in V \subset X\backslash\overline{U}$ and V meets at most one element of $\mathcal{B}_n^*$. Then $V \in \tau'$ so V witnesses the τ'-discreteness of $\mathcal{B}_n^*$ at the point x. If $x \in \overline{U}$ then there is a $C \in \mathcal{B}_m$ such that $x \in C,\ m > n$ and $\mathrm{cl}_{\tau_U}(C)$ meets at most one element of $\{\mathrm{cl}_{\tau_U}(B)\colon B \in \mathcal{B}_n\}$. Then, by (iv), the set $\overline{C^*}$ meets at most one element of $\{\overline{B^*}\colon B \in \mathcal{B}_n\}$. Therefore C^* meets at most one element of the family $\{\overline{B^*}\colon B \in \mathcal{B}_n\}$, which shows that $\mathcal{B}_n^* = \{B^*\colon B \in \mathcal{B}_n\}$ is τ'-discrete. □

Lemma 2.6. *Let (X,τ) be a metrizable space of weight $\kappa > \omega$ with a non-achievable extent. Denote by K the set $\{x \in X\colon$ any neighbourhood of x has weight $\kappa\}$. Suppose that H is a closed nowhere dense set in (X,τ). Then there is a weaker metrizable topology τ' of weight κ on X (and hence (X,τ') also has non-achievable extent) such that H is a closed nowhere dense subset of (X,τ') and the following conditions are fulfilled*:

(a) $\tau'|(X\backslash K) = \tau|(X\backslash K)$;

(b) $\tau'|H = \tau|H$;

(c) *there is a sequence of τ'-open finite covers $\{\mathcal{U}_n\colon n \in \omega\}$ of the subspace K such that the set $X\backslash \mathrm{cl}_{\tau'}(\bigcup\mathcal{U}_0)$ has a non-compact closure and we have*:

  (i) *for each $U \in \mathcal{U}_{n+1}$ there is a $V \in \mathcal{U}_n$ such that $\mathrm{cl}_{\tau'}(U) \subset V$, and, in particular, $\mathrm{cl}_{\tau'}(\bigcup\mathcal{U}_{n+1}) \subset \bigcup\mathcal{U}_n$*;

  (ii) *if $x \in U_n \in \mathcal{U}_n$ for all n, then $x \in K$ and $\{U_n\}_{n\in\omega}$ is a local base in τ' at the point x*;

  (iii) *for any $n \in \omega$, if we have a map $\sigma : n \to \exp(\bigcup_{i<n}\mathcal{U}_i)$ with $\sigma(i) \subset \mathcal{U}_i$, for each $i < n$, then $O = I(\sigma)\backslash\mathrm{cl}_{\tau'}(\bigcup\mathcal{U}_n) \neq \emptyset$ implies $\mathrm{cl}_{\tau'}(O)$ is not compact, where, for each family $\mathcal{V}_i \subset \mathcal{U}_i$, we let $I(\mathcal{U}_i,\mathcal{V}_i) = (\bigcap\mathcal{V}_i)\backslash\mathrm{cl}_{\tau'}(\bigcup(\mathcal{U}_i\backslash\mathcal{V}_i))$ and $I(\sigma) = \bigcap_{i<n} I(\mathcal{U}_i,\sigma(i))$.*

Proof. Note first that $K \cup H$ is τ-nowhere dense in X. Represent $X\backslash(K \cup H)$ as $\bigcup\{Q_n\colon n \in \omega\}$ where Q_n is τ-closed for each $n \in \omega$. If all Q_n's are compact then $X\backslash(K \cup H)$ is a second countable dense subspace of (X,τ) which is a contradiction because the weight of (X,τ) is uncountable. Thus, there exists an infinite discrete τ-closed set $E \subset X\backslash(K \cup H)$. Choose a faithfully enumerated set $D = \{d_n\colon n \in \omega\} \subset E$ such that $E\backslash D$ is infinite. Given an $n \in \omega$, let $\{C_{n,i}\colon i \in \omega\}$ be a sequence of open neighbourhoods of the point d_n such that $\bigcap\{C_{n,i}\colon i \in \omega\} = \{d_n\}$, $\mathrm{cl}_\tau(C_{n,i+1}) \subset C_{n,i}$ for all $n, i \in \omega$ and $\{C_{n,0}\colon n \in \omega\}$ is a discrete collection for which $\mathrm{cl}_\tau(\bigcup\{C_{n,0}\colon n \in \omega\}) \cap (K \cup H \cup (E\backslash D)) = \emptyset$.

Let $\{\mathcal{W}_n\colon\ n \in \omega\}$ be any sequence of finite open covers of K with the following properties:

(i′) for any $n \in \omega$ and any $W \in \mathcal{W}_{n+1}$ there is a $V \in \mathcal{W}_n$ such that $\mathrm{cl}_\tau(W) \subset V$.

(ii′) if $x \in W_n \in \mathcal{W}_n$ for each $n \in \omega$, then $x \in K$ and $\{W_n\}_{n\in\omega}$ is a local base at x.

(iii′) $((E\backslash D) \cup \mathrm{cl}_\tau(\bigcup\{C_{n,0}\colon\ n \in \omega\})) \cap \mathrm{cl}_\tau(\bigcup \mathcal{W}_0) = \emptyset$.

Let $\Delta = \{\delta\colon\ \delta$ is a function from some $n \in \omega$ into $\exp(\bigcup\{\mathcal{W}_i\colon\ i < n\})$ such that $\delta(i) \subset \mathcal{W}_i$ for every $i < n\}$. Given a $\delta \in \Delta$, let $J(\delta) = \bigcap\{J(\mathcal{W}_i, \delta(i))\colon\ i < n\}$, where $J(\mathcal{W}_i, \mathcal{V}_i) = (\bigcap \mathcal{V}_i)\backslash \mathrm{cl}_\tau(\bigcup(\mathcal{W}_i\backslash\mathcal{V}_i))$ for all $i < n$ and $\mathcal{V}_i \subset \mathcal{W}_i$. Now, choose any surjection $\theta : \omega \to \Delta' = \{\delta \in \Delta\colon\ J(\delta) \neq \emptyset\}$ such that $|\theta^{-1}(\delta)| = \omega$ for any $\delta \in \Delta'$. For each $i \in \omega$ and $W \in \mathcal{W}_i$, let $W^* = W \cup \bigcup\{C_{k,i}\colon\ W \in \theta(k)(i)\}$.

For $\mathcal{W} = \bigcup\{\mathcal{W}_i\colon\ i \in \omega\}$ and $\mu = \tau|(X\backslash K)$ let τ' be the topology generated by the family $\mu \cup \{W^*\colon\ W \in \mathcal{W}\}$. It is straightforward that τ' has the property (a). To show that τ' satisfies the rest of conditions, we will establish a series of “claims”.

Claim 1. *If $\delta \in \Delta,\ i \in \mathrm{dom}(\delta)$ and $W \in \mathcal{W}_i$, then $J(\delta) \subset W$ if $W \in \delta(i)$ and $J(\delta) \cap W = \emptyset$ if $W \notin \delta(i)$.*

Proof. Clear from the definition of $J(\delta)$.

Claim 2. *Suppose that $W \in \mathcal{W}_i,\ V \in \mathcal{W}_j$, where $i \leqslant j$ and $V \subset W$. Then $V^* \subset W^*$.*

Proof. By (iii′), it suffices to show that, for any natural k, if $V \in \theta(k)(j)$, then $W \in \theta(k)(i)$ (for then, if $C_{k,j}$ is put in V^*, the set $C_{k,i} \supset C_{k,j}$ will be put in W^*). Observe that if $V \in \theta(k)(j)$ and $W \notin \theta(k)(i)$, then $W \cap J(\theta(k)) = \emptyset$ and $J(\theta(k)) \subset V$ by Claim 1. Since $V \subset W$, we have $J(\theta(k)) = \emptyset$, which contradicts $\theta(k) \in \Delta'$.

Claim 3. *Suppose that $W \in \mathcal{W}_i,\ V \in \mathcal{W}_j$ and $x \in W^* \cap V^* \cap K$. Then there is an $O \in \mathcal{W}$ such that $x \in O^* \subset W^* \cap V^*$.*

Proof. Since $(\bigcup\{C_{n,0}\colon\ n \in \omega\}) \cap K = \emptyset$, we have $x \in W \cap V$. The point x is not isolated, and the family $\bigcup\{\mathcal{W}_k\colon\ k \leqslant i + j\}$ is finite so, by (ii′) there exists an $m \geqslant i + j$ and $O \in \mathcal{W}_m$ such that $O \subset U \cap V$. Now apply Claim 2 to conclude that $O^* \subset W^* \cap V^*$.

Remark. It is clear that Claim 3 implies that the family $\mu \cup \{W^*\colon\ W \in \mathcal{W}\}$ is a base for the topology τ'.

Claim 4. *If $W \in \mathcal{W}_i,\ V \in \mathcal{W}_j$ and $W \cap V = \emptyset$, then $W^* \cap V^* = \emptyset$.*

Proof. If $W^* \cap V^* \neq \emptyset$, then there is a $k \in \omega$ such that $C_{k,i} \subset W^*$ and $C_{k,j} \subset V^*$. This implies $W \in \theta(k)(i)$ and $V \in \theta(k)(j)$. Thus, by Claim 1, $J(\theta(k)) \subset W \cap V = \emptyset$ which is a contradiction.

Claim 5. *For any $W \in \mathcal{W}$ we have $\mathrm{cl}_{\tau'}(W^*) = \mathrm{cl}_\tau(W^*)$.*

Proof. We only have to show that $\mathrm{cl}_{\tau'}(W^*) \subset \mathrm{cl}_\tau(W^*)$. Suppose, for a contradiction, that $x \in \mathrm{cl}_{\tau'}(W^*)\backslash\mathrm{cl}_\tau(W^*)$. Then, clearly, $x \in K$, so there exists a $V \in \mathcal{W}$ such that $x \in V$ and $V \cap W^* = \emptyset$. Of course, this implies $V \cap W = \emptyset$, and, applying Claim 4, we obtain $V^* \cap W^* = \emptyset$, a contradiction.

Claim 6. *If $W \in \mathcal{W}_i$, and $i \notin \mathrm{dom}(\delta)$, where $\delta \in \Delta'$, then $W^* \cap C_{k,0} = \emptyset$ for any $k \in \theta^{-1}(\delta)$.*

Proof. It is clear from the definition of W^* that $W^* \cap C_{k,0} \neq \emptyset$ if and only if $W \in \theta(k)(i)$. Thus, $i \notin \mathrm{dom}(\theta(k))$ implies $W^* \cap C_{k,0} = \emptyset$.

Claim 7. *If $O \subset X\backslash K$ is a τ-open set, $\mathrm{cl}_\tau(O) \cap K = \emptyset$ and O meets $C_{k,0}$ only for finitely many k's, then $\mathrm{cl}_{\tau'}(O) = \mathrm{cl}_\tau(O)$.*

Proof. We only have to show that $\mathrm{cl}_{\tau'}(O) \subset \mathrm{cl}_\tau(O)$. Suppose, by way of contradiction, that $x \in \mathrm{cl}_{\tau'}(O)\backslash\mathrm{cl}_\tau(O)$. Then $x \in K$. Choose a $W \in \mathcal{W}_i$ with $x \in W$, $W \cap O = \emptyset$ and i sufficiently large so that $i \notin \mathrm{dom}(\theta(k))$ whenever $C_{k,0} \cap O \neq \emptyset$. Then $W^* \cap O = \emptyset$ by Claim 6, which is a contradiction.

Claim 8. *If $W \in \mathcal{W}_i$, $V \in \mathcal{W}_j$ for some $j > i$ and $\mathrm{cl}_\tau(V) \subset W$, then $\mathrm{cl}_{\tau'}(V^*) \subset W^*$.*

Proof. Given any $k \in \omega$, if $V \in \theta(k)(j)$, then, by Claim 1, $J(\theta(k)) \subset V \subset W$. Applying Claim 1 once more we see that $W \in \theta(k)(i)$. Thus, $C_{k,j} \subset V$ implies $C_{k,i} \subset W$ and therefore $\mathrm{cl}_\tau(C_{k,j}) \subset C_{k,i} \subset W^*$ for any $k \in \omega$ with $C_{k,j} \subset V$. This shows that $\mathrm{cl}_\tau(V^*) \subset W^*$. By Claim 5, we have $\mathrm{cl}_{\tau'}(V^*) = \mathrm{cl}_\tau(V^*) \subset W^*$.

Claim 9. *The topology τ' is regular.*

Proof. If $x \notin K$, then for any τ'-neighbourhood W of the point x there exists $V \in \tau$ such that $x \in V \subset X\backslash K$, the set V intersects at most one element of the family $\{C_{k,0}\colon k \in \omega\}$ and $\mathrm{cl}_\tau(V) \subset W$. Now, apply Claim 7 to conclude that $\mathrm{cl}_{\tau'}(V) \subset W$.

Suppose that $x \in K \cap W^*$ for some $W \in \mathcal{W}_i$. There exists a $j > i$ such that $x \in V \subset \mathrm{cl}_\tau(V) \subset W$ for some $V \in \mathcal{W}_j$. Then $\mathrm{cl}_{\tau'}(V^*) \subset W^*$ by Claim 8, and we are done.

Claim 10. *The topology τ' is metrizable and $w(X, \tau') = \kappa$.*

Proof. There is a family $\mathcal{B} = \bigcup\{\mathcal{B}_n\colon n \in \omega\} \subset \tau$ with the following properties:

(1) $\mathcal{B}$ contains a local base in τ for all points of $X\backslash K$ and each $\mathcal{B}_n$ is τ-discrete.

(2) For any $n \in \omega$ the set $\bigcup \mathcal{B}_n$ meets at most one of the sets $\{C_{k,0}\colon k \in \omega\}$.

(3) $\mathrm{cl}_\tau(\bigcup \mathcal{B}_n) \cap K = \emptyset$ for any $n \in \omega$.

Property (a) implies that $\mathcal{B}$ also contains a local base in τ' for all points of $X\backslash K$. If $x \in X\backslash K$ and $n \in \omega$, then there is a $W \in \tau$ such that $x \in W \subset X\backslash K$ and W intersects at most one element of $\mathcal{B}_n$. Since $W \in \tau'$, the family $\mathcal{B}_n$ is discrete in $X\backslash K$. If $x \in K$ then we can apply (2) and Claim 7 to conclude that $\mathrm{cl}_{\tau'}(\bigcup \mathcal{B}_n) = \mathrm{cl}_\tau(\bigcup \mathcal{B}_n) \subset X\backslash K$ and therefore

$V = X\setminus \mathrm{cl}_{\tau'}(\bigcup \mathcal{B}_n)$ is a τ'-neighbourhood of x which intersects no elements of $\mathcal{B}_n$. As a consequence, the family $\mathcal{B}_n$ is τ'-discrete in X and $\mathcal{B} \cup \{W^*\colon\ W \in \mathcal{W}\}$ is a σ-discrete base for τ'.

Since $w(X, \tau') \leqslant w(X, \tau) = \kappa$, we only have to prove that $w(X, \tau') \geqslant \kappa$. Note first that it is impossible that $w(X\setminus K) < \kappa$ because $w(K) = \omega$ and weight is additive in metrizable spaces. Hence $w(X\setminus K) = \kappa$. Since the topology on $X\setminus K$ was not changed, it is a subspace of (X, τ') of weight κ. This proves that $w(X, \tau') = \kappa$.

Claim 11. *The set H is closed and nowhere dense in (X, τ') and the topology τ' satisfies* (b)*, i.e., $\tau'|H = \tau|H$.*

Proof. If $x \in X\setminus(H \cup K)$ then $W = X\setminus(H \cup K)$ is a τ'-neighbourhood of x with $W \cap H = \emptyset$. If $x \in (X\setminus H) \cap K$ then there is a $W \in \mathcal{W}$ such that $x \in W$ and $W \cap H = \emptyset$. Now, we have $W^* \cap H = \emptyset$ because $H \cap (\bigcup_{k\in\omega} C_{k,0}) = \emptyset$. This proves that H is τ'-closed. Since a non-empty interior of H in (X, τ') implies H has a non-empty interior in (X, τ), the set H is nowhere dense in (X, τ').

Finally, $\tau'|H = \tau|H$ follows from the equality $\tau'|(X\setminus K) = \tau|(X\setminus K)$ and the fact that, for any $W \in \mathcal{W}$, we have $W^* \cap H = W \cap H$ (the last equality holds because $H \cap (\bigcup_{k\in\omega} C_{k,0}) = \emptyset$).

For any $n \in \omega$, let $\mathcal{U}_n = \{W^*\colon\ W \in \mathcal{W}_n\}$. We now proceed to show that the sequence $\{\mathcal{U}_n\colon\ n \in \omega\}$ satisfies (c).

Claim 12. *The subspace $X\setminus \mathrm{cl}_{\tau'}(\bigcup \mathcal{U}_0)$ of the space (X, τ') has a non-compact closure and, for each $U \in \mathcal{U}_{n+1}$, there exists a $V \in \mathcal{U}_n$ such that $\mathrm{cl}_{\tau'}(U) \subset V$, whence the property* (c)(i) *holds for the sequence $\{\mathcal{U}_n\colon\ n \in \omega\}$. In particular, $\mathrm{cl}_{\tau'}(\bigcup \mathcal{U}_{n+1}) \subset \bigcup \mathcal{U}_n$ for all $n \in \omega$.*

Proof. It is straightforward that $E\setminus D \subset X\setminus \mathrm{cl}_\tau(\bigcup \mathcal{U}_0) = X\setminus \mathrm{cl}_{\tau'}(\bigcup \mathcal{U}_0)$ is infinite, closed and discrete in (X, τ') which proves that no subspace containing $X\setminus \mathrm{cl}_\tau(\bigcup \mathcal{U}_0)$ is compact. The rest of the claim is an immediate consequence of (i') and Claim 8.

Claim 13. *If $x \in U_n \in \mathcal{U}_n$ for all $n \in \omega$ then $x \in K$ and $\{U_n\}_{n\in\omega}$ is a local base for τ' at x. Therefore the property* (c)(ii) *is fulfilled.*

Proof. Take a $W_n \in \mathcal{W}_n$ with $W_n^* = U_n$ for each $n \in \omega$. For any natural number k there is an $n \in \omega$ such that $i < n$ for all $i \in \mathrm{dom}(\theta(k))$. Apply Claim 6 to conclude that $C_{k,0} \cap U_n = C_{k,0} \cap W_n^* = \emptyset$. Therefore we must have $x \in K$ and hence $x \in W_n \in \mathcal{W}_n$ for each $n \in \omega$. By (ii$'$), the family $\{W_n\}_{n\in\omega}$ is a local base for τ at the point x. Now Claim 2 implies that $\{U_n\}_{n\in\omega} = \{W_n^*\}_{n\in\omega}$ is a local base for τ' at x.

It remains to show that (c)(iii) is fulfilled. Assume that we are given a mapping $\sigma : n \to \exp(\bigcup_{i<n} \mathcal{U}_i)$ with $\sigma(i) = \{U_1^i, \ldots, U_{k_i}^i\} \subset \mathcal{U}_i$ for each i. Choose $W_j^i \in \mathcal{W}_i$ so that $U_j^i = (W_j^i)^*$ for all $i < n$ and $j \in \{1, \ldots, k_i\}$. Given an $i < n$, consider the set $\delta_\sigma(i) = \{W_1^i, \ldots, W_{k_i}^i\} \subset \mathcal{W}_i$. It is clear that the last formula defines a map $\delta_\sigma \in \Delta$. We need to show that $O = I(\sigma)\setminus \mathrm{cl}_{\tau'}(\bigcup \mathcal{U}_n) \neq \emptyset$ implies that the τ'-closure of O is not compact.

Claim 14. *If $I(\sigma) \neq \emptyset$ then $J(\delta_\sigma) \neq \emptyset$.*

Proof. Suppose that $\mathrm{dom}(\sigma) = n$. Let us prove first that $I(\sigma) \subset J(\delta_\sigma) \cup (I(\sigma) \cap C)$, where $C = \bigcup\{C_{m,0}\colon m \in \omega\}$. To do this, take any $x \in I(\sigma) \setminus J(\delta_\sigma)$. Then, for some $i < n$, we have $x \notin J(\mathcal{W}_i, \delta_\sigma(i))$. But $x \in I(\mathcal{U}_i, \sigma(i))$ and therefore $x \notin \mathrm{cl}_{\tau'}(W^*) = \mathrm{cl}_\tau(W^*) \supset \mathrm{cl}_\tau(W)$, for any $W \in \mathcal{W}_i \setminus \delta_\sigma(i)$. This makes it impossible that $x \in \bigcap \delta_\sigma(i)$ and hence there is a $V \in \delta_\sigma(i)$ such that $x \notin V$. But $V^* \in \sigma(i)$, so we have $x \in V^* = V \cup (V^* \cap C)$ which implies $x \in C$ proving the promised inclusion.

Now, fix an $x \in I(\sigma) \subset J(\delta_\sigma) \cup (I(\sigma) \cap C)$. If $x \in J(\delta_\sigma)$ we are done, so we may assume that $x \in C_{k,0}$ for some $k \in \omega$. For any $W \in \delta_\sigma(i)$, we have $x \in W^* \cap C_{k,0}$ and hence $W \in \theta(k)(i)$ which in turn implies $W^* \cap C_{k,0} = C_{k,i}$. On the other hand, if $W \in \mathcal{W}_i \setminus \delta_\sigma(i)$ then $W^* \cap C_{k,i} = \emptyset$ and therefore $W \notin \theta(k)(i)$. As a consequence, $\theta(k)(i) = \delta_\sigma(i)$ for each $i < n$, which means that the map $\theta(k)$ extends δ_σ, i.e., $\theta(k)|n = \delta_\sigma$. Since $\theta(k) \in \Delta'$ we have $J(\theta(k)) \neq \emptyset$. Observe that $J(\theta(k)) \subset J(\delta_\sigma)$ and hence $J(\delta_\sigma) \neq \emptyset$.

Given a $\delta \in \Delta'$ and an $x \in K$, take any $n \in \omega \setminus \mathrm{dom}(\delta)$. If $x \in W \in \mathcal{W}_n$ then, by Claim 6, $W^* \cap C_{k,0} = \emptyset$ for any $k \in \theta^{-1}(\delta)$. This shows that the infinite collection $\{C_{k,0}\colon \theta(k) = \delta\}$ is τ'-discrete. Thus, the following fact will finish the proof that the sequence $\{\mathcal{U}_n\colon n \in \omega\}$ satisfies (c)(iii) and hence the proof of Lemma 2.6 will be complete.

Claim 15. *Let $\sigma : n \to \exp(\bigcup_{i<n} \mathcal{U}_i)$ be a map such that $\sigma(i) \subset \mathcal{U}_i$ for each $i < n$. Then $I(\sigma) \neq \emptyset$ implies $\delta_\sigma \in \Delta'$ and $I(\sigma) \setminus \mathrm{cl}_{\tau'}(\bigcup \mathcal{U}_n) \supset C_{k,n-1}$ for every $k \in \theta^{-1}(\delta_\sigma)$. Therefore $I(\sigma) \setminus \mathrm{cl}_{\tau'}(\bigcup \mathcal{U}_n)$ has non-compact τ'-closure.*

Proof. Suppose that $I(\sigma) \neq \emptyset$. Apply Claim 14 to conclude that $J(\delta_\sigma) \neq \emptyset$ and therefore $\delta_\sigma \in \Delta'$. Let $k \in \theta^{-1}(\delta_\sigma)$. Then, if $W \in \delta_\sigma(i)$ then, by definition of W^*, we have $W^* \supset C_{k,i} \supset C_{k,n-1}$. Now, if $W \in \mathcal{W}_i \setminus \delta_\sigma(i)$, then $W \notin \theta(k)(i)$, so $W^* \cap C_{k,0} = \emptyset$. As a consequence, $C_{k,n-1} \subset I(\sigma)$. Finally, for any $W \in \mathcal{W}_n$ we have $n \notin \mathrm{dom}(\sigma)$, so $W^* \cap C_{k,0} = \emptyset$ by Claim 6. This shows that

$$\bigcup\left\{C_{k,n-1}\colon k \in \theta^{-1}(\delta_\sigma)\right\} \subset I(\sigma) \setminus \mathrm{cl}_{\tau'}\Big(\bigcup \mathcal{U}_n\Big),$$

as required. □

Theorem 2.7. *Let (X, τ) be a metrizable non-compact space. Then, for any nowhere dense closed $H \subset X$, there exists a weaker nowhere locally compact metrizable topology τ' on X such that H is τ'-closed and $\tau'|H = \tau|H$.*

Proof. If $w(X) = \omega$ then the extent of X is achievable so our theorem holds by Corollary 2.3. Suppose that $w(X) = \kappa > \omega$ and the theorem is true for any metric non-compact Y with $w(Y) < \kappa$. By Corollary 2.3, if the extent of X is achieved, then the theorem holds, so let us assume that it is not achieved. Consider the compact set $K = \{x \in X\colon$ every neighbourhood of the point x has weight $\kappa\}$. By Lemma 2.6, by weakening the topology if necessary, we may assume that there is a sequence $\{\mathcal{U}_n\colon n \in \omega\}$ of finite open covers of K satisfying the conditions of Lemma 2.6(c). We will define a sequence $\{\tau_i\colon\ i\in\omega\}$ of successively weaker metrizable topologies on X satisfying the following properties (in this proof the bar is used to denote the τ-closure):

(i) $\tau_n|(X\backslash\overline{\bigcup\mathcal{U}_n})$ is nowhere locally compact;

(ii) $\tau_{n+1}|(X\backslash\bigcup\mathcal{U}_n)=\tau_n|(X\backslash\bigcup\mathcal{U}_n)$;

(iii) if $U\in\bigcup_{n\in\omega}\mathcal{U}_n$ then $U\in\bigcap\{\tau_n\colon\ n\in\omega\}$ and $\mathrm{cl}_{\tau_n}(U)=\mathrm{cl}_\tau(U)$ for any $n\in\omega$;

(iv) $\tau_n|(\overline{\bigcup\mathcal{U}_n})=\tau|(\overline{\bigcup\mathcal{U}_n})$ for any $n\in\omega$;

(v) the set H is τ_n-closed and $\tau_n|H=\tau|H$ for any $n\in\omega$.

To define τ_0, consider the set $W=X\backslash\overline{\bigcup\mathcal{U}_0}$. The subspace $\overline{W}$ is non-compact by condition (c) of Lemma 2.6 and has weight $<\kappa$ by Lemma 2.4. By the inductive hypothesis there is a weaker nowhere locally compact metrizable topology τ_W on $\overline{W}$ such that $H'=(\overline{W}\backslash W)\cup(H\cap\overline{W})$ is τ_W-closed and $\tau_W|H'=\tau|H'$. Note that any τ-closed $F\subset H'$ is τ_W-closed. Indeed, F is closed in $(H',\tau|H')$ and hence it is closed in $(H',\tau_W|H')$. Since H' is τ_W-closed, the set F has to be τ_W-closed. In particular, each one of the sets $\overline{W}\backslash W$ and $H\cap\overline{W}$ is τ_W-closed.

By Lemma 2.5, there is a weaker metrizable topology τ_0 on X such that the set $\overline{W}$ is τ_0-closed, $\tau_0|\overline{W}=\tau_W$ and $\tau_0|(X\backslash W)=\tau|(X\backslash W)$. The property (i) for τ_0 holds because τ_W is nowhere locally compact and $W=X\backslash\overline{\bigcup\mathcal{U}_0}$ is dense in $\overline{W}$. The property (ii) is vacuous, so let us check (iii). If $U\in\bigcup_{n\in\omega}\mathcal{U}_n$ then $U\subset X\backslash\overline{W}$. Thus $U\in\tau|(X\backslash\overline{W})=\tau_0|(X\backslash\overline{W})$. Since $X\backslash\overline{W}$ is τ_0-open, we have $U\in\tau_0$. The set $X\backslash W$ is τ_0-closed and $U\subset X\backslash W$, so the τ_0-closure of U coincides with its μ_0-closure for $\mu_0=\tau_0|(X\backslash W)$. Analogously, the τ-closure of U coincides with its μ-closure where $\mu=\tau|(X\backslash W)$. Since $\mu_0=\mu$, we have $\mathrm{cl}_\tau(U)=\mathrm{cl}_{\tau_0}(U)$. The property (iv) is a reformulation of the equality $\mu_0=\mu$, so let us prove (v).

Since H' and H are τ-closed, the set $H(1)=H\cap\overline{W}\subset H'$ is τ-closed and hence $H(1)$ is closed in $\tau|H'=\tau_W|H'=\tau_0|H'$. The set H' is τ_W-closed and hence it is closed in $(\overline{W},\tau_0|\overline{W})=(\overline{W},\tau_W)$. Since $\overline{W}$ is τ_0-closed, the set H' is also τ_0-closed, which implies that $H(1)$ is τ_0-closed. Applying once more the equality $\mu=\mu_0$ and the fact that $X\backslash W$ is τ_0-closed, we can conclude that the set $H(2)=H\cap(X\backslash W)$ is τ_0-closed. As a consequence, $H=H(1)\cup H(2)$ is τ_0-closed. Finally, $\tau_0|H=\tau|H$ because for any τ-closed $F\subset H$ we have $F=F_1\cup F_2$, where $F_i=F\cap H(i)$ and hence F_i is τ_0-closed for $i=1,2$.

Suppose that τ_n has been defined; we show how to define τ_{n+1}. Let $\{\sigma_i\colon\ i\leqslant k\}$ index all functions $\sigma:(n+1)\to\exp(\bigcup_{i\leqslant n}\mathcal{U}_i)$ such that $\sigma(i)\subset\mathcal{U}_i$ for any i and $I(\sigma)\backslash\overline{\bigcup\mathcal{U}_{n+1}}\neq\emptyset$ (see the definition of $I(\sigma)$ in Lemma 2.6(c)). Observe that $I(\sigma_i)\subset\bigcup\mathcal{U}_n$ for any $i\leqslant n$. Consider the set $S_0=I(\sigma_0)\backslash\overline{\bigcup\mathcal{U}_{n+1}}$. By (iv) the τ-closure of S_0 coincides with its τ_n-closure and therefore $\mathrm{cl}_{\tau_n}(S_0)$ is not compact by Lemma 2.6(c)(iii). Since $\overline{S}_0\cap K=\emptyset$, the weight of $\overline{S}_0$ is less than κ. By the inductive hypothesis, there is a weaker nowhere locally compact metrizable topology ρ_0 on $\mathrm{cl}_{\tau_n}(S_0)=\mathrm{cl}_\tau(S_0)$ such that the set $H_0=(\overline{S}_0\backslash S_0)\cup(\overline{S}_0\cap H)$ is ρ_0-closed and $\rho_0|H_0=\tau_n|H_0=\tau|H_0$.

Apply Lemma 2.5 to obtain a weaker metrizable topology $\tau_{n,0}\subset\tau_n$ such that $\tau_{n,0}|\overline{S}_0=\rho_0$, $\tau_{n,0}|(X\backslash S_0)=\tau_n|(X\backslash S_0)$ and for any τ_n-open set V the set $S_0\cup V$ is $\tau_{n,0}$-open and $\mathrm{cl}_{\tau_{n,0}}(S_0\cup V)=\mathrm{cl}_{\tau_n}(S_0\cup V)$.

Suppose that $m < k$ and we defined metrizable topologies $\tau_{n,-1}, \tau_{n,0}, \dots, \tau_{n,m}$ such that the following conditions are fulfilled for all $i = 0, \dots, m$:

(t_1) $\tau_{n,-1} = \tau_n \supset \tau_{n,0} \supset \cdots \supset \tau_{n,i-1} \supset \tau_{n,i}$;

(t_2) $\tau_{n,i}|\overline{S}_i$ is nowhere locally compact;

(t_3) $\tau_{n.i}|(X\backslash S_i) = \tau_{n,i-1}|(X\backslash S_i)$;

(t_4) the set $H_i = (\overline{S}_i\backslash S_i) \cup (\overline{S}_i \cap H)$ is $\tau_{n,i}$-closed;

(t_5) $\tau_{n,i}|H_i = \tau|H_i$;

(t_6) for any $V \in \tau_{n,i-1}$ the set $S_i \cup V$ is $\tau_{n,i}$-open and $\mathrm{cl}_{\tau_{n,i}}(S_i \cup V) = \mathrm{cl}_{\tau_{n,i-1}}(S_i \cup V)$.

Observe first that the family $\{S_i = I(\sigma_i)\backslash\overline{\bigcup\mathcal{U}_{n+1}}\colon\ i \leqslant k\}$ consists of disjoint open subsets of $\bigcup\mathcal{U}_n\backslash\overline{\bigcup\mathcal{U}_{n+1}}$ and that $\tau_{n,m}$ restricted to $\overline{S}_i$ is equal to $\tau_n|\overline{S}_i$ for all $i > m$.

Consider the set $S_{m+1} = I(\sigma_{m+1})\backslash\overline{\bigcup\mathcal{U}_{n+1}}$. By (iv) and ($t_3$) the τ-closure of S_{m+1} coincides with its $\tau_{n,m}$-closure and therefore $\mathrm{cl}_{\tau_{n,m}}(S_{m+1})$ is not compact by Lemma 2.6(c)(iii). Since $\overline{S}_{m+1} \cap K = \emptyset$, the weight of $\overline{S}_{m+1}$ is less than κ. By the inductive hypothesis, there is a weaker nowhere locally compact metrizable topology ρ_{m+1} on $\mathrm{cl}_{\tau_{n,m}}(S_{m+1}) = \mathrm{cl}_\tau(S_{m+1})$ such that the set $H_{m+1} = (\overline{S}_{m+1}\backslash S_{m+1}) \cup (\overline{S}_{m+1} \cap H)$ is ρ_{m+1}-closed and $\rho_{m+1}|H_{m+1} = \tau_n|H_{m+1} = \tau|H_{m+1}$.

Apply Lemma 2.5 to obtain a weaker metrizable topology $\tau_{n,m+1} \subset \tau_{n,m}$ such that $\tau_{n,m+1}|\overline{S}_{m+1} = \rho_{m+1}$, $\tau_{n,m+1}|(X\backslash S_{m+1}) = \tau_{n,m}|(X\backslash S_{m+1})$ and for any $V \in \tau_{n,m}$, the set $S_{m+1} \cup V$ is also $\tau_{n,m+1}$-open and we have $\mathrm{cl}_{\tau_{n,m+1}}(S_{m+1} \cup V) = \mathrm{cl}_{\tau_{n.m}}(S_{m+1} \cup V)$. It is clear that the conditions (t_1)–(t_6) are fulfilled for $i \leqslant m + 1$. Therefore our inductive construction can continue until $m = k$.

We claim that the topology $\tau_{n+1} = \tau_{n,k}$ satisfies the conditions (i)–(v). Condition (iv) is clear since $\tau_{n,i}$ does not alter the topology on $X\backslash S_i \supset \overline{\bigcup\mathcal{U}_{n+1}}$.

We will show that τ_{n+1} satisfies (v). Note that, by (iv), we have $\mathrm{cl}_{\tau_n}(S_i) = \overline{S}_i$ for every $i \in \{0, \dots, k\}$. Observe that, letting $V = \emptyset$ in (t_6), we obtain $\mathrm{cl}_{\tau_{n,i}}(S_i) = \mathrm{cl}_{\tau_{n,i-1}}(S_i)$ for each $i \in \{0, \dots, k\}$. It is clear that by induction we can conclude that $\mathrm{cl}_{\tau_{n,i}}(S_i) = \mathrm{cl}_{\tau_n}(S_i) = \overline{S}_i$ for every $i \leqslant k$. Now let us establish by induction on i that $\tau_{n,i}$ has property (v) for every $i \leqslant k$. The case of $i = -1$ is clear. Suppose that, for every $j < i \leqslant k$, the topology $\tau_{n,j}$ satisfies (v).

The set $H(1) = \overline{S}_i \cap H \subset H_i$ is τ-closed and hence it is closed in the topology $\tau|H_i = \tau_{n,i}|H_i$. Since H_i is $\tau_{n,i}$-closed by (t_4), the set $H(1)$ is $\tau_{n,i}$-closed. Now the set H is $\tau_{n,i-1}$-closed by the inductive hypothesis, so $H(2) = H\backslash S_i \subset X\backslash S_i$ is also $\tau_{n,i-1}$-closed. Consequently, $H(2)$ is closed in the topology $\tau_{n,i-1}|(X\backslash S_i) = \tau_{n,i}|(X\backslash S_i)$. Thus, $H(2)$ is closed in $\tau_{n,i}|(X\backslash S_i)$ while $X\backslash S_i$ is $\tau_{n,i}$-closed. Therefore $H(2)$ is $\tau_{n,i}$-closed as well as $H = H(1) \cup H(2)$. Finally, if F is a τ-closed subspace of H then $F = F_1 \cup F_2$, where $F_i = F \cap H(i)$ for $i = 1, 2$. Since $F(1)$ is a closed subspace of H_i, the property (t_5) implies that $F(1)$ is $\tau_{n,i}$-closed. The inductive hypothesis easily implies that $F(2)$ is $\tau_{n,i-1}$-closed and hence $\tau_{n,i}$-closed because $\tau_{n,i}$ and $\tau_{n,i-1}$ coincide on $H\backslash S_i$. Therefore any τ-closed subset of H is $\tau_{n,i}$-closed which proves that $\tau_{n,i}|H = \tau|H$. Thus, τ_{n+1} satisfies (v).

Since $S_i \subset \bigcup\mathcal{U}_n$ and $\tau_{n,i}|(X\backslash S_i) = \tau_{n,i-1}|(X\backslash S_i)$, the topology outside of $X\backslash\bigcup\mathcal{U}_n$ has not been altered from $\tau_n|(X\backslash\bigcup\mathcal{U}_n)$ and the condition (ii) follows.

For (iii) we need to show that if $U \in \bigcup_{m\in\omega} \mathcal{U}_m$, then $U \in \tau_{n+1}$ and $\mathrm{cl}_{\tau_{n+1}}(U) = \mathrm{cl}_\tau(U)$. It suffices to prove by induction that this holds for $\tau_{n,i}$ for any $i \leqslant k$. Let $U \in \mathcal{U}_m$ and $i \in \{0, \ldots, k\}$. By the inductive hypothesis, we have $U \in \tau_{n,i-1}$. Suppose that $U \cap S_i = \emptyset$. Then $\mathrm{cl}_{\tau_{n,i}}(U) = \mathrm{cl}_{\tau_{n,i-1}}(U)$ because $\tau_{n,i}|(X\backslash S_i) = \tau_{n,i-1}|(X\backslash S_i)$. Applying the inductive assumption once more we can conclude that $\mathrm{cl}_{\tau_{n,i}}(U) = \mathrm{cl}_{\tau_{n,i-1}}(U) = \overline{U}$. Now $U \cap \overline{S}_i = \emptyset$ and hence $U \cap \mathrm{cl}_{\tau_{n,i}}(S_i) = \emptyset$ due to the fact that $\mathrm{cl}_{\tau_{n,i}}(S_i) = \overline{S}_i$. Since $S_i \cup U$ is $\tau_{n,i}$-open, it is immediate by (t_6), that $U \in \tau_{n,i}$. If $U \cap S_i \neq \emptyset$, then $m \leqslant n$ and hence $S_i \subset U$ by definition of $I(\sigma_i)$. Then $S_i \cup U = U \in \tau_{n,i}$ and applying (t_6) once more, we obtain $\mathrm{cl}_{\tau_{n,i}}(U) = \mathrm{cl}_{\tau_{n,i-1}}(U) = \overline{U}$.

It remains to show that the property (i) holds, i.e., that $\tau_{n+1}|(X\backslash\overline{\bigcup \mathcal{U}_{n+1}})$ is nowhere locally compact. By the construction, τ_{n+1} is nowhere locally compact on $X\backslash\overline{\bigcup \mathcal{U}_n}$ and on each S_i. So we are done if we show that $\bigcup\{S_i\colon i \leqslant k\}$ is dense in $\overline{\bigcup \mathcal{U}_n}\backslash\overline{\bigcup \mathcal{U}_{n+1}}$. To this end, let O be a τ_{n+1}-open set with $O \cap (\overline{\bigcup \mathcal{U}_n}\backslash\overline{\bigcup \mathcal{U}_{n+1}}) \neq \emptyset$. Let $\{U_i\colon i \leqslant m\}$ be a listing of the members of $\bigcup\{\mathcal{U}_i\colon i \leqslant (n+1)\}$. There exists an $O' \in \tau_{n+1}$ such that $O' \subset O \cap (\bigcup \mathcal{U}_n\backslash\overline{\bigcup \mathcal{U}_{n+1}})$. Now define inductively O_i, $i \leqslant m$ so that $O_0 \subset O'$, $O_i \cap U_i = \emptyset$ or $O_i \subset U_i$ and $O_i \neq \emptyset$. Then O_m is either contained in or misses each $U \in \bigcup_{i\leqslant n} \mathcal{U}_i$. For any $i \leqslant n$ let $\sigma(i) = \{U \in \mathcal{U}_i\colon O_m \subset U\}$. Then $O_m \subset I(\sigma)\backslash\overline{\bigcup \mathcal{U}_{n+1}}$. Hence $\sigma = \sigma_i$ for some $i \leqslant k$ and $O_m \subset S_i$ which implies $O \cap (\bigcup_{i\leqslant m} S_i) \neq \emptyset$.

Having constructed the sequence $\{\tau_n\colon n \in \omega\}$, we are ready to define the topology τ'. Let $\mathcal{U} = \bigcup\{\mathcal{U}_n\colon n \in \omega\}$. For any $x \in K$ let $\mathcal{B}_x = \{U \in \mathcal{U}\colon x \in U\}$. If $x \notin K$ then let $n(x) = \min\{n \in \omega\colon x \notin \overline{\bigcup \mathcal{U}_n}\}$. Then we define $\mathcal{B}_x$ to be the family $\{V \in \tau_{n(x)}\colon x \in V$ and $V \cap \overline{\bigcup \mathcal{U}_{n(x)}} = \emptyset\}$. It is an easy consequence of (ii) and (iii) that the families $\{\mathcal{B}_x\colon x \in X\}$ satisfy the usual axioms of generating a topology as local bases (see [3, Proposition 1.2.3]). Let τ' be the topology for which $\mathcal{B}_x$ is a local base at x for every $x \in X$.

It is evident that $\tau' \subset \tau$ and immediate from (i) that τ' is nowhere locally compact. Property (v) guarantees that H is τ'-closed and $\tau'|H = \tau|H$. We will show that τ' is metrizable. It is an easy consequence of properties (ii) and (iii) that the topology τ' is regular. Since $\mathcal{U}$ is a countable base for the points of K, it suffices to show that there is a σ-discrete base in τ' for the points of $X\backslash K$. For each $n, m \in \omega$ there exists a τ_n-discrete family $\mathcal{B}_{nm}$ such that $(\bigcup \mathcal{B}_{nm}) \cap \overline{\bigcup \mathcal{U}_n} = \emptyset$ and $\mathcal{B}_n = \bigcup\{\mathcal{B}_{nm}\colon m \in \omega\}$ contains local bases for the points of $X\backslash\overline{\bigcup \mathcal{U}_n}$. It is clear from the definition of the topology τ', that $\bigcup\{\mathcal{B}_n\colon n \in \omega\}$ is a base in τ' for the points of $X\backslash K$. Thus, it remains to prove that each $\mathcal{B}_{nm}$ is τ'-discrete.

Let $x \in X$. If $x \in \bigcup \mathcal{U}_n$, then $W = \bigcup \mathcal{U}_n$ is a neighbourhood of x which intersects no elements of $\mathcal{B}_{nm}$. If $x \in X\backslash\bigcup \mathcal{U}_n$ take a τ_n-neighbourhood W of the point x which meets at most one element of $\mathcal{B}_{nm}$. Since $x \in X\backslash\overline{\bigcup \mathcal{U}_{n+1}}$ and $\tau_{n+1}|(X\backslash\bigcup \mathcal{U}_n) = \tau_n|(X\backslash\bigcup \mathcal{U}_n)$, there exists a τ_{n+1}-open subset V of $X\backslash\overline{\bigcup \mathcal{U}_{n+1}}$ such that $V \cap (X\backslash\bigcup \mathcal{U}_n) = W \cap (X\backslash\bigcup \mathcal{U}_n)$. Since $V \cap \overline{\bigcup \mathcal{U}_{n+1}} = \emptyset$ and $V \in \tau_{n+1}$, we have $V \in \tau'$ and V meets at most one element of $\mathcal{B}_{nm}$. $\square$

Theorem 2.8. *Any metrizable non-compact space has a weaker connected Hausdorff topology.*

Proof. Let X be a metrizable non-compact space. By Theorem 2.7 we can assume that X is nowhere locally compact, weakening its topology if necessary. Fix a σ-discrete base $\mathcal{B}$ in X. Let D be an infinite closed discrete subset of X. The family $\mathcal{B}' = \{U \in \mathcal{B}\colon U \cap D = \emptyset\}$ is a σ-discrete π-base in X and $(\bigcup \mathcal{B}') \cap D = \emptyset$.

In [10] it was proved that every Hausdorff space X with a σ-discrete π-base $\mathcal{B}'$ and a free open ultrafilter on each non-empty open set can be densely embedded into a connected Hausdorff space Y in such a way that $Y \setminus X$ is a countable closed and discrete subspace of Y and $\mathrm{cl}_Y(F) \cap (Y \setminus X) = \emptyset$, where $F = X \setminus \bigcup \mathcal{B}'$. Since no closure of a non-empty open subset of X is compact, every open non-empty subset of X is contained in a free open ultrafilter on X. Therefore we can apply the above mentioned result from [10] and use the π-base $\mathcal{B}'$ constructed above to find such a space Y. It is immediate that D is closed and discrete in Y. Now apply Proposition 2.4 of the paper [7] to conclude that X can be condensed onto a connected Hausdorff space. □

Corollary 2.9. *Any paracompact non-compact space with a G_δ-diagonal has a weaker Hausdorff connected topology. In particular, every stratifiable non-compact space has a weaker Hausdorff connected topology.*

Proof. Every paracompact space with a G_δ-diagonal has a weaker metrizable topology [1]. We leave it to the reader to follow the line of any of many well-known proofs of this fact to assure that every paracompact non-compact space with a G_δ-diagonal condenses onto a non-compact metric space. Theorem 2.8 completes the proof. □

Example 2.10. Not every non-compact first countable Tychonoff space condenses onto a nowhere locally compact space. In fact, there are non-compact locally compact spaces with all their Tychonoff continuous images locally compact. An example of such a space is $X = \omega_1$, for which $\beta X = \omega_1 + 1$ and hence $\beta X \setminus X$ consists of only one point. Now, if $f : X \to Y$ is a surjective continuous map, then $\beta Y \setminus Y$ can not have more than one point being a subset of a continuous image of $\beta X \setminus X$ which consists of only one point. Therefore Y is locally compact.

Not every first countable non-compact Tychonoff space has a weaker connected regular topology: a trivial example is any non-compact countable space. A much less evident one is the free union of countably many copies of the Cantor set [7]. However, a first countable non-compact space without a weaker Hausdorff connected topology is not so easy to construct. Recall that a space is called *almost H-closed* if no Hausdorff extension of this space can have more than one point in the remainder. It is an easy exercise that any continuous image of an almost H-closed space is an almost H-closed space.

Example 2.11. Under CH there is a non-compact first countable Tychonoff space which has no weaker Hausdorff connected topology.

Proof. Porter and Woods [6] constructed under CH an example of a connected normal first countable almost H-closed space Y. Let X be obtained from Y by adding two isolated points a and b. Then X can not be condensed onto a connected Hausdorff space. Indeed, if $f: X \to Z$ is such a condensation, then $f(Y)$ is an almost H-closed dense subspace of Z such that $Z \setminus f(Y)$ consists of two points, a contradiction. □

3. Condensations of Hausdorff spaces with a σ-locally finite base

This group of results is concerned with Hausdorff non-regular spaces. If we could prove that any Hausdorff non-H-closed space with a σ-locally finite base has a weaker connected Hausdorff topology, this would be a natural extension of Theorem 2.8. We will establish this fact for the case when the weight of the space is a successor cardinal.

Recall that if a Hausdorff space with a σ-locally finite base is feebly compact, then it is second countable and hence H-closed. Furthermore, each Hausdorff space X with a σ-locally finite π-base has a dense strongly σ-discrete subspace and, as a consequence, the set of isolated points of X is the union of a countable family of closed discrete subsets of X.

Lemma 3.1. *For each infinite cardinal κ there is a connected Hausdorff space H_κ of cardinality and weight κ with a σ-locally finite base which possesses a locally finite disjoint family of cardinality κ of non-empty open subsets.*

Proof. Note that if $\kappa \geqslant \mathfrak{c}$, then the hedgehog with κ spines is such a space. For the general case, let Y be a connected first countable, countable Hausdorff space with a distinguished point p. Let $Z = Y \times D_\kappa$, where D_κ is the discrete space of cardinality κ. Define an equivalence relation $\sim$ on Z by $(x,\alpha) \sim (y,\beta)$ if and only if $x = y$ and $\alpha = \beta$ or $x = y = p$.

Consider the space $T = Z/{\sim}$, and let $f: Z \to T$ be the relevant quotient map. Denote by q the image of each point (p,α) under f. Fix a local base $\{B_n\colon n \in \omega\}$ at p in the space Y and define a topology μ on T as follows:

If $q \notin U$ then $U \in \mu$ if and only if $f^{-1}(U)$ is open in Y; if $q \in U$, then $U \in \mu$ if and only if for some $n \in \omega$, we have $B_n \times D_\kappa \subset U$.

We leave to the reader the trivial verification that the space $H_\kappa = (T,\mu)$ has the required properties. Note that H_κ is first countable and may be thought of as a hedgehog constructed using the space Y instead of the unit interval. □

Lemma 3.2. *Given an infinite cardinal κ, suppose that (X,τ) is a Hausdorff space of density $\leqslant\kappa$ which has a closed set of isolated points of cardinality κ. Then there exists a connected Hausdorff topology $\mu \subset \tau$.*

Proof. Denote by D the set of isolated points of X and let $E \subset D$ be a closed subset of cardinality κ. Take any connected Hausdorff topology ρ on E which has the properties specified in Lemma 3.1. In particular, the space (E,ρ) possesses a locally finite family of cardinality κ of non-empty mutually disjoint open subsets.

Take a dense subset $\{d_\alpha\colon \alpha \in \nu \leqslant \kappa\}$ of $X\setminus E$ and choose a family $\{\omega_\alpha\colon \alpha \in \kappa\}$ of mutually disjoint countably infinite subsets of κ. Denote by $\mathcal{G}_\alpha$ the cofinite filter on ω_α and let $\mathcal{F}_\alpha$ be the open filter generated by the family $\{\bigcup\{U_n\colon n \in G\}\colon G \in \mathcal{G}_\alpha\}$. The promised topology μ on the set X is defined by requiring that $U \in \mu$ if and only if

(μ_1) U is open in $(X\setminus E, \tau|(X\setminus E)) \oplus (E, \rho)$, and

(μ_2) for each $\alpha \in \kappa$ such that $d_\alpha \in U$, there is $F \in \mathcal{F}_\alpha$ such that $F \subset U$.

It is now routine to verify that (X, μ) is a Hausdorff space which is connected since (E, ρ) is connected and dense in (X, μ). □

Let us call a topological space (X, τ) *equipotent* if every non-empty element of the topology τ has the same cardinality. Theorem 2.6 of [9], states that each Hausdorff first countable space with a σ-disjoint π-base has a dense metrizable subspace and clearly every metrizable space has a strongly σ-discrete dense subspace. Furthermore, by a result of Medvedev [5], for each infinite cardinal ξ, there is, up to homeomorphism, precisely one strongly σ-discrete equipotent metrizable space of weight ξ which we denote by Q_ξ. The main reason why it is possible to condense any Hausdorff space X of weight $\kappa = \lambda^+$ with a σ-locally finite base onto a connected space, is that we can find a locally finite disjoint family $\gamma \subset \tau(X)$ such that $|\gamma| = \kappa$ and some Q_ξ dense in each element of γ.

Proposition 3.3. *Let X be a Hausdorff first countable dense-in-itself space of π-weight κ with a σ-disjoint π-base. Then there is a metrizable subspace $Y \subset X$ such that $Y \subset \mathrm{Int}(\overline{Y})$ and Y is homeomorphic to Q_ξ for some $\xi \leqslant \kappa$.*

Proof. Apply a theorem of White [9, Theorem 2.6] to find a dense metrizable subspace M of the space X. Let N be a dense strongly σ-discrete subspace of M. Of course, N is dense in X as well. Let Y be a non-empty open subset of the space N of minimal cardinality. Since N is dense-in-itself, the subspace Y is infinite. It is immediate that Y is an equipotent strongly σ-discrete metric space. Now, we can apply the above mentioned result of Medvedev [5] to conclude that Y is homeomorphic to the space Q_ξ for some $\xi \leqslant \kappa$. □

Suppose $\mathcal{S} = \{S_\alpha\colon \alpha \in I\}$ is a family of open sets; a *disjoint shrinking* of $\mathcal{S}$ is a family $\mathcal{T} = \{T_\alpha\colon \alpha \in I\}$ of non-empty open sets such that for each α, $T_\alpha \subset S_\alpha$ and $T_\beta \cap T_\gamma = \emptyset$ if $\beta \neq \gamma$. If $\mathcal{S}$ is locally finite, then it follows from Lemma 2.1 of [10] that $\mathcal{S}$ has a disjoint shrinking. Furthermore, a disjoint shrinking of a base is a π-base and so if X is a Hausdorff space with a σ-locally finite base, then it has a π-base which is σ-disjoint.

Lemma 3.4. *Let κ be a cardinal of uncountable cofinality. If (X, τ) is a first countable Hausdorff space of π-weight κ with a σ-locally finite π-base, then there is a dense-in-itself Hausdorff topology $\mu \subset \tau$, such that either (X, μ) is connected or it is a first countable space of π-weight κ with a σ-disjoint π-base and has a locally finite family of cardinality κ of non-empty disjoint open sets.*

Proof. Denote by D the set of isolated points of (X,τ). By the remark preceding Lemma 3.1, we have $D=\bigcup\{D_n\colon n\in\omega\}$ where each D_n is closed (and discrete) in X. Since the space X has a σ-locally finite π-base of cardinality κ and $\mathrm{cf}(\kappa)>\omega$, there is a locally finite family $\mathcal{U}'$ of non-empty open subsets of X with $|\mathcal{U}'|=\kappa$. By Lemma 2.1 of [10], the family $\mathcal{U}'$ has a disjoint shrinking $\mathcal{U}=\{U_\alpha\colon \alpha<\kappa\}$. There are two cases to consider.

(1) If $|\{U\in\mathcal{U}\colon U\cap D\neq\emptyset\}|=\kappa$, then there is a closed subset of D of cardinality κ and the result follows from Lemma 3.2.

(2) If $|\{U\in\mathcal{U}\colon U\cap D\neq\emptyset\}|<\kappa$, then without loss of generality, we can assume that if $U\in\mathcal{U}$, then $U\subset X\backslash \mathrm{cl}(D)$ and hence for each $\alpha\in\kappa$, the set U_α is dense-in-itself. Let $\{\omega_\alpha\colon \alpha\in\kappa\}$ be a mutually disjoint countably infinite subsets of κ, such that $|\kappa\backslash\bigcup\{\omega_\alpha\colon \alpha\in\kappa\}|=\kappa$. Clearly, $\{U_m\colon m\in\omega_\alpha\}$ is a locally finite disjoint family for each $\alpha\in\kappa$. Denote by $\mathcal{G}_\alpha$ the cofinite filter on ω_α and enumerate its elements as $\{G_{n,\alpha}\colon n\in\omega\}$. For each $\alpha\in\kappa$, we define an open filter $\mathcal{F}_\alpha$ on X in the following way:
$F\in\mathcal{F}_\alpha$ if and only if $F\in\tau$ and $F\supset\bigcup\{U_m\colon m\in A\}$ for some $A\in\mathcal{G}_\alpha$.

Clearly the filters $\mathcal{F}_\alpha$ have mutually disjoint bases. For later use, we define $B(n,\alpha)=\bigcup\{U_m\colon m\in G_{n,\alpha}\}$. Since $|D|\leqslant\kappa$ it is possible to choose a cardinal $\nu\leqslant\kappa$ and a faithful ($\equiv$one-to-one) enumeration $\{d_\alpha\colon \alpha<\nu\}$ of the set D. For any $W\in\tau$ let $O(W,n)=W\cup\bigcup\{B(n,\alpha)\colon d_\alpha\in W\}$. The required dense-in-itself topology μ is now generated by the family $\{O(W,n)\colon W\in\tau,\ n\in\omega\}$ as a base, i.e.,

$$\mu=\bigl\{U\in\tau\colon \text{ for any } x\in U \text{ there are } W\in\tau \text{ and } n\in\omega \text{ such that } x\in O(W,n)\subset U\bigr\}.$$

Note that,

$$\text{for any } n\in\omega \text{ and } W\in\tau \text{ with } W\cap D=\emptyset, \text{ we have } O(W,n)=W \text{ and hence } W\in\mu. \qquad (*)$$

The Hausdorff property of (X,μ) follows immediately from the fact that for any disjoint $U,V\in\tau$ the sets $O(U,n)$ and $O(V,m)$ are disjoint for all $m,n\in\omega$. It is also clear that the space (X,μ) is dense-in-itself and and $\{U_\xi\colon \xi\in\kappa\backslash\bigcup\{\omega_\alpha\colon \alpha\in\kappa\}\}$ is a locally finite family of disjoint non-empty open subsets of (X,μ). It remains to show that (X,μ) is first countable and has a σ-disjoint π-base of cardinality κ.

To show that (X,μ) is first countable, note that for any $\alpha<\nu$ the family $\{\{d_\alpha\}\cup B(n,\alpha)\colon n\in\omega\}$ is a countable local base for μ at d_α. If $x\in X\backslash D$, fix a local base $\{W_n\colon n\in\omega\}$ of the point x in the topology τ so that $W_{n+1}\subset W_n$ for all $n\in\omega$. It is easy to see that $\{O(W_n,n)\colon n\in\omega\}$ is a local base at x in μ.

Let $\mathcal{B}=\bigcup\{\mathcal{B}_n\colon n\in\omega\}$ be a σ-locally finite π-base of cardinality $\leqslant\kappa$ for $X\backslash\mathrm{cl}_\tau(D)$ considered with the topology $\tau|(X\backslash\mathrm{cl}_\tau(D))$. For each $n\in\omega$, take a disjoint shrinking $\mathcal{V}_n$ of the family $\mathcal{B}_n$. Observe that each element of $\mathcal{V}=\bigcup\{\mathcal{V}_n\colon n\in\omega\}$ is τ-open and hence μ-open by $(*)$. As a consequence, the family $\mathcal{V}$ is a π-base for $(X\backslash\mathrm{cl}_\tau(D),\mu|(X\backslash\mathrm{cl}_\tau(D)))$. Furthermore, let $\mathcal{C}_m=\{\{d_\alpha\}\cup B_{m,\alpha}\colon d_\alpha\in D\}$. We leave to the reader the straightforward verification that the family $\bigcup\{\mathcal{C}_m\colon m \in \omega\} \cup \mathcal{V}$ is a σ-disjoint π-base of cardinality κ for (X, μ). $\square$

Lemma 3.4 shows that if we construct condensations of Hausdorff first countable spaces with σ-disjoint π-bases onto connected Hausdorff spaces, we can restrict attention to those topologies which are dense-in-themselves.

Theorem 3.5. *Let κ be a successor cardinal. Suppose that (X, τ) is a dense-in-itself first countable Hausdorff space of π-weight κ with a σ-disjoint π-base. If (X, τ) has a locally finite family of cardinality κ, which consists of non-empty open sets, then (X, τ) is T_2-subconnected, i.e., has a weaker connected Hausdorff topology.*

Proof. Since $\kappa > \omega$, the space X is not H-closed. Taking a disjoint shrinking if necessary, we can find a locally finite family $\mathcal{U} = \{U_\alpha\colon \alpha \in \kappa\}$ of disjoint non-empty open subsets of X. Apply Proposition 3.3 to conclude that each of the sets U_α has a metrizable subspace D_α such that $D_\alpha \subset \operatorname{Int}(\overline{D_\alpha})$ and $D_\alpha \simeq Q_{\xi_\alpha}$ for some cardinal $\xi_\alpha \leqslant \kappa$. Since κ is a non-limit cardinal, the set $\{\xi\colon \xi \in \kappa \text{ and } \xi \text{ is a cardinal}\}$ has cardinality less than κ. Now use the regularity of κ to see that there is a cardinal $\nu \leqslant \kappa$ such that $\xi_\alpha = \nu$ for κ many α.

The reasoning in the preceding paragraph shows that replacing, if necessary, each set U_α by $\operatorname{Int}_\tau(\operatorname{cl}_\tau(D_\alpha)) \cap U_\alpha$ and choosing an appropriate subfamily of $\mathcal{U}$ of cardinality κ, we can assume that each element of the family $\mathcal{U}$ has a dense subspace D_α, homeomorphic to some fixed strongly σ-discrete equipotent metrizable space (Q, ν).

For each $\alpha \in \kappa$, we let $h_\alpha : D_\alpha \to Q$ be a homeomorphism and let ρ be a connected Hausdorff topology on κ having the properties specified in Lemma 3.1. For ease of notation we denote $\bigcup\mathcal{U}$ by Z.

Given $T \in \nu$ and $W \in \rho$, we define

$$O(T, W) = \bigcup\bigl\{\operatorname{Int}_\tau\bigl(\operatorname{cl}_\tau\bigl(h_\alpha^{-1}(T)\bigr)\bigr)\colon \alpha \in W\bigr\}.$$

For each $T \in \nu$ and $W \in \rho$, it is immediate that $O(T, W) \in \tau$. We define a topology μ on X as follows:

$$\mu = \bigl\{U \in \tau\colon \text{ if } x \in U \cap Z \text{ then } x \in O(T, W) \subset U \text{ for some } T \in \nu \text{ and } W \in \rho\bigr\}.$$

We leave to the reader the straightforward verification that μ is indeed a topology and $\mu \subset \tau$. Let us proceed to prove that (X, μ) is a Hausdorff space. To this end, suppose x, y are distinct elements of X.

(1) $x, y \in Z$. If $x \in U_\alpha$ and $y \in U_\gamma$, where $\alpha \neq \gamma$, then there are mutually disjoint $W_1, W_2 \in \rho$ such that $\alpha \in W_1$ and $\gamma \in W_2$. The sets $O(Q, W_1)$ and $O(Q, W_2)$ are μ-open disjoint neighbourhoods of x and y, respectively. On the other hand, if $x, y \in U_\alpha$ for some $\alpha \in \kappa$, then there are disjoint τ-open neighbourhoods $U, V \subset U_\alpha$ of x, y, respectively. If we define $T_1 = h_\alpha(U \cap D_\alpha)$ and $T_2 = h_\alpha(V \cap D_\alpha)$ it turns out that $x \in O(T_1, \kappa) \in \mu$, $y \in O(T_2, \kappa) \in \mu$ and $O(T_1, \kappa) \cap O(T_2, \kappa) = \emptyset$.

(2) $x \in Z$, say $x \in U_\alpha$ and $y \in X \setminus Z$. There exist disjoint τ-open sets U, V with the following properties: $x \in U \subset U_\alpha$, $y \in V$ and the set $M = \{\beta \in \kappa\colon V \cap U_\beta \neq \emptyset\}$ is

finite, say $M=\{\beta_1,\dots,\beta_k\}$. If $\alpha\in M$, say $\alpha=\beta_j$, then since (κ,ρ) is Hausdorff, there are disjoint sets $W_1,\dots,W_k\in\rho$ such that $\beta_i\in W_i$ for each $i\in 1,\dots,k$. Then the sets $G=O(h_\alpha(U\cap D_\alpha),W_j)$ and $H=V\cup\bigcup\{O(V\cap D_{\beta_i},W_i)\colon 1\leqslant i\leqslant k\}$ have the property that $H,G\in\mu$, $H\cap G=\emptyset$ and $x\in G$, $y\in H$.

If, on the other hand, $\alpha\notin M$, then we choose disjoint sets $W_0,W_1,\dots,W_k\in\rho$ such that $\alpha\in W_0$ and $\beta_i\in W_i$ for $i=1,\dots,k$. The sets $G=O(Q,W_0)$ and $H=V\cup\bigcup\{O(Q,W_i)\colon 1\leqslant i\leqslant k\}$ are disjoint μ-open neighbourhoods of x and y, respectively.

(3) If $x,y\in X\setminus Z$, then choose disjoint τ-open sets U and V such that $x\in U$, $y\in V$ and the sets $A=\{\alpha\in\kappa\colon U\cap U_\alpha\neq\emptyset\}$ and $B=\{\alpha\in\kappa\colon V\cap U_\alpha\neq\emptyset\}$ are both finite. For each $\gamma\in A\cup B$, choose a ρ-open set W_γ in such a way that if $\gamma\neq\xi$, then $W_\gamma\cap W_\xi=\emptyset$. The sets $G=U\cup\bigcup\{O(h_\gamma(U\cap D_\gamma),W_\gamma)\colon \gamma\in A\}$ and $H=V\cup\bigcup\{O(h_\xi(V\cap D_\xi),W_\xi)\colon \xi\in B\}$ are disjoint τ-open neighbourhoods of x and y, respectively.

For each $d\in Q$, consider the set $Y_d=\{h_\alpha^{-1}(d)\colon \alpha\in\kappa\}$. It is clear that $(Y_d,\mu|Y_d)$ is homeomorphic to (κ,ρ) and hence is connected. There exists a locally finite family $\mathcal{W}=\{W_\alpha\colon \alpha\in\kappa\}$ of disjoint open subsets of (κ,ρ). The family $\mathcal{V}=\{O(Q,W_\alpha)\colon \alpha\in\kappa\}$ of μ-open sets is then mutually disjoint and we claim that $\mathcal{V}$ is locally finite.

To prove our claim, suppose $x\in X$. If $x\in Z$, say $x\in U_\alpha$, then since $\mathcal{W}$ is locally finite, there is some open ρ-neighbourhood V of α which meets only finitely many elements of $\mathcal{W}$. Now $O(Q,V)$ is a μ-neighbourhood of x which meets only finitely many elements of $\mathcal{V}$. If, on the other hand, $x\in X\setminus Z$, then x has a τ-neighbourhood V which meets only finitely many elements of $\mathcal{U}$, say $\{U_{\beta_1},\dots,U_{\beta_k}\}$. Since $\mathcal{W}$ is locally finite there are disjoint ρ-neighbourhoods $\{V_{\beta_1},\dots,V_{\beta_k}\}$ of $\beta_1,\dots,\beta_k$, respectively, each meeting only a finite number of elements of $\mathcal{W}$. Now $V\cup\bigcup\{O(Q,V_{\beta_j})\colon 1\leqslant j\leqslant k\}$ is a μ-neighbourhood of x meeting only finitely many elements of $\mathcal{V}$, and our claim is proved.

Let $\{\omega_\alpha\colon \alpha\in\kappa\}$ be a family of mutually disjoint countable subsets of κ and let $E=\{e_\alpha\colon \alpha\in\kappa\}$ be a dense subspace of $X\backslash\mathrm{cl}(Z)$ of cardinality (at most) κ. Let $\mathcal{G}_\alpha$ be the Frechet filter on ω_α and let $\mathcal{F}_\alpha$ be the open filter generated by $\{\bigcup\{O(Q,W_\beta)\colon \beta\in G\}\colon G\in\mathcal{G}_\alpha\}$. We define

$$\theta=\{U\in\mu\colon \text{for each } \alpha\in\kappa \text{ such that } e_\alpha\in U, \text{ there is } F\in\mathcal{F}_{\alpha+1} \text{ such that } F\subset U\}.$$

It is now routine to verify that (X,θ) is a Hausdorff space in which (Z,θ) (and hence $\bigcup\{Y_d\colon d\in Q\}$) is dense and $\theta|Z=\mu|Z$. Furthermore, since $\{O(Q,W_\beta)\colon \beta\in\omega_0\}$ is a locally finite family of θ-open sets, the space (X,θ) is not feebly compact.

Let $\mathcal{F}_0$ be the θ-open filter generated by $\{\bigcup\{O(Q,W_\beta)\colon \beta\in G\}\colon G\in\mathcal{G}_0\}$. Choose any point $x\in X$ and define a new topology η on X as follows:

$$\eta=\{U\in\theta\colon \text{ if } x\in U \text{ then } F\subset U \text{ for some } F\in\mathcal{F}_0\}.$$

It is clear that the filter $\mathcal{F}_0$ converges to x in the topology θ. Now, $F\cap Y_d\neq\emptyset$ for each $F\in\mathcal{F}_0$ and $d\in Q$. As a consequence, $x\in\mathrm{cl}_\theta(Y_d)$ for each $d\in Q$. Each space Y_d is θ-connected, whence $\{x\}\cup\bigcup\{Y_d\colon d\in Q\}$ is η-connected. Since $\{x\}\cup\bigcup\{Y_d\colon d\in Q\}$ is dense in (X,η), this latter space is also connected. $\square$

Theorem 3.6. *Let (X, τ) be a Hausdorff space of weight κ with a σ-locally finite base. If κ is a successor cardinal, then X is T_2-subconnected, i.e., there exists a condensation of X onto a connected Hausdorff space.*

Proof. It follows from regularity of κ that X has a locally finite family of cardinality κ of non-empty open sets. A disjoint shrinking of this family provides a disjoint family of cardinality κ of non-empty open sets. Thus the π-weight of X is also equal to κ and Lemma 3.4 is applicable. If its conclusion is the connectedness of a weaker topology, we are done. If not, then there exists a Hausdorff dense-in-itself topology $\mu \subset \tau$ such that (X, μ) is a first countable space of π-weight κ with a σ-disjoint π-base which has a locally finite family of cardinality κ of non-empty open sets. Now apply Theorem 3.5 to see that there is a connected Hausdorff topology $\nu \subset \mu \subset \tau$. □

Corollary 3.7. *Let X be a disconnected Hausdorff space of weight κ with a σ-locally finite base. If κ is a successor cardinal, or $\kappa = \omega$ then X is T_2-subconnected if and only if X is not H-closed.*

Proof. For $\kappa = \omega$ this was proved in [8]. If κ is uncountable then X can not be H-closed, so Theorem 3.6 is applicable. □

Question 3.8. Is it possible to omit the Continuum Hypothesis in Example 2.11?

Question 3.9. Let X be a non-H-closed Hausdorff space with a σ-locally finite base. Is it true that X has a weaker connected Hausdorff topology?

Question 3.10. Does every paracompact non-compact space X have a weaker Hausdorff connected topology? What happens if X is hereditarily paracompact or perfect?

References

[1] C.R. Borges, On stratifiable spaces, Pacific J. Math. 17 (1966) 1–16.
[2] J. Dugundji, An extension of Tietze's theorem, Pacific J. Math. 1 (1951) 353–367.
[3] R. Engelking, General Topology, PWN, Warszawa, 1977.
[4] B.J. Fitzpatrick, G. Gruenhage, J.W. Ott, Topological completions of metrizable spaces, Proc. Amer. Math. Soc. 117 (1993) 259–267.
[5] S.V. Medvedev, Topological characteristics of the spaces $Q(\kappa)$ and $Q \times B(\kappa)$, Moscow Univ. Math. Bull. 41 (1) (1981) 42–45.
[6] J.R. Porter, R.G. Woods, Subspaces of connected spaces, Topology Appl. 68 (1996) 113–131.
[7] M.G. Tkačenko, V.V. Tkachuk, V.V. Uspenskij, R.G. Wilson, In quest of weaker connected topologies, Comment. Math. Univ. Carolin. 37 (4) (1996) 825–841.
[8] V.V. Tkachuk, R.G. Wilson, Weaker connected Hausdorff topologies on spaces with a countable network, Preprint.
[9] H.E. White, First countable spaces that have special pseudo-bases, Canad. Math. Bull. 21 (1) (1978) 103–112.
[10] R.G. Wilson, S. Watson, Embeddings in connected spaces, Houston J. Math. 19 (3) (1993) 469–481.