Baire and Volterra Spaces

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Abstract: In this paper we describe broad classes of spaces for which the Baire space property is equivalent to the assertion that any two dense G_δ-sets have dense intersection. We also provide examples of spaces where the equivalence does not hold. Finally, our techniques provide an easy proof of a new internal characterization of perfectly meager subspaces of $[0,1]$ and characterize metric spaces that are always of first category.

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1. Introduction.

Recall that a topological space X is a *Baire space* if the intersection of any sequence of dense open subsets of X is dense. It follows immediately that the intersection of countably many dense G_δ-subsets of a Baire space X must be dense in X. A weaker condition is the that intersection of any *two* dense G_δ-sets of X must be dense in X, and that is the definition of a *Volterra* space [GP], [GGP]. Obviously, any Baire space is Volterra, and in this paper we study when the converse holds.

The term "Volterra space" was first used in [GP]. That name was chosen in the light of an 1881 paper by V. Volterra [V] who proved that if $f : R \to R$ is any function such that both $C(f) = \{x \in R : f \text{ is continuous at } x\}$ and $D(f) = R - C(f)$ are dense in R, then there cannot be a function $g : R \to R$ such that $C(g) = D(f)$ and $D(g) = C(f)$. The key idea in Volterra's proof was that the intersection of two dense G_δ subsets of R must be dense in R.

In Section 2 we give properties of a space X which guarantee that X is a Baire space if and only if X is Volterra. In Corollary 2.8, we apply our conditions to answer a question of Piotrowski concerning metric spaces. We show that the equality "Baire = Volterra" holds (i) for any space that has a dense metrizable subspace, (ii) for any Lasnev space, and (iii) for any metacompact Moore space. In Section 3 we give three examples of spaces that are Volterra but not Baire. The first is countable and regular; the second is

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a Lindelöf, hereditarily paracompact, linearly ordered topological space; and the third is first countable and paracompact. We do not know whether the equality "Baire = Volterra" holds for arbitrary Moore spaces.

The final section of our paper uses a lemma from Section 2 to provide a short proof of a new internal characterization of perfectly meager subspaces of $[0,1]$ that was originally obtained as a corollary of more technical constructions in [BHL], and to characterize metric spaces that are always of first category. Relevant definitions appear in Section 4.

All spaces in our paper are assumed to be regular and T_1, unless otherwise noted. Our terminology and notation follow [E] and [G1].

2. Spaces in which "Volterra = Baire."

Clearly, any Baire space is Volterra. The goal of this section is to give a large class of spaces in which the converse holds. Recall from [H] that a space X is *resolvable* if X contains two disjoint dense subsets. The next result is due to Pytkeev [P].

2.1 **Lemma**: *Any dense-in-itself subspace of a sequential space is resolvable.* $\square$

2.2 **Lemma**: *Let M be a dense subset of a regular T_1-space Y, such that every point of M is G_δ in Y. Suppose $\{p\} \cup M$ is homeomorphic to a subspace of a Hausdorff sequential space S (we are not assuming $p \in Y$), and $p \in cl_S(M)$. Then there is a countable subset $C(p, M)$ of M with $p \in cl_S(C(p, m))$ and such that $C(p, M)$ is G_δ in Y.* $\square$

Proof: For a subset A of S, let $seq(A)$ be the set of all limits in the space S of convergent sequences $\{a_n : n \in \omega\} \subset A$. Then define $A^0 = A$, $A^{\alpha+1} = seq(A^\alpha)$, and $A^\beta = \bigcup\{A^\alpha : \alpha < \beta\}$ if β is a limit ordinal. Recall that S sequential means that $cl_S(A) = \bigcup\{A^\alpha : \alpha < \omega_1\}$.

Let α be least such that $p \in M^\alpha$. Call α the "sequential order of p w.r.t. M", and note that α is a successor. If $\alpha = 1$, then there is a sequence $J = \{m_k : k \in \omega\}$ of points in M converging to p. Since J is relatively discrete in M, it follows from the regularity of Y that there is a disjoint collection $\{U_k : k \in \omega\}$ of open subsets of Y with $m_k \in U_k$ for every k. Since each point of M is G_δ in Y, it follows that J is G_δ in Y. Thus we can take $C(p, M) = J$.

Suppose $\alpha > 1$ and the lemma holds whenever the sequential order of p in M is less than α. Let $\alpha = \gamma + 1$. Then there is a sequence $\{m_k : k \in \omega\} \subset M^\gamma$ converging to p in S. Since S is Hausdorff, we may assume that there are disjoint open (in S) sets V_k, $k \in \omega$, with $m_k \in V_k$. Note that the sequential order of m_k is the same w.r.t. $V_k \cap M$ as w.r.t. M. For each k let V_k^* be open in Y such that $V_k^* \cap M = V_k \cap M$, and note that the V_k^*'s are also disjoint (since M is dense in Y). Now apply the induction hypothesis with $p = m_k$, $Y = V_k^*$, and $M = M \cap V_k$ to obtain a countable $C(m_k, M \cap V_k)$ contained in $M \cap V_k$ which is G_δ in V_k^* (and hence in Y). Let $C(p, M) = \bigcup_{k \in \omega} C(m_k, M \cap V_k)$. Clearly $p \in cl_S(C(p, M)$, and since the V_k^*'s are disjoint open sets in Y, it follows that $C(p, M)$ is G_δ in Y. $\square$

2.3 **Lemma**: *Suppose $\mathcal{U}$ is a point-countable collection of open subsets of a space X and that for each $U \in \mathcal{U}$ we have a G_δ-subset $G(U) \subset U$. Then $S = \bigcup\{G(U) : U \in \mathcal{U}\}$ is a G_δ-subset of X.* $\square$

Proof: For each $U \in \mathcal{U}$, write $G(U) = \bigcap\{G(n,U) : n \geq 1\}$ where each $G(n,U)$ is an open subset of X with $G(n+1,U) \subset G(n,U)$. Define $H_n = \bigcup\{G(n,U) : U \in \mathcal{U}\}$. Then point-finiteness of $\mathcal{U}$ yields that $\bigcap\{H_n : n \geq 1\} = S$. $\square$

2.4 **Lemma**: *Suppose X is regular, points are G_δ, and X has a dense subspace $D = \bigcup\{D_n : n \geq 1\}$ satisfying:*

- *a) D is homeomorphic to a subspace of a Hausdorff sequential space;*
- *b) for each $n \geq 1$ there is a collection $\{V(d,n) : d \in D_n\}$ of open subsets of X that is point-finite in X and has $\{d\} = V(d,n) \cap D_n$.*

If X is of the first category in itself, then D contains a subspace E that is dense in X and is a G_δ-subset of X.

Proof: Because X is a first category space, there is a sequence $\{G_n : n \geq 1\}$ of dense open subsets of X such that $\cap\{G_n : n \geq 1\} = \emptyset$. We may assume that $G_{n+1} \subset G_n$. Then X has no isolated points so that neither does the dense subspace D.

We will show that D contains a dense, G_δ subset E of X. Let $\mathcal{V}_n = \{V(n,d) : d \in D_n\}$ be the point- finite collection given by (b). Because $G_n \cap D$ is dense in X, for each $d \in D_n$ the conditions of Lemma 2.2 are satisfied with $Y = G_n \cap (V(d,n))$, $M = D \cap G_n \cap (V(d,n) - \{d\})$, S any Hausdorff sequential space containing D, and $p = d$. Thus there is a countable subset $K(d,n)$ of $G_n \cap D \cap (V(d,n) - \{d\})$ which is G_δ in Y and so also in X, and $d \in \mathrm{cl}_S(K(d,n))$. Since $\{d\} \cup K(d,n) \subset D$ and D has the same topology in X as in S, we also have $d \in \mathrm{cl}_X(K(d,n)$. Now by (2.3), each set $H_n = \bigcup\{K(d,n) : d \in D_n\}$ is a G_δ-subset of X. Because the collection $\{G_n : n \geq 1\}$ is point-finite in X, it follows again from (2.3) that the set $E = \bigcup\{H_n : n \geq 1\}$ is a G_δ-subset of X. Observe that $E \subset D$ and that each point of each set D_n is a limit point of E, showing that E is dense in X, as required. $\square$

2.5 **Corollary**: *If X is a metric space that is first category in itself, and if Y is a dense subset of X, then Y contains a dense, G_δ-subset of X that is σ-closed discrete in X.*

Proof: The dense subspace Y contains a dense subset D that is σ- closed discrete in X. Now apply (2.4) to the subspace D. $\square$

2.6 **Proposition**: *Suppose X is regular and has a dense subspace $D = \bigcup\{D_n : n \geq 1\}$ satisfying:*

- *a) D is homeomorphic to a subspace of a Hausdorff sequential space;*
- *b) for each $n \geq 1$ there is a collection $\{V(d,n) : d \in D_n\}$ of open subsets of X that is point-finite in X and has $V(d,n) \cap D_n = \{d\}$ for each $d \in D_n$.*

Then X is a Baire space if and only if X is Volterra.

Proof: Any Baire space is Volterra, so it is enough to prove the converse. Suppose X is Volterra and yet there is a sequence $\{G_n : n \geq 1\}$ of dense open subsets of X such that $G_{n+1} \subset G_n$ and $\bigcap\{G_n : n \geq 1\}$ is not dense. Then there is a non-empty open subset $Y \subset X$ such that $Y \cap \bigcap\{G_n : n \geq 1\} = \emptyset$. Observe that the set $D \cap Y$ is dense in Y and satisfies both (a) and (b) above. Replacing X by its subspace Y if necessary, we may assume that $\bigcap\{G_n : n \geq 1\} = \emptyset$. It follows that X has no isolated points. Hence neither does the dense subspace D.

Apply (2.1) to D to find two disjoint, dense subspaces D_1, D_2 of D. Apply (2.4) to each D_i to find a subspace $E_i \subset D_i$ that is dense in X and is a G_δ-subset of X. But then we have two disjoint dense G_δ-subsets of X, and that is impossible because X is Volterra. □

2.7 **Remark**: Condition (b) in (2.6) can be weakened to "For each $n \geq 1$ there is a collection $\{V(d,n) : d \in D_n\}$ of open subsets of X with $d \in V(d,n) \cap D_n$ and such that for each $x \in X$, the set $\{d \in D_n : x \in V(d,n)\}$ is finite."

Our next result gives several broad classes $\mathcal{S}$ of spaces such that any $X \in \mathcal{S}$ is a Baire space if and only if X is Volterra. In particular, (2.8) gives an affirmative answer to a question posed by Gauld, Greenwood, and Piotrowski [GGP2] who asked whether a Volterra metric space must be a Baire space.

2.8 **Corollary**: *A Volterra space X is Baire if X belongs to any one of the following classes:*

a) X has a dense subspace Y that is a strongly collectionwise Hausdorff, sequential, and has a relatively σ-closed discrete dense subset;

b) X has a dense metrizable subspace;

c) X is a Lasnev space, i.e., a closed continuous image of a metric space;

d) X is a metacompact sequential space that has a σ-closed-discrete dense set;

e) X is a metacompact Moore space or, more generally, a metacompact semi-stratifiable sequential space;

f) X is separable and sequential.

Proof: Case (a) follows from the fact that if Y is as described in (b), then Y has a dense subset $D = \bigcup\{D_n : n \geq 1\}$ such that each D_n is a relatively closed, discrete subspace of Y. Because Y is strongly collectionwise Hausdorff, for each n we may find a collection $\{W(d,n) : d \in D_n\}$ of pairwise disjoint, relatively open subsets of Y, with $W(d,n) \cap D_n = \{d\}$ for each $d \in D_n$. Write $W(d,n) = Y \cap V(d,n)$ where $V(d,n)$ is an open subset of X. Because Y is dense in X, the collection $\{V(d,n) : d \in D_n\}$ must be pairwise disjoint and hence point-finite in X. Now (2.6) applies to complete the proof of (a). Given (a), assertion (b) is immediate and (c) follows from (a) by letting $Y = X$.

Assertion (d) is an immediate consequence of (2.6) because metacompactness of X allows us to expand each level of the σ-closed discrete dense set $D = \bigcup\{D_n : n \geq 1\}$ to a point-finite open collection in X, as required in (2.6). To prove assertion (e), recall that any semi-stratifiable space has a dense σ-closed discrete subspace so that (d) yields the desired conclusion. Finally, assertion (f) follows from (a) by letting Y be any countable dense subset of X. $\square$

2.9 **Remark**: Arhangel'skii and Nedev [AN] noted that if $V = L$ then every normal semi-metric space contains a dense metrizable subspace. Hence, $V = L$ yields that if X is a normal semi-metric space, then X is a Baire space if and only if X is Volterra.

2.10 **Remark**: Example 3.1 provides a countable, regular space that is Volterra but not Baire, showing that the sequentiality conditions in 2.6 and 2.8 cannot be eliminated entirely.

2.11 **Questions**:
(a) Is it true that any Volterra Moore space must be a Baire space?
(b) Is it true that a space X must be a Baire space provided X is Volterra and has a dense subspace that is developable and metacompact? (If the dense subspace is developable and screenable, then the answer is affirmative.[He])
(c) Must X be Baire if X is stratifiable and Volterra?
(d) Suppose X is a σ-space and first category in itself. Must every dense subset D of X contain a dense subset E which is G_δ in X? (By 2.4, the answer is positive if X is metacompact and sequential.)

An earlier draft of this paper pointed out that if a Moore space has a dense subspace D as in (2.6), then D is itself a metacompact Moore space, and asked whether every Moore space has a dense, metacompact subspace. G.M. Reed and D. McIntyre have provided a counterexample in [RM].

3. Examples.

3.1 **Example**: *There is a countable regular space that is Volterra but not Baire.*
Proof: In [vD], van Douwen constructed a space X with the following properties:

a) X is countable, regular, and T_1;
b) every non-empty open subset of X is infinite;
c) if $A, B \subset X$ have no isolated points in their relative topologies, and if $A \cap B = \emptyset$, then no point of X is a limit point of both A and B so that $\mathrm{cl}(A) \cap \mathrm{cl}(B) = \emptyset$. (Van Douwen called such a space "ultra-disconnected." Note that this property implies that X contains no convergent sequences.)

Being a countable space with no isolated points, X is first category in itself. However, X is Volterra. Indeed, even more is true, for suppose that S and T are *any* two dense subsets of X. If $S \cap T$ is not dense in X, then for some open $U \subset X$ we have $S \cap T \cap U = \emptyset$. Let $A = S \cap U$ and $B = T \cap U$. Because $A \cap B = \emptyset$, property (c) yields $\emptyset = \mathrm{cl}(A) \cap \mathrm{cl}(B) \supseteq U$ which is impossible.

Note that while this X shows that Volterra and Baire are not equivalent in the class of σ-spaces, it does not answer question 2.11(e), since every subset of X is G_δ. □

3.2 **Example**: *There is a Lindelöf, hereditarily paracompact linearly ordered topological space that is Volterra but not Baire.*

Proof: This space is a slight modification of [G2, Ex. 1.1]. Let Z be the set of integers and let $X = \{f : \omega_1 \to Z : \{\alpha < \omega_1 : f(\alpha) \neq 0\}$ is finite $\}$. Endow X with the lexicographic order and the open interval topology of that ordering. For $f \in X$ and $\alpha < \omega_1$, let $B(f, \alpha) = \{g \in X : g(\beta) = f(\beta)$ for each $\beta \leq \alpha\}$. One can check that $\{B(f, \alpha) : \alpha < \omega_1\}$ is a neighborhood base at f for any $f \in X$. It follows that every G_δ-subset of X is open and therefore that X is Volterra. Further, as in [G2], X is Lindelöf and hereditarily paracompact. However, X is not a Baire space because $X = \bigcup\{X_n : n \geq 1\}$ where $X_n = \{f \in X : |\{\alpha < \omega_1 : f(\alpha) \neq 0\}| \leq n\}$, and each X_n is closed and nowhere dense. □

3.3 **Example**: *There is a first countable, completely regular, paracompact space that is a Volterra space and is not a Baire space.*

Proof: The underlying set of this example will be $X = Q \times \mathcal{B}$ where Q is the usual space of rational numbers and $\mathcal{B}$ is a certain branch space, described below. The topology τ for X will be constructed inductively, starting with the usual product topology τ_0 of X. Below we present a series of Claims that verify properties of (X, τ). Claim 3 shows that (X, τ) is first countable; Claim 4 shows that the space is completely regular and T_1, and Claim 5 shows that it is paracompact; Claim 6 shows that (X, τ) is not Baire; and Claim 7 shows that the space is Volterra.

Let B be a bistationary subset of $[0, \omega_1)$ (i.e., both B and $[0, \omega_1) - B$ are stationary) and let $\mathcal{T} = \{T \subset B : T$ is a closed subset of $[0, \omega_1)\}$. Each member of $\mathcal{T}$ is countable and contains its supremum. Further, $|\mathcal{T}| = 2^\omega = \underline{c}$. Partially order $\mathcal{T}$ by "end extension", i.e. $S \prec T$ if and only if $[0, \sup(S)] \cap T = S$. For each $T \in \mathcal{T}$, let $V_T = \{S \in \mathcal{T} : T \prec S\}$ and let Φ be the topology on $\mathcal{T}$ for which the collection $\{V_T : T \in \mathcal{T}\}$ is a base. This auxilliary space is not even T_1, but it does have one crucial property that we will need later, namely that the intersection of countably many dense open subsets of $(\mathcal{T}, \Phi)$ is a dense, open set. One proof appears in [T, Lemma 9.12]; direct proofs are also possible, using the fact that B is stationary.

By a "branch" of $\mathcal{T}$ we mean a maximal, linearly ordered (by $\prec$) subset of $\mathcal{T}$. Because $[0, \omega_1) - B$ is stationary, each branch is countable. Let $\mathcal{B} = \{b : b$ is a branch of $\mathcal{T}\}$ and define $[T] = \{b \in \mathcal{B} : T \in b\}$ for each $T \in \mathcal{T}$. Topologize $\mathcal{B}$ by using $\{[T] : T \in \mathcal{T}\}$ as a subbase. Each set $[T]$ is clopen in this topology and, because each branch is countable, the branch space is first countable. In addition the branch space is T_1. Later in the proof, we will need the following easily verified claim:

Claim 1: Suppose $\mathcal{C} \cup \mathcal{D}$ is a dense open subset of $\mathcal{B}$. Then $\mathcal{H} = \{S \in \mathcal{T} : [S] \subset \mathcal{C}$ or $[S] \subset \mathcal{D}\}$ is a dense open subset of $(\mathcal{T}, \Phi)$.

Claim 2: There is a set $\{(A_\alpha, D_\alpha, T_\alpha) : \alpha < \underline{c}\}$ such that

a) $A_\alpha \subset D_\alpha \subset Q$ and $T_\alpha \in \mathcal{T}$ for each $\alpha < \underline{c}$;
b) if $\alpha \neq \beta$, then $T_\alpha \neq T_\beta$;
c) whenever $\beta < \underline{c}$ and $A \subset D \subset Q$ and $T \in \mathcal{T}$, there is an α with $\beta < \alpha < \underline{c}$ such that $A_\alpha = A,\ D_\alpha = D$ and $T \prec T_\alpha$.

To show that such an indexing can be found, let $\mathcal{C} = \{(A, D, T^{'}) : A \subset D \subset Q, T^{'} \in \mathcal{T}\}$ and let $\mathcal{C} = \{(A_\alpha, D_\alpha, T^{'}_\alpha) : \alpha < \underline{c}\}$ be any indexing of $\mathcal{C}$. Choose any $T_0 \in \mathcal{T}$ with $T^{'}_0 \prec T_0$ and for $\alpha > 0$ replace $T^{'}_\alpha$ by some set T_α having $T^{'}_\alpha \prec T_\alpha$ and $T_\alpha \notin \{T_\beta : \beta < \alpha\}$. Notice that for fixed A, D, and T with $A \subset D \subset Q$, their are $\underline{c}$-many triples of the form (A, D, T') with $T \prec T'$ so that, given β, A, D, and T as in (c), there is an $\alpha > \beta$ with $A_\alpha = A$, $D_\alpha = D$, and $T \prec T^{'}_\alpha$. Replacing $T^{'}_\alpha$ by T_α as described produces the required $(A_\alpha, D_\alpha, T_\alpha)$.

Next we describe a recursion that produces the topology τ on $X = Q \times \mathcal{B}$. We will define a hierarchy $\{\tau_\alpha : \alpha < \underline{c}\}$ of topologies on X as follows. Begin by letting τ_0 be the usual product topology on $Q \times \mathcal{B}$. Next, if $0 < \alpha < \underline{c}$ and we have defined τ_β for each $\beta < \alpha$, we consider two cases. In case α is a limit ordinal, let τ_α be the topology for which $\bigcup\{\tau_\beta : \beta < \alpha\}$ is a base. In case $\alpha = \gamma + 1$, consider the triple $(A_\gamma, D_\gamma, T_\gamma)$ and let $\tau_\alpha = \tau_\gamma$ unless each of the following holds:

i) $D_\gamma \times [T_\gamma]$ is clopen in τ_γ;
ii) $\mathrm{Int}_{\tau_\gamma}(A_\gamma \times [T_\gamma]) = \emptyset$; and
iii) $\mathrm{Int}_{\tau_\gamma}((D_\gamma - A_\gamma) \times [T_\gamma]) = \emptyset$

and in that case, let τ_α be the topology having $\tau_\gamma \cup \{A_\gamma \times [T_\gamma],\ (D_\gamma - A_\gamma) \times [T_\gamma]\}$ as a subbase. This recursion defines τ_α for each $\alpha < \underline{c}$ and we let τ be the topology having $\bigcup\{\tau_\alpha : \alpha < \underline{c}\}$ as a base.

Claim 3: The topololgy τ has a subbase consisting of sets of the form $C \times [T]$ where one of the following holds:

i) C is clopen in the usual topology of Q;
ii) for some $\gamma < \underline{c}$, $C \times [T] = A_\gamma \times [T_\gamma]$;
iii) for some $\gamma < \underline{c}$, $C \times [T] = (D_\gamma - A_\gamma) \times [T_\gamma]$.

Consequently, X is first countable, because each $(q, b) \in X$ belongs to only countably many of the above subbasic open sets.

Claim 4: The topology τ has a base consisting of clopen sets of the form $D \times [T]$. One can verify this assertion by an inductive proof that each τ_α has a base of such sets, each being

τ_α-clopen. Hence the space (X, τ) is completely regular. Further, it is a T_1 space because τ_0 is a T_1-topology on X and $\tau_0 \subset \tau$.

Claim 5: (X, τ) is paracompact.

To verify Claim 5, let $\mathcal{U}$ be an open cover of X. For $q \in Q$, let $\mathcal{T}(q)$ be the set of all $T \in \mathcal{T}$ such that $D_T \times [T] \subset U$ for some $U \in \mathcal{U}$ and for some clopen set $D_T \times [T]$ with $q \in D_T$. Let $\mathcal{M}(q)$ be the mimimal members (in the tree order) of $\mathcal{T}(q)$. Let $\mathcal{V}(q) = \{D_T \times [T] : T \in \mathcal{M}(q)\}$. Since clopen sets of the form $D \times [T]$ form a base, it is easy to check that $\{[T] : T \in \mathcal{M}(q)\}$ covers the branch space $\mathcal{B}$. By minimality, $\mathcal{M}(q)$ is an antichain in $\mathcal{T}$, so it follows that $\mathcal{V}(q)$ is a disjoint open partial refinement of $\mathcal{U}$ which covers $\{q\} \times \mathcal{B}$. Now for any point $(q', b) \in X$, b is in a unique member T of $\mathcal{M}(q)$, whence $Q \times [T]$ is an open neighborhood of (q', b) that meets only one member of $\mathcal{V}(q)$. Thus $\mathcal{V}(q)$ is discrete, and so $\mathcal{U}$ has a σ-discrete open refinement. Hence (X, τ) is paracompact.

Claim 6: If $\mathcal{U}$ is a non-empty open subset of (X, τ) and if $\pi : X \to Q$ is first coordinate projection, then $\pi[\mathcal{U}]$ is infinite. Consequently, each set $\{q\} \times \mathcal{B}$ is closed and nowhere dense in X. Thus X is not a Baire space because $X = \bigcup\{\{q\} \times \mathcal{B} : q \in Q\}$.

To verify Claim 6, suppose there is a first ordinal α and a non-empty $U \in \tau_\alpha$ with $\pi[U]$ finite. Then $\alpha > 0$ and α is not a limit ordinal, so $\alpha = \gamma + 1$ for some γ. Note that $\tau_\alpha \neq \tau_\gamma$. Hence the three conditions of the inductive construction of τ_α are satisfied and sets of the form V, $V \cap (A_\gamma \times [T_\gamma])$ and $V \cap ((D_\gamma - A_\gamma) \times [T_\gamma])$, where $V \in \tau_\gamma$, form a base for τ_α. Minimality of α yields that $U \notin \tau_\gamma$. Consider the case where $U = V \cap (A_\gamma \times [T_\gamma])$, the other case being analogous. Because of condition (i) in Claim 2, we know that $D_\gamma \times [T_\gamma] \in \tau_\gamma$ so that $V \cap (D_\gamma \times [T_\gamma]) \in \tau_\gamma$. Minimality of α yields that $\pi[V \cap (D_\gamma \times [T_\gamma])]$ is infinite. Because $\pi[U]$ is finite, the set $\pi[U] \times \mathcal{B}$ is closed in (X, τ_γ) so that the set $W = V \cap (D_\gamma \times [T_\gamma]) - (\pi[U] \times \mathcal{B})$ belongs to τ_γ and is non-empty. But $W \subset (D_\gamma - A_\gamma) \times [T_\gamma]$ so that $\mathrm{Int}_{\tau_\gamma}((D_\gamma - A_\gamma) \times [T_\gamma]) \neq \emptyset$, contradicting part (iii) in the inductive construction of τ_α. Thus, Claim 6 holds.

Claim 7: If $\mathcal{G}_0$ and $\mathcal{G}_1$ are dense open subsets of (X, τ), then $\mathcal{G}_0 \cap \mathcal{G}_1$ is dense in X. Hence X is a Volterra space.

To verify Claim 7, suppose (for contradiction) that some non-empty $\mathcal{U} \in \tau$ has $\mathcal{U} \cap \mathcal{G}_0 \cap \mathcal{G}_1 = \emptyset$. We may assume that $\mathcal{U} = D \times [T]$ (see Claim 4).

For $q \in Q$ and $e \in \{0, 1\}$, let $\mathcal{G}(q, e) = \{b \in \mathcal{B} : (q, b) \in \mathcal{G}_e\}$. Then $\mathcal{G}(q, e)$ is a G_δ-subset of the branch space $\mathcal{B}$. Find open sets $\mathcal{V}(q, e, n)$ in $\mathcal{B}$ such that $\mathcal{V}(q, e, n+1) \subset \mathcal{V}(q, e, n)$ and $\mathcal{G}(q, e) = \bigcap\{\mathcal{V}(q, e, n) : n \geq 1\}$. Let $\mathcal{O}_q = \mathcal{B} - \mathrm{cl}_{\mathcal{B}}(\mathcal{G}(q, 0) \cup \mathcal{G}(q, 1))$. Then for each $q \in Q$ and $n \geq 1$, the set $\mathcal{O}_q \cup \mathcal{V}(q, 0, n) \cup \mathcal{V}(q, 1, n)$ is a dense open subset of $\mathcal{B}$. According to Claim 1, each set $\mathcal{H}(q, n) = \{S \in \mathcal{T} : [S] \subset \mathcal{O}_q$ or $[S] \subset \mathcal{V}(q, 0, n)$ or $[S] \subset \mathcal{V}(q, 1, n)\}$ is dense and open in the auxiliary space $(\mathcal{T}, \Phi)$. Hence so is $\mathcal{H} = \bigcap\{\mathcal{H}(q, n) : q \in Q, n \geq 1\}$.

Recall that $\mathcal{U} = D \times [T]$. Density allows us to choose $S \in \mathcal{H}$ with $T \prec S$. Then

$[S] \subset \mathcal{O}_q$ or else for each $n \geq 1$, $[S] \subset \mathcal{V}(q, e, n)$ for some $e \in \{0, 1\}$. Consider the case where $[S] \subset \mathcal{V}(q, 1, n_i)$ for infinitely many $n_i < n_{i+1}$, the other case being analogous. Then $[S] \subset \mathcal{G}(q, 1)$. Thus, either $[S] \subset \mathcal{O}_q$ or $[S] \subset \mathcal{G}(q, 0)$ or $[S] \subset \mathcal{G}(q, 1)$.

With D as above, let $A = \{q \in D : [S] \subset \mathcal{G}(q, 0)\}$. It is easy to see that $A \times [S] \subset \mathcal{G}_0$ and $(A \times [S]) \cap \mathcal{G}_1 = \emptyset$. We assert that $((D - A) \times [S]) \cap \mathcal{G}_0 = \emptyset$. To verify that assertion, suppose

(*) $\exists (q, b) \in ((D - A) \times [S]) \cap \mathcal{G}_0$.

Because $q \in D - A$ we know that $[S] \not\subset \mathcal{G}(q, 0)$. Hence $[S] \subset \mathcal{G}(q, 1)$ or $[S] \subset \mathcal{O}_q$. In case $[S] \subset \mathcal{G}(q, 1)$, we have $(q, b) \in \mathcal{G}_1$ so that $(q, b) \in \mathcal{G}_0 \cap \mathcal{G}_1$. Hence $(q, b) \notin \mathcal{U} = D \times [S]$ because $\mathcal{U} \cap \mathcal{G}_0 \cap \mathcal{G}_1 = \emptyset$. Because $q \in D - A \subset D$, we are forced to conclude that $b \notin [T]$. However, $b \in [S] \subset [T]$ because $T \prec S$, and that contradiction shows that $[S] \subset \mathcal{G}(q, 1)$ cannot occur. In the remaining case, $[S] \subset \mathcal{O}_q$, so that our definition of $\mathcal{O}_q$ yields $[S] \cap \mathcal{G}(q, 0) = \emptyset$. Hence $b \notin \mathcal{G}(q, 0)$, i.e., $(q, b) \notin \mathcal{G}_0$, contrary to assumption (*). Therefore $((D - A) \times [S]) \cap \mathcal{G}_0 = \emptyset$ as asserted.

Now apply the special properties of our indexing $\{(A_\alpha, D_\alpha, T_\alpha) : \alpha < \underline{c}\}$ as in Claim 2 to find some $\gamma < \underline{c}$ so large that the triple $(A_\gamma, D_\gamma, T_\gamma)$ has:

i) $D \times [T]$ is clopen in τ_γ; and
ii) $A_\gamma = A$, $D_\gamma = D$ and $S \prec T_\gamma$.

Then, at stage $\gamma + 1$ of the construction of the topology τ, either:

iii) $\mathrm{Int}_{\tau_\gamma}(A_\gamma \times [T\gamma]) \neq \emptyset$; or
iv) $\mathrm{Int}_{\tau_\gamma}((D_\gamma - A_\gamma) \times [T\gamma]) \neq \emptyset$; or
v) we added the set $A_\gamma \times [T_\gamma]$ to the topology $\tau_{\gamma+1}$.

In any of these cases, we have a non-empty τ-open set that is disjoint from either $\mathcal{G}_0$ or $\mathcal{G}_1$ and that is impossible because both $\mathcal{G}_0$ and $\mathcal{G}_1$ are dense in (X, τ). Thus, Claim 7 is established. □

4. Application to special first category sets.

A subspace X of a space Y is said to be *perfectly meager in* Y if $X \cap K$ is a first-category subset of K whenever K is a closed dense-in-itself subset of Y. A space X is *always of first category* if every dense-in-itself subset A of X is first category in itself. It is well-known that for subsets X of a complete separable metric space Y, X is perfectly meager in Y if and only if X is always of first category ([K, Theorem 1, page 516]).

Techniques from Section 2 can give a self-contained proof of a characterization of perfectly meager subsets of the unit interval $I = [0, 1]$ that first appeared in [BHL].

4.1 **Theorem**: *The following properties of a space X are equivalent:*

a) X is homeomorphic to a perfectly meager subset of I;
b) X is homeomorphic to a subspace of I and whenever $A \subset X$ there is a countable set $B \subset A$ such that B is dense in A and B is a G_δ-subset of X;

c) X is a zero-dimensional separable metric space and whenever $A \subset X$ there is a countable set $B \subset A$ such that B is dense in A and B is a G_δ-subset of X.

Proof: Because assertions (b) and (c) are clearly equivalent, it is enough to show the equivalence of (a) and (b).

Suppose (a) holds and suppose $A \subset X$ is given. First consider the case where A is dense in itself. Fix a countable dense subset $\{a_n : n \geq 1\}$ of A. Let $Y = \text{cl}_X(A)$ and let $K = \text{cl}_I(Y)$. Because X is perfectly meager, so is Y, and K is a closed, dense-in-itself subset of I. Hence $Y \cap K$ is first category in K so there are relatively closed subsets C_n of the space K with $Y = Y \cap K \subset \bigcup\{C_n : n \geq 1\}$ and $\text{Int}_K(C_n) = \emptyset$.

Consider the relatively open subset $G_n = Y - C_n$ of Y. We claim that G_n is dense in Y. If not, then there is a point $p \in Y$ and an interval $(a, b) \subset I$ such that $p \in (a, b) \cap Y \subset C_n$. Because Y is dense in K, we have $p \in (a, b) \cap K \subset \text{cl}_K((a, b) \cap Y)) \subset \text{cl}_K(C_n) = C_n$ showing that $p \in \text{Int}_K(C_n)$ which is impossible. Thus, G_n is a dense relatively open subset of Y.

Because A is also dense in Y, the set $A \cap G_n$ is dense in Y. Thus, for each a_n in the countable dense set chosen above, we can find a sequence $\{a(n, j) : j \geq 1\}$ of points of $A \cap G_n$ that converges to a_n. Then the set $H_n = \{a(n, j) : j \geq 1\}$ is a G_δ-subset of Y, $H_n \subset G_n$, and $H_n \subset A$. Because $\bigcap\{G_n : n \geq 1\} = \emptyset$, Lemma 2.2 shows that the set $B = \bigcup\{H_n : n \geq 1\}$ is a G_δ-subset of the space Y. But, because Y is closed in X, Y is a G_δ-susbet of X so that B is also a G_δ-subset of X. Finally, note that each point a_n is a limit point of B, so that B is dense in A.

Now consider the general case, where the subset A of X might not be dense-in-itself. Let $A_0 = \{a \in A : a \text{ is an isolated point of the set } A\}$. Then A_0 is countable and is a G_δ in X. Let $A_1 = A - \text{cl}_X(A_0)$. Then A_1 is dense in itself so that the first part of our argument yields a countable set $B_1 \subset A_1$ that is dense in A_1 and is a G_δ-subset of X. Letting $B = A_0 \cup B_1$ we obtain the desired set B. Thus (a) implies (b).

To prove that (b) implies (a), suppose that X satisfies (b) of the theorem, and suppose K is a dense-in-itslf subset of I. We must show that the set $A = K \cap X$ is first category in K. Find the countable set $B \subset A$ as described in (b), and write $B = \bigcap\{X \cap V_n : n \geq 1\}$ where each V_n is open in I. It is easy to see that $A \subset B \cup \bigcup\{X - V_n : n \geq 1\}$ so that, with $C = \text{cl}_K(A)$, we have $A \subset B \cup \bigcup\{C - V_n : n \geq 1\}$. Because $B \subset V_n$ and B is dense in A, which is dense in C,we see that V_n is dense in C. Hence $C - V_n$ is a closed, nowhere dense subset of C. Therefore each $C - V_n$ is also a closed nowhere dense subset of K. Because K is dense-in- itself, the set $\{b\}$ is closed and nowhere dense in K for each $b \in B$. Thus $A \subset B \cup \bigcup\{C - V_n : n \geq 1\}$ shows that $A = K \cap X$ is first category in K as required. $\square$

It is natural to ask whether the hypothesis of zero-dimensionality can be dropped from (4.1(c)). The answer is “No” as can be seen from the fact that there exist λ-sets (i.e., separable metric spaces every countable subset of which is a relative G_δ) of every possible dimension ([K, Theorem 5, p. 520]) and any λ-set clearly satisfies all of the conditions of 4.1(c) except for 0-dimensionality. However, the remaining conditions in 4.1(c) turn out to decsribe exactly the class of separable metric spaces that are always of first category. In fact, from the results in Section 2 we obtain the more general characterization of metric spaces that are always of first category, as follows.

4.3 **Proposition**: *For any metric space X, the following are equivalent:*

a) X is always of first category;

b) For every subset $A \subseteq X$, there is a σ-closed discrete subset $B \subset A$ that is dense in A and is a G_δ-subset of X.

Proof: It is clear that (b) implies (a). To prove the converse, suppose X satisfies (a) and $A \subset X$. Let B_0 be the set of isolated points of A and let $A_1 = A - \mathrm{cl}_X(B_0)$. Then B_0 is σ-closed discrete in X. Further, A_1 is dense in itself, whence so is $\mathrm{cl}_X(A_1)$, so that by (a), $\mathrm{cl}_X(A_1)$ is of first category in itself. Now apply Corollary 2.5 to find a dense subset B_1 of A_1 that is a σ-closed discrete G_δ-subset of $\mathrm{cl}_X(A_1)$. Then B_1 is also a σ-closed discrete G_δ-subset of X. Now let $B = B_0 \cup B_1$. □

4.4 **Remark**: In case X is separable metric, then the subset B in 4.3(b) will be countable.

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