

# SPACES HAVING A SMALL DIAGONAL

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ABSTRACT. We obtain several results and examples concerning the general question “When must a space with a small diagonal have a  $G_\delta$ -diagonal?”. In particular, we show (1) every compact metrizable fibered space with a small diagonal is metrizable; (2) there are consistent examples of regular Lindelöf (even hereditarily Lindelöf) spaces with a small diagonal but no  $G_\delta$ -diagonal; (3) every first-countable hereditarily Lindelöf space with a small diagonal has a  $G_\delta$ -diagonal; (4) assuming  $CH$ , every Lindelöf  $\Sigma$ -space with a small diagonal has a countable network; (5) the statement “countably compact spaces with a small diagonal are metrizable” is consistent with and independent of  $ZFC$ ; (6) there is in  $ZFC$  a locally compact space with a small diagonal but no  $G_\delta$  diagonal.

## 0. INTRODUCTION

According to M. Hušek [H<sub>2</sub>] (see also [H<sub>1</sub>]), a space  $X$  has a *small diagonal*<sup>1</sup> if for every uncountable subset  $Y$  of  $X^2 \setminus \Delta$ , there is an open set  $U \supset \Delta$  such that  $Y \setminus U$  is uncountable. Clearly a space with a  $G_\delta$ -diagonal has a small diagonal; the question is, for what classes of spaces does small diagonal imply  $G_\delta$ -diagonal?

This question for compact  $T_2$ -spaces is a well-known and still not completely solved problem of Hušek, who proved that, assuming the Continuum Hypothesis ( $CH$ ), compact spaces of countable tightness having a small diagonal are metrizable (recall metrizability is equivalent to  $G_\delta$ -diagonal for compact or even countably compact spaces—see, e.g., [Gr]). Dow [D] showed that this result holds in any model obtained by adding Cohen reals over a model of  $CH$ , and Zhou [Z] proved, assuming  $CH$  plus what he called “Fleissner’s Axiom”, that compact spaces having a small diagonal are metrizable. Later, Juhasz and Szentmiklóssy [JS] showed in  $ZFC$  that non-countably-tight compact spaces cannot have a small diagonal (because they have convergent  $\omega_1$ -sequences). This implies that the countable tightness assumption in the above results of Hušek and Dow can be omitted, i.e., compact spaces having a small diagonal are metrizable in models of  $CH$  or in their extensions by Cohen forcing.

Hušek also asked the corresponding question for  $\omega_1$ -compact and other spaces. Zhou [Z] studied the question for Lindelöf, countably compact, and locally compact spaces, and obtained, under  $MA + \neg CH$ , a locally compact example, and a

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<sup>1</sup>Hušek actually used the more descriptive “ $\omega_1$ -inaccessible diagonal”, but the term “small diagonal”, which was suggested by E. van Douwen, seems to have become more popular.

hereditarily Lindelöf, but non-regular, example of a space with a small diagonal and no  $G_\delta$ -diagonal. Arhangel'ski and Bella [AB] generalized the afore-mentioned  $CH$  result for compact spaces to Lindelöf spaces which are perfect pre-images of metrizable spaces.

Here we show: (1) every compact metrizable fibered space with a small diagonal is metrizable; (2) there are consistent examples of regular Lindelöf (even hereditarily Lindelöf) spaces with a small diagonal but no  $G_\delta$ -diagonal; (3) every first-countable hereditarily Lindelöf space with a small diagonal has a  $G_\delta$ -diagonal; (4) assuming  $CH$ , every Lindelöf  $\Sigma$ -space with a small diagonal has a countable network; (5) whether countably compact spaces with a small diagonal are metrizable depends on your set theory; (6) there is a locally compact space with a small diagonal but no  $G_\delta$  diagonal. The Lindelöf  $\Sigma$ -space result (4) answers a question of Arhangel'skii, statement (5) answers questions of Zhou and Shakhmatov, and (2) and (6) answer questions left open by Zhou.

In the sequel we mention several questions which remain open, including:

- (1) Is it true in  $ZFC$  that every compact space (or Lindelöf  $\Sigma$ -space) with a small diagonal is metrizable?
- (2) Is there in  $ZFC$  a Lindelöf space with a small diagonal but no  $G_\delta$ -diagonal?
- (3) Can there be a first-countable countably compact, or first-countable Lindelöf, space with a small diagonal but no  $G_\delta$ -diagonal<sup>2</sup>?

Unless stated otherwise, all spaces are assumed to be regular and  $T_1$ .

## 1. PRELIMINARIES

It will be helpful in the sequel to make some observations about small diagonals that are probably well known to anyone who has considered this property. First, note that any open superset of the diagonal in  $X^2$  contains an open set of the form  $\bigcup_{U \in \mathcal{U}} U^2$  for some open cover  $\mathcal{U}$  of  $X$ . The following is then easy to prove:

**Lemma 1.1.** *The following are equivalent for a space  $X$ :*

- (a)  $X$  has a small diagonal;
- (b) Whenever  $\mathcal{D}$  is a uncountable collection of doubletons in  $X$ , there is an open cover  $\mathcal{U}$  such that, for uncountably many  $d = \{d_1, d_2\} \in \mathcal{D}$ ,

$$\forall U \in \mathcal{U} (d \not\subseteq U).$$

We recall here that it is well-known and easy to see that a space with a small diagonal cannot contain a convergent  $\omega_1$ -sequence, i.e., a sequence  $\langle x_\alpha \rangle_{\alpha < \omega_1}$  such that every neighborhood of some point  $x$  contains all but countably many  $x_\alpha$ 's. For then the set of points  $(x_\alpha, x)$ ,  $\alpha < \omega_1$ , in  $X^2 \setminus \Delta$  is readily seen to witness the failure of the small diagonal property (from the definition or from Lemma 1.1(b)). It is also clear that the small diagonal property is hereditary.

**Proposition 1.2.** *Of the statements below, (c)  $\Rightarrow$  (b)  $\Rightarrow$  (a). If  $X$  is Lindelöf, all are equivalent.*

- (a)  $X$  has a small diagonal;

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<sup>2</sup> O. Pavlov[P] has recently obtained a positive solution to the countably compact question.

- (b) Whenever  $\{\{d_\alpha^0, d_\alpha^1\} : \alpha < \omega_1\}$  is a collection of doubletons of  $X$ , there are disjoint closed sets  $H_0$  and  $H_1$  with  $d_\alpha^0 \in H_0$  and  $d_\alpha^1 \in H_1$  for uncountably many  $\alpha < \omega_1$ ;
- (c) Whenever  $\{\{d_\alpha^0, d_\alpha^1\} : \alpha < \omega_1\}$  is a collection of doubletons of  $X$ , there is a co-zero (if  $X$  is 0-dimensional we can say clopen) set  $U$  such that  $d_\alpha^0 \in U$  and  $d_\alpha^1 \notin U$  for uncountably many  $\alpha \in \omega_1$ .

*Proof.* That (c) implies (b) is clear from the fact that co-zero sets are  $F_\sigma$ . For (b) implies (a), note that the complement of  $H_0 \times H_1$  is an open superset of the diagonal.

Now assume  $X$  is Lindelöf. We show (a) implies (c). Let  $\mathcal{D} = \{\{d_\alpha^0, d_\alpha^1\} : \alpha < \omega_1\}$  be a collection of doubletons of  $X$ . There is an open cover  $\mathcal{U}$  of  $X$  satisfying the conditions of Lemma 1.1(b) with respect to  $\mathcal{D}$ . Since  $X$  is regular and Lindelöf, hence completely regular, we may assume  $\mathcal{U}$  is countable and consists of co-zero sets (clopen sets if  $X$  is 0-dimensional). For each  $\alpha < \omega_1$ , there is some  $U_\alpha \in \mathcal{U}$  with  $d_\alpha^0 \in U_\alpha$ . Note that  $d_\alpha^1 \notin U_\alpha$ . Now the result follows since  $U_\alpha$  must be the same member of  $\mathcal{U}$  for uncountably many  $\alpha$ .  $\square$

## 2. LINDELÖF $\Sigma$ -SPACES AND COMPACT METRIZABLY FIBERED SPACES

A space  $X$  is a Lindelöf  $\Sigma$ -space [N] if it is a continuous image of a perfect pre-image of a separable metric space; equivalently, there is a countable collection  $\mathcal{F}$  of closed sets and a cover  $\mathcal{K}$  by compact sets such that, whenever  $U$  is an open superset of some  $K \in \mathcal{K}$ , then  $K \subset F \subset U$  for some  $F \in \mathcal{F}$ . The class of Lindelöf  $\Sigma$ -spaces can be viewed as a common generalization of the class of compact spaces and separable metric spaces. Every  $K$ -analytic space (see, e.g., [RJ] for the definition) is in this class. A Lindelöf  $\Sigma$ -space has a  $G_\delta$ -diagonal iff it has a countable network (and hence iff it is a continuous image of a separable metric space).

As a generalization of the result for compact spaces, Arhangel'skii and Bella [AB] proved, assuming  $CH$ , that if  $X$  is a perfect pre-image of a separable metric space, and has a small diagonal, then  $X$  is metrizable. (Bennett and Lutzer [BL] showed, however, that there are paracompact –but necessarily non-Lindelöf – perfect pre-images of metric spaces having a small diagonal but no  $G_\delta$ -diagonal.)

Thus the following question, due to Arhangel'skii [A, Problem 70], is natural: Is it true, or at least consistent, that a Lindelöf  $\Sigma$ -space  $X$  with a small diagonal must have a countable network (equivalently, must be a continuous image of a separable metric space)? The question was repeated by Tkachuk [T<sub>1</sub>], who answered it affirmatively in case  $X$  is a space of the form  $C_p(Y)$ , i.e., all continuous real-valued valued functions on  $Y$  with the topology of pointwise convergence.

In this section, we solve part of Arhangel'skii's question by showing that the answer is positive under  $CH$ . The main result here is in fact the following theorem of ZFC, which has the  $CH$  result as a corollary.

**Theorem 2.1.** *Suppose  $X$  is a regular Lindelöf  $\Sigma$ -space, witnessed by the countable collection  $\mathcal{F}$  of closed sets and cover  $\mathcal{K}$  by compact sets. If every member of  $\mathcal{K}$  is metrizable, and  $X$  has a small diagonal, then  $X$  has a countable network.*

Before embarking on the proof, we first establish the following useful fact.

**Lemma 2.2.** *Suppose  $X$  is a regular space, and that  $\mathcal{F}$  and  $\mathcal{K}$  satisfy the conditions for  $X$  to be a Lindelöf  $\Sigma$ -space, where  $\mathcal{F}$  is closed under finite intersections. Let  $\mathcal{K}^*$  be the collection of all non-empty intersections from the collection  $\{K \cap F : K \in \mathcal{K}, F \in \mathcal{F}\}$ . Then  $\mathcal{F}$  and  $\mathcal{K}^*$  also satisfy these conditions.*

*Proof.* Let  $\mathcal{H}$  be the collection of all closed sets  $H$  such that for any open superset  $U$  of  $H$ , there is some  $F \in \mathcal{F}$  with  $H \subset F \subset U$ . Note that both  $\mathcal{K}$  and  $\mathcal{F}$  are contained in  $\mathcal{H}$ . We need to show  $\mathcal{K}^* \subset \mathcal{H}$ .

We first show that if  $H_0, H_1 \in \mathcal{H}$ , then  $H_0 \cap H_1 \in \mathcal{H}$ . Suppose  $H_0 \cap H_1 \subset U$ , where  $U$  is open. Since  $X$  is regular Lindelöf, hence normal, there are disjoint open sets  $V_0, V_1$  containing  $H_0 \setminus U$  and  $H_1 \setminus U$ , respectively. Now there are  $F_0, F_1 \in \mathcal{F}$  containing  $H_0, H_1$ , respectively, and contained in  $U \cup V_0, U \cup V_1$ , respectively. Then  $F_0 \cap F_1$  contains  $H_0 \cap H_1$ , and it is easy to check that  $F_0 \cap F_1 \subset U$ .

It follows that we may assume  $\mathcal{K}$  is closed under intersections with members of  $\mathcal{F}$ . Thus it remains to check that every non-empty intersection of members of  $\mathcal{K}$  is in  $\mathcal{H}$ . By the above paragraph, this is true for finite intersections. Suppose  $\kappa$  is an infinite cardinal and every non-empty intersection of fewer than  $\kappa$  members from  $\mathcal{K}$  is in  $\mathcal{H}$ . The lemma follows if we can show that whenever  $\{K_\alpha : \alpha < \kappa\} \subset \mathcal{K}$  and  $\emptyset \neq \bigcap_{\alpha < \kappa} K_\alpha$ , then  $\bigcap_{\alpha < \kappa} K_\alpha \in \mathcal{H}$ . But this holds because any open superset of  $\bigcap_{\alpha < \kappa} K_\alpha$  contains  $\bigcap_{\alpha < \beta} K_\alpha$  for some  $\beta < \alpha$ , and  $\bigcap_{\alpha < \beta} K_\alpha \in \mathcal{H}$  by the inductive assumption.  $\square$

*Proof of Theorem 2.1.* Assume  $X, \mathcal{F}$ , and  $\mathcal{K}$  satisfy the hypotheses of the theorem.

For each  $K \in \mathcal{K}$ , let  $\mathcal{F}_K = \{F \in \mathcal{F} : K \subset F\}$ . If  $p \in X$  and  $\mathcal{N}$  is a collection of sets, let us say that  $\mathcal{N}$  generates a network at  $p$  if the collection of all finite intersections of members of  $\{N \in \mathcal{N} : p \in N\}$  is a network at  $p$ . First we show:

*Claim.* *For each  $K \in \mathcal{K}$ , there is a countable collection  $\mathcal{U}_K$  of cozero sets such that  $\mathcal{U}_K \cup \mathcal{F}_K$  generates a network at each point of  $K$ .*

*Proof of Claim.* Take  $K \in \mathcal{K}$ . Since  $K$  is separable metric, we may let  $\mathcal{B}_K$  be a countable (relative) base for the subspace  $K$ . For each pair  $B_0, B_1 \in \mathcal{B}_K$  having disjoint closures, since  $X$  is normal we can choose disjoint cozero sets in  $X$  containing them. Let  $\mathcal{U}_K$  be the collection of these chosen cozero sets. Suppose  $p \in K$ , and consider an open neighborhood  $U$  of  $p$ . Let  $N_0, N_1, \dots$  list all members of  $\mathcal{U}_K \cup \mathcal{F}_K$  which contain  $p$ . If no finite intersection of  $N_i$ 's is contained in  $U$ , then choose  $x_n \in \bigcap_{i \leq n} N_i \setminus U$ . Since the  $x_n$ 's diagonalize through members of  $\mathcal{F}_K$ , and  $\mathcal{F}_K$  is an outer network for the compact set  $K$ , the  $x_n$ 's must have some limit point, say  $q$ , in  $K$ . Certainly  $q \notin U$ . Thus there are  $B_0, B_1 \in \mathcal{B}_K$  having disjoint closures and containing  $p$  and  $q$ , respectively. Then there is a cozero set  $V$  in  $\mathcal{U}_K$  containing  $p$  whose closure misses  $q$ . But all but finitely many  $x_n$ 's are in  $V$ , contradiction. Thus  $\mathcal{U}_K \cup \mathcal{F}_K$  generates a network at  $p$ , which proves the claim.

We now observe that to complete the proof of the theorem, it suffices to show that there is a countable collection  $\mathcal{U}$  of cozero sets such that  $\mathcal{U} \cup \mathcal{F}$  separates points in the  $T_1$  sense. For, if such  $\mathcal{U}$  exists, then for each  $U \in \mathcal{U}$  we can add to  $\mathcal{F}$  a countable collection of closed sets whose union is  $U$ , and close up under finite intersections. Then every singleton is in the collection  $\mathcal{K}^*$  defined in Lemma 2.2, whence  $\mathcal{F}$  is a countable network for  $X$ .

Suppose then that no such collection  $\mathcal{U}$  exists. Pick  $K_0 \in \mathcal{K}$ , and let  $\mathcal{U}_{K_0}$  be as in the Claim. By assumption,  $\mathcal{U}_{K_0} \cup \mathcal{F}$  is not  $T_1$ -separating, so there are distinct

points  $x_1, y_1$  such that every member of  $\mathcal{U}_{K_0} \cup \mathcal{F}$  which contains  $x_1$  also contains  $y_1$ . It follows that every member of  $\mathcal{K}$  which contains  $x_1$  also contains  $y_1$ ; in particular, there is some  $K_1 \in \mathcal{K}$  with  $x_1, y_1 \in K_1$ . Then let  $\mathcal{U}_{K_1}$  be as in the Lemma 2.2.

Suppose  $\alpha < \omega_1$  and we have defined  $K_\beta$  for all  $\beta < \alpha$ , and points  $x_\beta \neq y_\beta \in K_\beta$  for  $0 < \beta < \alpha$ , such that every member of  $\mathcal{F} \cup (\bigcup_{\gamma < \beta} \mathcal{U}_{K_\gamma})$  which contains  $x_\beta$  also contains  $y_\beta$ . Since the collection  $\mathcal{F} \cup (\bigcup_{\gamma < \alpha} \mathcal{U}_{K_\gamma})$  is not  $T_1$ -separating, we can find  $x_\alpha \neq y_\alpha$  such that every member of the collection which contains  $x_\alpha$  also contains  $y_\alpha$ . Then choose some  $K_\alpha \in \mathcal{K}$  containing  $x_\alpha$ , and note that  $K_\alpha$  must contain  $y_\alpha$  too.

Thus we can define  $x_\alpha, y_\alpha$ , and  $K_\alpha$  as above for all  $\alpha < \omega_1$ . Since  $X$  has a small diagonal and is Lindelöf, there are disjoint closed sets  $H_0, H_1$  and an uncountable subset  $W$  of  $\omega_1$  such that  $x_\alpha \in H_0$  and  $y_\alpha \in H_1$  for all  $\alpha \in W$ .

Now consider  $\alpha \in W$ . For each  $p \in K_\alpha$ , assuming as we may that the collections  $\mathcal{F}$  and  $\mathcal{U}_{K_\alpha}$  are closed under finite intersections, by the Claim there is some  $U_p^\alpha \in \mathcal{U}_{K_\alpha}$  and some  $F_p^\alpha \in \mathcal{F}_{K_\alpha}$  such that  $p \in U_p^\alpha$  and  $U_p^\alpha \cap F_p^\alpha$  misses either  $H_0$  or  $H_1$ . By compactness, there are finite subcollections  $U_0^\alpha, U_1^\alpha, \dots, U_{n_\alpha}^\alpha$  of  $\mathcal{U}_{K_\alpha}$  and  $F_0^\alpha, F_1^\alpha, \dots, F_{n_\alpha}^\alpha$  of  $\mathcal{F}_{K_\alpha}$  such that the  $U_i^\alpha$ 's cover  $K_\alpha$  and each  $U_i^\alpha \cap F_i^\alpha$  misses either  $H_0$  or  $H_1$ . Choose  $F^\alpha \in \mathcal{F}_{K_\alpha}$  such that  $K_\alpha \subset F^\alpha \subset \bigcup_{i \leq n_\alpha} U_i^\alpha$  and  $F^\alpha \subset \bigcap_{i \leq n_\alpha} F_i^\alpha$ .

There are  $\alpha_1 < \alpha_2$  with  $F^{\alpha_1} = F^{\alpha_2} = F$ . Then  $K_{\alpha_1} \cup K_{\alpha_2} \subset F \subset \bigcup_{i \leq n_{\alpha_1}} U_i^{\alpha_1}$ . Choose  $i \leq n_{\alpha_1}$  with  $x_{\alpha_2} \in U_i^{\alpha_1}$ . Since  $U_i^{\alpha_1} \cap F_i^{\alpha_1}$  misses either  $H_0$  or  $H_1$ , and  $x_{\alpha_2} \in H_0$ , it must miss  $H_1$ . Thus  $x_{\alpha_2}$  is in  $U_i^{\alpha_1} \cap F_i^{\alpha_1}$  but  $y_{\alpha_2}$  is not, contradicting the way  $x_{\alpha_2}$  and  $y_{\alpha_2}$  were chosen. This contradiction completes the proof of the theorem.  $\square$

Since compact spaces with a small diagonal are metrizable under  $CH$ , the following corollary is immediate.

**Corollary 2.3 (CH).** *Every regular Lindelöf  $\Sigma$ -space with a small diagonal has a countable network.*

Fremlin [F<sub>1</sub>] showed that, assuming  $MA + \neg CH$ , if every compact subset of a  $K$ -analytic space  $X$  is metrizable, then  $X$  is analytic. Fremlin's result fails under  $CH$ , but we have the following corollary to our theorem.

**Corollary 2.4(CH).** *Every regular  $K$ -analytic space with a small diagonal is analytic.*

*Proof.* Let  $X$  be regular  $K$ -analytic space with a small diagonal. Then  $X$  is a Lindelöf  $\Sigma$ -space, hence has a countable network. A  $K$ -analytic space with a countable network is analytic (see, e.g., [RJ; Theorem 5.5.1]).  $\square$

Our theorem can also be applied to the class of compact metrizable fibered spaces. According to Tkachuk [T<sub>2</sub>],  $X$  is *metrizable fibered* if there is a continuous mapping  $f : X \rightarrow M$  for some metrizable space  $M$ , such that each point-inverse is metrizable. The class of metrizable fibered compacta contains the Alexandroff duplicate of the interval, the Alexandroff double arrow space, and many variations of these spaces. This class has been a rich source of examples in topology (see, e.g., [W] or [GN]). The following corollary, this time a *ZFC* result, shows that no member of this class can provide an answer to Hušek's question about compact spaces with a small diagonal.

**Corollary 2.5.** *A metrizable fibered compact space with a small diagonal must be metrizable.*

*Proof.* Let  $X$  be compact, and let  $f : X \rightarrow M$  be a continuous map from  $X$  onto the metrizable space  $M$ , with  $f^{-1}(m)$  metrizable for each  $m \in M$ . Then  $M$  has a countable base  $\mathcal{B}$ . Let  $\mathcal{F} = \{f^{-1}(B) : B \in \mathcal{B}\}$  and  $\mathcal{K} = \{f^{-1}(m) : m \in M\}$ . Then  $X$ ,  $\mathcal{F}$ , and  $\mathcal{K}$  satisfy all the hypotheses of Theorem 2.1. So  $X$  has a countable network, hence is metrizable.  $\square$

**Remark 2.6.** Some well-known members of the class of compact metrizable fibered spaces are perfectly normal, equivalently, hereditarily Lindelöf (e.g., the double arrow space). We note here that it is a corollary to Theorem 3.6 that in fact any perfectly normal compact space with a small diagonal is metrizable (in *ZFC*).

### 3. GENERAL LINDELÖF SPACES

Zhou [Z] gave an example, under  $MA + \neg CH$ , of a Hausdorff, non-regular, (hereditarily) Lindelöf space which has a small diagonal but no  $G_\delta$ -diagonal. It has remained unsolved whether or not there could be a regular example. In this section, we give two different constructions of consistent regular examples. One exists in a model of  $\neg CH$ , and is hereditarily Lindelöf, the other exists in some models of  $CH$  (including  $V = L$ ). The latter example shows a contrast with the compact case, where with  $CH$  small diagonal does imply  $G_\delta$ -diagonal.

Our example consistent with  $CH$  is obtained by modifying a construction due to Shelah for building an example of a Lindelöf space of cardinality  $\omega_2$  ( $= c^+$  since  $CH$  holds) in which each point is a  $G_\delta$ . The space cannot have a  $G_\delta$ -diagonal, for if it did, it would have a weaker separable metrizable topology (see, e.g., [Gr; Corollary 2.9]) and hence could not have cardinality greater than  $\mathfrak{c}$ . We don't know if the space as defined by Shelah always has a small diagonal, but we will show that it is possible to modify the forcing to make sure the diagonal will be small. Let us also remark that an easier construction of a large size Lindelöf points  $G_\delta$  space due to Gorelic [Go] does not seem to lend itself to a similar modification.

We will closely follow the presentation due to Juhász [J<sub>1</sub>] of Shelah's example. We recall the following definition:

**Definition.** A map  $f : X^2 \rightarrow 2$  is called *flexible* if for any distinct  $x, y \in X$  and  $i, j \in 2$  there is  $z \in X$  such that  $f(z, x) = i$  and  $f(z, y) = j$ .

For any  $x \in X$  and  $i \in 2$ , put

$$A_x^i = \{y \in X : y \neq x \text{ and } f(x, y) = i\}.$$

Let  $\tau_f^i$  be the topology on  $X$  having as subbase sets of the form  $A_x^i \cup \{x\}$  and their complements.

Let us call a map  $f : \omega_2^2 \rightarrow 2$  *very flexible* if it is flexible, and

(\*) whenever  $\alpha < \beta \leq \gamma < \omega_2$ , there exists  $\delta \in (\gamma, \gamma + \omega)$  with  $f(\delta, \alpha) \neq f(\delta, \beta)$ .

Shelah shows that there is a countably closed  $\omega_2$ -c.c. poset forcing a flexible function  $f : \omega_2^2 \rightarrow 2$  such that the topologies  $\tau_f^i$  are Lindelöf with points  $G_\delta$ . We will see that if  $f$  is very flexible, then the resulting space has a small diagonal. We then show that it is possible to modify the forcing so that  $f$  is very flexible.

**Lemma 3.1.** *If  $f : \omega_2^2 \rightarrow 2$  is very flexible, then  $\omega_2$  with either of the topologies  $\tau_f^i$  has a small diagonal.*

*Proof.* Suppose  $\{\alpha_\mu, \beta_\mu\}_{\mu < \omega_1} \subset [\omega_2]^2$ . Choose  $\gamma < \omega_2$  with  $\gamma > \alpha_\mu + \beta_\mu$  for every  $\mu < \omega_1$ . Since  $f$  is very flexible, for each  $\mu < \omega_1$  there is  $\delta_\mu \in (\gamma, \gamma + \omega)$  with  $f(\delta_\mu, \alpha_\mu) \neq f(\delta_\mu, \beta_\mu)$ . Thus for some  $\delta$ ,  $f(\delta, \alpha_\mu) \neq f(\delta, \beta_\mu)$  for uncountably many  $\mu$ . For these  $\mu$ ,  $A_\delta^i \cup \{\delta\}$  contains exactly one of  $\alpha_\mu, \beta_\mu$  and is clopen in  $\tau_f^i$ . Thus  $(\omega_2, \tau_f^i)$  has a small diagonal by Proposition 1.2.  $\square$

We continue to follow [J<sub>1</sub>]. Let  $\text{Fn}(I, J)$  denote the set of finite partial functions from  $I$  into  $J$ . For  $s \in \text{Fn}(\omega_2, 2)$ , put

$$U_s = \bigcap \{A_\alpha^{s(\alpha)} \cup \{\alpha\} : \alpha \in \text{dom}(s)\}$$

and let  $\mathcal{U}_f = \{U_s : s \in \text{Fn}(\omega_2, 2)\}$ .  $\mathcal{U}_f$  is said to be *Lindelöf* if every cover of  $\omega_2$  by members of  $\mathcal{U}_f$  has a countable subcover. As is shown in [J<sub>1</sub>], if  $\mathcal{U}_f$  is Lindelöf, then the topologies  $\tau_f^i$  are Lindelöf with points  $G_\delta$ . So it remains to prove the following Theorem 3.2, which is precisely Theorem 1.6 of [J<sub>1</sub>] with “very flexible” in place of “flexible”. The proof is also similar to that given in [J<sub>1</sub>], with one extra condition on members of the poset so that  $F$  will be sure to be very flexible. However, it’s not completely obvious that the same proof works with this extra condition, so it will be necessary to define the poset and give several key parts of the argument. But we will not repeat here the parts that are clearly not affected by the extra condition.

**Theorem 3.2.**  *$\text{Con}(ZF) \Rightarrow \text{Con}(ZFC + CH + \exists$  a very flexible  $F : \omega_2^2 \rightarrow 2$  for which  $\mathcal{U}_F$  is Lindelöf).*

*Proof.* Assume the ground model  $V$  satisfies  $ZFC + CH$ . A condition  $p$  will determine a countable subset  $A^p$  of  $\omega_2$  and a countable fragment  $f^p : (A^p)^2 \rightarrow 2$  of  $F$ . Now for  $s \in \text{Fn}(\omega_2, 2)$ , put

$$U_s^p = \{x \in A : \forall z \in \text{dom}(s)(z \neq x \Rightarrow f^p(z, x) = s(z))\}.$$

A condition  $p \in P$  is a triple  $p = \langle A, f, T \rangle$  satisfying:

- (i)  $A \in [\omega_2]^\omega$ ;
- (ii)  $f : A^2 \rightarrow 2$ ;
- (iii)  $|T| \leq \omega$  and  $\forall B \in T(B \subset \text{Fn}(A, 2) \ \& \ \cup \{U_s^p : s \in B\} = A)$ ;
- (iv)  $\forall B \in T \forall \delta \in A \forall y \in (A \setminus \delta) \forall h \in \text{Fn}(A \setminus \delta, 2) \exists s \in B (y \in U_{s \upharpoonright \delta}^p \ \& \ s \cup h \in \text{Fn}(A, 2))$ ;
- (v) whenever  $\alpha < \beta \leq \gamma \in A$ , there exists  $\delta \in A \cap (\gamma, \gamma + \omega)$  with  $f(\delta, \alpha) \neq f(\delta, \beta)$ .

Define a 3-place relation  $E^p$  on  $A$  by

$$E^p(\delta, y, z) \iff [\delta \leq y, z \ \& \ \forall x \in A \cap \delta (f(x, y) = f(x, z))].$$

Then if  $p' = \langle A', f', T' \rangle \in P$ , we say  $p' \leq p$  iff  $A \subset A'$ ,  $f \subset f'$ ,  $T \subset T'$ , and  $E^p \subset E^{p'}$ .

This poset is the same as the poset  $P$  given in [J<sub>1</sub>] except for the additional condition (v) (and the quite trivial but technically useful change in (i) disallowing

$|A| < \omega$ ). Clearly condition (v) will in the end give us that  $F$  is very flexible, once we have shown that everything else goes through as before.

Note that condition (v) does not affect whether or not  $E^p \subset E^{p'}$ , since  $E^p$  is determined by values  $f(x, y)$  for  $x < y$ , i.e., values above the diagonal, while condition (v) is determined by values below the diagonal. Let us note also that condition (iv) is determined by values of  $f$  above the diagonal, since  $\delta \leq y$  there and the truth of  $y \in U_{s \upharpoonright \delta}^p$  depends on what  $f(z, y)$  is for  $z \in s \upharpoonright \delta$ . Keeping this in mind will greatly simplify our task ahead.

The proof that  $P$  is  $\omega_1$ -complete is easy and the same as in [J<sub>1</sub>]. What will require some work is showing that Lemmas 1.9 and 1.10 of [J<sub>1</sub>] still hold. Juhász's Lemma 1.9 is:

**Lemma 3.3.**  *$P$  satisfies the  $\omega_2$ -c.c..*

*Proof.* As in [J<sub>1</sub>], by a  $\Delta$ -system argument, the proof boils down to showing that two conditions  $p = \langle A, f, T \rangle$  and  $p' = \langle A', f', T' \rangle$  are compatible whenever:

- (a)  $\Delta = A \cap A' < A \setminus \Delta < A' \setminus \Delta$ ;
- (b)  $f \upharpoonright \Delta^2 = f' \upharpoonright \Delta^2$ ;
- (c)  $\text{type}(A \setminus \Delta) = \text{type}(A' \setminus \Delta)$ ;
- (d) if  $z \in A \setminus \Delta$  and  $z' \in A' \setminus \Delta$  are such that  $\text{type}(A \cap z) = \text{type}(A' \cap z')$ , then  $f(x, z) = f(x, z')$  for all  $x \in \Delta$ ;
- (e) The natural bijection  $\theta : A \rightarrow A'$  induces a bijection from  $T$  to  $T'$ .

To this end, a function  $g : (A \cup A')^2 \rightarrow 2$  is constructed which extends both  $f$  and  $f'$ , and so that  $q = \langle A \cup A', g, T \cup T' \rangle$  is in  $P$  and extends both  $p$  and  $p'$ . The function  $g$  needs to be defined on

$$[(A \setminus \Delta) \times (A' \setminus \Delta) \cup (A' \setminus \Delta) \times (A \setminus \Delta)].$$

We can define  $g$  on  $(A \setminus \Delta) \times (A' \setminus \Delta)$  as in [J<sub>1</sub>], since as we noted above the extra condition (v) only depends on values of  $g$  below the diagonal. Note that this will also get condition (iii) holding for  $B \in T$ .

*Definition of  $g$  on  $(A' \setminus \Delta) \times (A \setminus \Delta)$ .* We are below the diagonal here, so we don't need to worry about condition (iv) or  $E^q$ . Satisfying condition (iii) for  $B' \in T'$ , and condition (v), are our only concern.

Enumerate in type  $\omega$  all 4-tuples  $\langle B', a, a', a'' \rangle$  where  $B' \in T'$ ,  $a \in A \setminus \Delta$ ,  $a' \in A' \setminus \Delta$ , and  $a'' \in (A \cup A') \cap (a' + 1)$ . The function  $g$  is defined by induction, finitely many values at a time. Suppose at stage  $n$ , we are given the  $n^{\text{th}}$  4-tuple  $\langle B', a, a', a'' \rangle$ . Let  $k'(x, a) = g(x, a)$  for the finitely many  $x \in A' \setminus \Delta$  for which  $g(x, a)$  has been defined. Let  $B = \theta^{-1}(B')$ , and let  $k = k' \circ \theta \in \text{Fn}(A \setminus \Delta, 2)$ . Apply condition (iv) for  $p$  with this  $B$ , and with  $\delta = \min(A \setminus \Delta)$ ,  $y = a$ , and  $h = k$  to get  $s \in B$  satisfying

$$a \in U_{s \upharpoonright \delta}^p \text{ and } s \cup k \in \text{Fn}(A, 2).$$

Let  $s' = s \circ \theta^{-1} \in T'$  and note that  $s' \in B'$ . Since  $s$  and  $k$  are compatible, so are  $s'$  and  $k'$ . Also,  $s \upharpoonright \delta = s \upharpoonright \Delta = s' \upharpoonright \Delta$ . Thus  $a \in U_{s \upharpoonright \delta}^p$  implies  $g(z, a) = f(z, a) = s(z) = s'(z)$  for all  $z \in \Delta \cap \text{dom}(s)$ . We can extend  $g$  so that now  $g(x, a) = (s' \cup k')(x)$  for every  $x \in (A' \setminus \Delta) \cap \text{dom}(s' \cup k')$ . This is consistent (by

the use of  $k'$ ) with how  $g$  was defined at previous steps of the induction, and it puts  $a \in U_s^q$ . So, when we're done, the conclusion of (iii) will hold for  $B'$ .

To make sure (v) will hold, at this same stage look at  $(a, a', a'')$  and choose some  $\delta \in (a', a' + \omega) \cap A'$  such that  $g(\delta, a)$  has not yet been defined. (Since (v) holds for  $p'$ ,  $(a', a' + \omega) \cap A'$  must be infinite.) Then simply make  $g(\delta, a)$  different from  $g(\delta, a'')$ .

Finally, let  $g(a', a) = 0$  if it is not yet defined.

This completes the definition of  $g$ , and the verification that  $q$  satisfies conditions (iii) and (v). Verification of the other conditions and that  $q$  extends  $p$  and  $p'$  is the same as in  $[J_1]$ , and for the most part is easily observed from the fact that above the diagonal this  $g$  is the same as the  $g$  there.  $\square$

Essentially what remains to show now is the following analogue of Lemma 1.10 of  $[J_1]$ . The difference is that here we use  $A \cup [z, z + \omega)$  instead of  $A \cup \{z\}$ ; we can't put the latter because condition (v) implies that  $(z, z + \omega) \cap A$  is infinite whenever  $z \in A$ . As in  $[J_1]$ , it's part (b) which implies that the resulting generic  $F$  is flexible.

**Lemma 3.4.** *Suppose  $p = \langle A, f, T \rangle \in P$ . Then*

- (a) *for every  $z \in \omega_2 \setminus A$  there is an extension  $g : (A \cup [z, z + \omega))^2 \rightarrow 2$  of  $f$  such that  $q = \langle A \cup [z, z + \omega), g, T \rangle \in P$  and  $q \leq p$ ;*
- (b) *If  $z \in \omega_2 \setminus (\cup A + 1)$ ,  $\delta, y \in A$  with  $\delta \leq y$  and  $h \in \text{Fn}(A \setminus \delta, 2)$ , there is an extension  $g = (A \cup [z, z + \omega))^2 \rightarrow 2$  of  $f$  such that  $q = \langle A \cup [z, z + \omega), g, T \rangle \in P$ ,  $q \leq p$ ,  $g(x, z) = h(x)$  for every  $x \in \text{dom}(h)$  and moreover  $E^q(\delta, y, z)$  holds.*

*Proof.* (a) We add elements of  $[z, z + \omega) \setminus A$  one at a time, starting with  $z$  itself. The first line of Lemma 1.10 in  $[J_1]$  says "put  $g(z, y) = 0$  for all  $y \in A$ ". This we do for  $y > z$ , i.e., for points above the diagonal. We define  $g(z, y)$  for  $y < z$  later as necessary to obtain condition (v) of the partial order.

Juhász defines  $g(y, z)$  for all  $y \in A$  in an induction which takes care of finitely many  $y$  at each stage. Note that no harm is done to the argument to at each stage arbitrarily define  $g(y, z)$  for finitely many "extra"  $y$ . In the following proof, of course we won't do this arbitrarily, but rather to, once again, ensure condition (v).

*Case a-1.*  $|(z, z + \omega) \cap A| = \omega$ . In this case, follow the proof in  $[J_1]$  (here we can let  $g(z, y) = 0$  for all  $y \in A$ ) to first extend to  $A \cup \{z\}$ . However, to obtain (v) also index the triples  $(\alpha, \beta, \gamma) \in (A \cup \{z\})^3$  with  $\alpha < \beta \leq \gamma$  and  $z \in \{\alpha, \beta\}$  as  $\{(\alpha_n, \beta_n, \gamma_n)\}_{n \in \omega}$ . Note that condition (v) holding for  $p$  implies that  $(\gamma_n, \gamma_n + \omega) \cap A$  is infinite whenever  $\gamma \in A \setminus [z, z + \omega)$ , and the hypothesis of Case a-1 implies the same for  $\gamma \in [z, z + \omega)$ . Then, at each stage of the definition of  $g(y, z)$ , after doing the same thing as in  $[J_1]$ , also find some  $y \in (\gamma_n, \gamma_n + \omega) \cap A$  such that  $g(y, z)$  is undefined, and define it so that  $g(y, \alpha_n) \neq g(y, \beta_n)$ . This suffices to ensure (v). (Note that triples  $(\alpha, \beta, z)$  with  $z \notin \{\alpha, \beta\}$  are taken care of by the fact that (v) holds for  $p$  with respect to triples  $(\alpha, \beta, \gamma)$  where  $\gamma \in A \cap [z, z + \omega)$ .)

Finally repeat the above process for  $z_1, z_2, \dots$ , where the  $z_i$ 's enumerate  $(z, z + \omega) \setminus A$ .

*Case a-2.*  $|(z, z + \omega) \cap A| < \omega$ . List  $[z, z + \omega) \setminus A$  in increasing order as  $z_0, z_1, \dots$  and add them in one at a time as in Case a-1. Triples  $(\alpha, \beta, \gamma)$  with

$\gamma \in A \setminus [z, z + \omega)$  can be taken care of as before. For triples with  $\gamma \in [z, z + \omega)$ , do the following. Let  $\{\{\alpha_n, \beta_n\}\}_{n \in \omega}$  list all doubletons from  $(A \cap z) \cup [z, z + \omega)$  with each one listed infinitely often. Then when considering  $z_n$ , if  $\alpha_n, \beta_n < z_n$ , make  $g(z_n, \alpha_n) \neq g(z_n, \beta_n)$  (which we may do by the comment in the first paragraph of the proof).

(b) Add in  $z + n$ ,  $n = 0, 1, \dots$  one at a time, defining  $g(x, z + n)$  as in  $[J_1]$  (with  $z = z + n$ ). Define  $g(z + n, y)$  for  $y < z + n$  so that in the end (v) will hold for any  $\gamma \in [z, z + \omega)$ .  $\square$

The remainder of the proof of Theorem 3.2 follows as in  $[J_1]$ , so that completes our argument.  $\square$

**Remark.** As with Shelah's example, the above construction can be done using an  $(\omega_1, 1)$  morass with built-in  $\diamond$  sequence, which exists in Godel's constructible universe  $L$  (see [V], Theorem 5.3.2); essentially what goes on is that under these assumptions there is a filter  $G$  on the partial order  $P$  meeting enough dense sets to obtain the desired function  $F : \omega_2^2 \rightarrow 2$ .

We now turn to the hereditarily Lindelöf example, which is built from an  $HFC^2$  in  $2^{\omega_2}$ . Recall (see  $[J_2]$  or  $[J_3]$ ) that an uncountable subset  $F$  of  $2^{\omega_2}$  is  $HFC^k$  ( $k \in \omega$ ) if for every uncountable subset  $W$  of  $F^k$ , and for every collection  $\{\langle \sigma_n^i \rangle_{i < k}\}_{n \in \omega} \subset (\text{Fn}(\omega_2, 2))^k$  with  $\text{dom}(\sigma_n^i) = H_n$  for all  $i < k$  and  $n \in \omega$ , where the  $H_n$ 's are disjoint and have the same cardinality, there is some  $n \in \omega$  and some  $k$ -tuple  $\langle g^i \rangle_{i < k} \in W$  with  $\sigma_n^i \subset g^i$  for all  $i < k$ . Recall also that any  $HFC(= HFC^1)$  is hereditarily Lindelöf, and there is an  $HFC^2$  (in fact a strong  $HFC$ ) of cardinality  $\omega_2$  in  $2^{\omega_2}$  in any model obtained by adding  $\omega_2$ -many Cohen reals.

**Theorem 3.5.** *Suppose there is an  $HFC^2$   $F$  in  $2^{\omega_2}$  of cardinality  $\omega_2$ . Then there is a hereditarily Lindelöf space with a small diagonal but no  $G_\delta$ -diagonal.*

*Proof.* For convenience, we may index  $F$  as  $\{f_\alpha^e : \alpha < \omega_2, e < 2\}$ . Now define  $g_\alpha : \omega_2 \rightarrow 2$  by  $g_\alpha(\beta) = f_\alpha^0(\beta)$  if  $\beta < \alpha$  and  $g_\alpha(\beta) = f_\alpha^1(\beta)$  if  $\beta \geq \alpha$ . Observe that, by the  $HFC^2$  property,  $f_\alpha^0$  and  $f_\alpha^1$  can agree on  $\omega_2 \setminus \alpha$  for at most countably many  $\alpha$ . Thus  $g_\alpha = f_\alpha^0$  for at most countably many  $\alpha$ .

Our example is the subspace  $X = \{g_\alpha : \alpha < \omega_2\} \cup \{f_\alpha^0 : \alpha < \omega_2\}$  of  $2^{\omega_2}$ . We'll prove  $X$  is  $HFC$  and hence hereditarily Lindelöf. Suppose  $\{\sigma_n\}_{n \in \omega} \subset \text{Fn}(\omega_2, 2)$  is such that the  $\sigma_n$ 's have disjoint domains of the same cardinality, and  $W \in [\omega_2]^{\omega_1}$ . Then, by  $F$  being  $HFC^2$ , there are  $\alpha \in W$  and  $n \in \omega$  such that  $f_\alpha^e \supset \sigma_n$  for each  $e < 2$ . Note that this implies that  $g_\alpha \supset \sigma_n$ . It follows that both  $\{f_\alpha^0 : \alpha < \omega_2\}$  and  $\{g_\alpha : \alpha < \omega_2\}$  are  $HFC$ -spaces, thus  $X$  is too and is hereditarily Lindelöf.

We show  $X$  has no  $G_\delta$ -diagonal. Suppose on the contrary that  $(\mathcal{G}_n)_{n \in \omega}$  is a  $G_\delta$ -diagonal sequence for  $X$ . Since  $X$  is Lindelöf, we may assume  $\mathcal{G}_n = \{[\sigma] : \sigma \in \Sigma_n\}$ , where  $\Sigma_n \in [\text{Fn}(\omega_2, 2)]^\omega$ . Choose  $\alpha < \omega_2$  with  $g_\alpha \neq f_\alpha^0$  and

$$\alpha > \bigcup \{\text{dom}(\sigma) : \sigma \in \bigcup_{n \in \omega} \Sigma_n\}.$$

Then since  $g_\alpha \upharpoonright \alpha = f_\alpha^0 \upharpoonright \alpha$ , we have  $g_\alpha \in [\sigma]$  whenever  $\sigma \in \Sigma_n$  and  $f_\alpha^0 \in [\sigma]$ . Thus  $g_\alpha \in st(f_\alpha^0, \mathcal{G}_n)$  for all  $n$ , contradiction.

It remains to show that  $X$  has a small diagonal. To this end, let  $\{h_\gamma^0, h_\gamma^1\}_{\gamma < \omega_1} \subset [X]^2$ . There is  $\mu < \omega_2$  sufficiently large so that if  $h_\gamma^e = g_\alpha$ , then  $\alpha < \mu$ . It follows

that

$$\{h_\gamma^0 \upharpoonright (\omega_2 \setminus \mu), h_\gamma^1 \upharpoonright (\omega_2 \setminus \mu)\} = \{f_\alpha^i \upharpoonright (\omega_2 \setminus \mu), f_\beta^j \upharpoonright (\omega_2 \setminus \mu)\}$$

where either  $\alpha \neq \beta$  or  $i \neq j$ . Let  $\omega_1 = \bigcup_{\beta < \omega_1} W_\beta$ , where the  $W_\beta$ 's are disjoint. Applying  $HFC^2$  to each  $W_\beta$  and  $\{\langle \mu + n, i \rangle_{i < 2}\}_{n \in \omega}$ , we see that there are  $\alpha_\beta \in W_\beta$  and  $n_\beta \in \omega$  such that  $h_{\alpha_\beta}^0(\mu + n) = 0$  and  $h_{\alpha_\beta}^1(\mu + n) = 1$ . It follows that for some  $\delta \in (\mu, \mu + \omega)$ ,  $h_\gamma^0(\delta) \neq h_\gamma^1(\delta)$  for uncountably many  $\gamma$ . So  $X$  has a small diagonal by Proposition 1.2.  $\square$

**Remark.** Any hereditarily Lindelöf regular space has cardinality not greater than  $2^\omega$ , so any model which contains an  $HFC$  of cardinality  $\omega_2$  cannot satisfy  $CH$ . We don't know if there is a space having the properties of the above example which is consistent with  $CH$ , or even one which exists in  $ZFC$ !

The example of 3.5 is clearly not first-countable. In fact it can't be because of the following result:

**Theorem 3.6.** *If  $X$  is a first-countable hereditarily Lindelöf space with a small diagonal, then  $X$  has a  $G_\delta$ -diagonal.*

*Proof.* Suppose  $X$  satisfies the hypotheses but not the conclusion. Note that  $X$  cannot have a countable  $T_0$ -separating open cover  $\mathcal{U}$ , for otherwise

$$\Delta_X = \bigcap_{U \in \mathcal{U}, n \in \omega} [U^2 \cup (X \setminus U_n)^2]$$

where  $\{U_n : n \in \omega\}$  is a countable closed cover of  $U$ .

Let  $B(x, n)$ ,  $n < \omega$ , be a countable base at  $x$ . Then we can construct doubletons  $\{x_\alpha^0, x_\alpha^1\}$ ,  $\alpha < \omega_1$ , such that  $\{x_\alpha^0, x_\alpha^1\}$  is not separated by any member of  $\{B(x_\beta^e, n) : \beta < \alpha, n < \omega, e < 2\}$ . By Proposition 1.2(c) there is an open set  $V$  with  $W = \{\alpha : x_\alpha^0 \in V, x_\alpha^1 \notin V\}$  uncountable. For each  $\alpha \in W$ , choose  $n_\alpha \in \omega$  with  $B(x_\alpha^0, n_\alpha) \subset V$ . There is  $\delta < \omega_1$  such that  $\{B(x_\beta^0, n_\alpha) : \beta \in W \cap \delta\}$  covers  $\{x_\alpha^0 : \alpha \in W\}$ . But now if  $\alpha \in W \setminus \delta$ , then  $\{x_\alpha^0, x_\alpha^1\}$  is separated by  $B(x_\beta^0, n_\alpha)$  for some  $\beta \in W \cap \delta$ , contradiction.  $\square$

**Remark.** We don't know if the above result remains true with "hereditarily Lindelöf" weakened to "Lindelöf". Zhou[Z] showed that the answer is "yes" under  $CH$ . Also, Bennett and Lutzer [BL] showed in  $ZFC$  that any Lindelöf space with a small diagonal which is a subspace of a linearly ordered space must have a  $G_\delta$ -diagonal.

#### 4. COUNTABLY COMPACT SPACES

Zhou mentions in [Z] that it is unknown if countably compact spaces with a small diagonal must be metrizable (equivalently, have a  $G_\delta$ -diagonal), and he obtains some partial results related to this question. Shakhmatov [Sh] asks if one can at least show that they must be compact. In this section, we show that the statement "Countably compact spaces with a small diagonal are metrizable" is consistent with and independent of  $ZFC$ .

Let us note that the space  $\omega_1$  of countable ordinals does not have a small diagonal, for the sequence  $\langle (\alpha, \alpha + 1) \rangle_{\alpha < \omega_1}$  in  $\omega_1^2$  converges to the diagonal. More generally,

Bennett and Lutzer [BL] have shown that countably compact suborderable spaces having a small diagonal are metrizable.

The positive consistency result follows easily from the following recent powerful result of Eisworth and Nyikos [EN]:

**Theorem 4.1[EN].** *The following statement (\*) is relatively consistent with ZFC + CH:*

(\*) *A countably compact first-countable space is either compact or contains a copy of  $\omega_1$ .*

**Theorem 4.2.** *If CH + (\*) holds, then countably compact spaces with a small diagonal are metrizable.*

*Proof.* Suppose CH + (\*) holds and  $X$  is a countably compact space with a small diagonal which is not metrizable. By CH and the Juhász- Szentmiklóssy result mentioned in the Introduction,  $X$  is not compact.

*Case 1.*  $X$  has a separable closed non-compact subspace  $Y$ . By CH,  $Y$  has character not greater than  $\omega_1$ . Suppose some point  $y \in Y$  has character exactly  $\omega_1$ . The point  $y$  cannot be a  $G_\delta$ -point of  $Y$ , for if  $U_n$ ,  $n < \omega$ , is a sequence of neighborhoods of  $y$  with  $\{y\} = \bigcap_{n < \omega} U_n$  and  $\overline{U_{n+1}} \subset U_n$  for all  $n$ , then it follows from countable compactness that  $\{U_n\}_{n < \omega}$  is a (countable) base at  $y$ . So, if now  $V_\alpha$ ,  $\alpha < \omega_1$ , is a base at  $y$ , we can choose  $y_\alpha \in \bigcap_{\beta < \alpha} V_\beta$ ,  $y_\alpha \neq y$ , and then  $\langle y_\alpha \rangle_{\alpha < \omega_1}$  is a convergent  $\omega_1$ -sequence, contradicting the small diagonal property.

Thus  $Y$  must be first-countable. Since  $Y$  is not compact, by (\*) we have that  $Y$  contains a copy of  $\omega_1$ . But  $\omega_1$  has no small diagonal, contradiction.

*Case 2.* *Every separable closed subset of  $X$  is compact.* Let  $\mathcal{U}$  be an open cover of  $X$  with no finite (hence countable) subcover. Using the hypothesis of Case 2, one can easily construct points  $y_\alpha$ ,  $\alpha < \omega_1$ , and finite subcollections  $\mathcal{U}_\alpha$  of  $\mathcal{U}$ , such that  $\mathcal{U}_\alpha$  covers  $\overline{\{y_\beta : \beta < \alpha\}}$  and  $y_\alpha \notin \bigcup_{\beta < \alpha} \mathcal{U}_\beta$ . Let  $Y = \bigcup_{\alpha < \omega_1} \overline{\{y_\beta : \beta < \alpha\}}$ . Then  $Y$  is countably compact, non-compact ( $\bigcup_{\alpha < \omega_1} \mathcal{U}_\alpha$  is an open cover with no finite subcover), and each  $\overline{\{y_\beta : \beta < \alpha\}}$  is a compact, hence metrizable (by CH), open subspace of  $Y$ . It follows that  $Y$  is first-countable. Now (\*) implies that  $Y$  contains a copy of  $\omega_1$ , contradiction.  $\square$

On the other hand, there are in some models countably compact spaces with a small diagonal which are not metrizable. Recall that a space is *initially  $\omega_1$ -compact* if every open cover of cardinality  $\omega_1$  or less has a finite subcover; equivalently, every subset of cardinality  $\omega$  or  $\omega_1$  has a complete accumulation point.

**Theorem 4.3.** *Suppose  $2^{\omega_1} = \omega_2$  and there is a subset  $X$  of  $2^{\omega_2}$  of cardinality  $\omega_2$  satisfying:*

- (a) *For every infinite subset  $Y$  of  $X$ , there is  $\gamma < \omega_2$  such that  $\{y \upharpoonright (\omega_2 \setminus \gamma) : y \in Y\}$  is dense in  $2^{\omega_2 \setminus \gamma}$ ;*
- (b) *Every  $\omega_1$ -sequence  $\langle x_\alpha^0, x_\alpha^1 \rangle_{\alpha < \omega_1}$  in  $X^2 \setminus \Delta$  satisfies: for all sufficiently large  $\gamma < \omega_2$ , there is  $\alpha < \omega_1$  and  $n < \omega$  with  $x_\alpha^0(\gamma + n) \neq x_\alpha^1(\gamma + n)$ .*

*Then there is an initially  $\omega_1$ -compact non-compact space with a small diagonal but no  $G_\delta$ -diagonal.*

**Remark.** A set  $X$  satisfies the hypotheses of the above theorem, if, e.g., it is both *HFD* (to get (a)) and *HFD* $_w^2$  (to get (b)) in  $2^{\omega_2}$ . Such a set, along with

$2^{\omega_1} = \omega_2$ , can be obtained by adding  $\omega_2$ -many Cohen reals to a model of  $2^{\omega_1} = \omega_2$ . See [J<sub>2</sub>] or [J<sub>3</sub>] for information on *HFD*'s.

*Proof.* Let  $\{x_\alpha : \alpha < \omega_2\}$  index  $X$ , and let  $\{g_\alpha : \alpha < \omega_2\}$  index all functions from ordinals less than  $\omega_2$  into 2 (by  $2^{\omega_1} = \omega_2$ , there are not more than  $\omega_2$  of them). Then define  $z_\alpha \in 2^{\omega_2}$  by  $z_\alpha \upharpoonright \text{dom}(g_\alpha) = g_\alpha$  and  $z_\alpha \upharpoonright (\omega_2 \setminus \text{dom}(g_\alpha)) = x_\alpha \upharpoonright (\omega_2 \setminus \text{dom}(g_\alpha))$ . We claim that the subspace  $Z = \{z_\alpha : \alpha < \omega_2\}$  of  $2^{\omega_2}$  is the desired example.

It is easy to see that any infinite subset of  $2^{\omega_2}$  satisfying (a), which is satisfied by  $Z$  since each  $z_\alpha$  is the same as  $x_\alpha$  beyond some ordinal  $< \omega_2$ , cannot be compact. Initial  $\omega_1$ -compactness of  $Z$  also follows from (a). The proof of this is the same as the proof of Theorem 5.4 of [BG], which in turn is a mild generalization of the proof due to Juhász (see [J<sub>2</sub>]) that an analogous construction in  $2^{\omega_1}$  yields a countably compact space. For the benefit of the reader, we repeat the argument here.

We need to show that every subset of  $Z$  of cardinality  $\omega$  or  $\omega_1$  has a complete accumulation point. To this end, let  $\kappa \in \{\omega, \omega_1\}$ , and suppose  $Y \in [Z]^\kappa$ . Using property (a), it is easy to see that for each  $W \in [Z]^\kappa$  there is  $\delta_W < \omega_2$  such that  $\{z \upharpoonright (\omega_2 \setminus \delta_W) : z \in Z\}$  is  $\kappa$ -dense in  $2^{\omega_2 \setminus \delta_W}$  (consider splitting  $W$  into  $\kappa$ -many infinite disjoint pieces).

Now we can find  $\gamma < \omega_2$  satisfying:

- (i) The projection  $\pi_\gamma : Y \rightarrow 2^\gamma$  is one-to-one;
- (ii) For each  $\sigma \in \text{Fn}(\gamma, 2)$ , if  $|Y \cap [\sigma]| = \kappa$  then  $\delta_{Y \cap [\sigma]} < \gamma$ .

( $\text{Fn}(\gamma, 2)$  is the set of all finite functions from a subset of  $\gamma$  into 2, and  $[\sigma] = \{x \in 2^{\omega_2} : x \text{ extends } \sigma\}$ .)

Choose a complete accumulation point  $g \in 2^\gamma$  of  $\pi_\gamma(Y)$ , and let  $z \in Z$  be an extension of  $g$ . We claim that  $z$  is a complete accumulation point of  $Y$ . Suppose  $\sigma \in \text{Fn}(\omega_2, 2)$  with  $z \in [\sigma]$ . Let  $W = Y \cap [\sigma \upharpoonright \gamma]$ ; then  $|W| = \kappa$ . By (ii),  $\delta_W < \gamma$ , so  $|W \cap [\sigma \upharpoonright (\omega_2 \setminus \gamma)]| = \kappa$ . Thus  $|Y \cap [\sigma]| = \kappa$ . Thus  $Z$  is initially  $\omega_1$ -compact.

Finally, that  $Z$  has a small diagonal follows from (b), which is satisfied by  $Z$  since it is satisfied by  $X$ . For, suppose  $\{z_\alpha^0, z_\alpha^1\}_{\alpha < \omega_1}$  is an  $\omega_1$ -sequence of doubletons of  $Z$ . Write  $\omega_1$  as the union of  $\omega_1$ -many disjoint uncountable sets  $W_\alpha$ ,  $\alpha < \omega_1$ . For each  $\alpha$ , there is  $\gamma_\alpha < \omega_2$  such that the condition of (b) for the sequence  $\langle z_\beta^0, z_\beta^1 \rangle_{\beta \in W_\alpha}$  is satisfied for all  $\gamma > \gamma_\alpha$ . Choose  $\delta < \omega_2$ ,  $\delta > \sup\{\gamma_\alpha : \alpha < \omega_1\}$ . Then for each  $\alpha < \omega_1$ , there is some  $n_\alpha < \omega$  and some  $\beta_\alpha \in W_\alpha$  with  $z_{\beta_\alpha}^0(\delta + n_\alpha) \neq z_{\beta_\alpha}^1(\delta + n_\alpha)$ . Then for some  $n < \omega$ , uncountably many doubletons of the original sequence differ on  $\delta + n$ . Thus  $Z$  has a small diagonal by Proposition 1.2.  $\square$

We conclude this section with some questions:

I. Does *CH* imply that countably compact spaces having a small diagonal are metrizable? What about *PFA* or the statement (\*) of Theorem 3.1?

II. Can there exist a non-metric countably compact space with a small diagonal which is countably tight, or first-countable? Dow[D] has shown initially  $\omega_1$ -compact, countably tight spaces with a small diagonal are metrizable in models obtained by Cohen forcing over a model of *CH*. The example of Theorem 4.3, which does exist in some Cohen models, is not countably tight.

Since Ostaszewski spaces (i.e., countably compact, locally compact, locally countable spaces in which every closed subset is either countable or co-countable) are particularly interesting examples of first-countable countably compact non-compact spaces, we ask:

III. Can there be an Ostaszewski space with a small diagonal?

IV. Can there be a first-countable perfect pre-image of  $\omega_1$  with a small diagonal?

The answer to Question III is “no” under  $MA_{\omega_1}$ , which kills Ostaszewskii spaces, and the answer to Question IV is “no” under  $PFA$ , for then such a space would have to contain a copy of  $\omega_1$  [F<sub>2</sub>]. Of course, by our Theorem 4.1 the answer to II, III, and IV is “no” under  $CH$  together with the statement (\*).

**Remark.** O. Pavlov [P] recently obtained a positive solution to Question IV, namely, that there is such an example assuming axiom  $\diamond^+$ . This also gives a positive answer to Question II, and answers in the negative the part of Question I about  $CH$ . He has also shown that on the other hand there is no finite-to-one perfect preimage of  $\omega_1$  with a small diagonal.

## 5. LOCALLY COMPACT SPACES

Must locally compact spaces with a small diagonal have a  $G_\delta$ -diagonal? Bennett and Lutzer [BL] showed that the answer is “yes” for linearly ordered spaces (they actually use a stronger assumption than small diagonal, but it turns out a little tweaking of the argument gets it for small diagonal). Also, Zhou [Z] obtained an example under  $MA_{\omega_1}$  showing that the answer can be “no”. The purpose of this section is to show that the answer is “no” in  $ZFC$ , i.e., we construct in  $ZFC$  a locally compact space with a small diagonal but no  $G_\delta$ -diagonal.

Both Zhou’s example and ours are locally countable; their existence depends essentially on the existence of almost disjoint families of countable sets having certain combinatorial properties. This makes the problem for locally compact quite different than for compact; but the combinatorics involved are natural and perhaps have some interest in their own right.

If  $\mathcal{A}$  is a collection of sets, let us say that a set  $B$  is *orthogonal* to  $\mathcal{A}$  if  $B \cap A$  is finite for every  $A \in \mathcal{A}$ .

**Lemma 5.1.** *There is a locally countable, locally compact  $T_2$ -space  $X$  with a small diagonal but no  $G_\delta$ -diagonal if there is an almost disjoint family  $\mathcal{A} \subset [\kappa]^\omega$  for some cardinal  $\kappa$  satisfying:*

- (a) *For every uncountable subset  $H$  of  $\kappa$ , there is an uncountable subset  $H'$  of  $H$  orthogonal to  $\mathcal{A}$ ;*
- (b)  *$\kappa$  is not the union of countably many subsets orthogonal to  $\mathcal{A}$ .*

*Proof.* Let  $X = (\kappa \times 2) \cup \mathcal{A}$ , where  $\kappa \times 2$  is a set of isolated points, and a basic neighborhood of  $A \in \mathcal{A}$  is  $\{A\}$  together with a cofinite subset of  $A \times 2$ .

Let us see that  $X$  has no  $G_\delta$ -diagonal. Suppose  $\mathcal{G}_n$ ,  $n < \omega$ , is a sequence of open covers of  $X$ . For each  $n$ , let

$$B_n = \{\alpha \in \kappa : \forall G \in \mathcal{G}_n (\{\langle \alpha, 0 \rangle, \langle \alpha, 1 \rangle\} \not\subseteq G)\}.$$

Since for each  $A \in \mathcal{A}$ ,  $\mathcal{G}_n$  has an element  $G$  containing all but finitely many points of  $A \times 2$ , it follows  $B_n$  is orthogonal to  $\mathcal{A}$ . Hence there is  $\alpha \in \kappa \setminus \bigcup_{n \in \omega} B_n$ . Then for each  $n$ ,  $\langle \alpha, 1 \rangle \in st(\langle \alpha, 0 \rangle, \mathcal{G}_n)$ , whence  $\mathcal{G}_n$  cannot be a  $G_\delta$ -diagonal sequence for  $X$ .

Now let us see that  $X$  has a small diagonal. Suppose  $\{\langle x_\alpha, y_\alpha \rangle\}_{\alpha < \omega_1}$  is an uncountable subset of  $X^2 \setminus \Delta$ . W.l.o.g., the  $x_\alpha$ ’s are distinct.

Suppose uncountably many  $x_\alpha$ 's are in  $\mathcal{A}$ . For each  $A \in \mathcal{A}$ , let  $N(A)$  be a basic neighborhood of  $A$  not containing  $y_\alpha$  if  $x_\alpha = A$ . Then

$$U = (\cup\{N(A)^2 : A \in \mathcal{A}\}) \cup \{\langle z, z \rangle : z \in \kappa \times 2\}$$

is an open neighborhood of  $\Delta$  missing the uncountably many points with  $x_\alpha \in \mathcal{A}$ .

It remains to consider the case where uncountably many  $x_\alpha$ 's are in  $\kappa \times 2$ . Let  $H = \{\gamma \in \kappa : \exists \alpha < \omega_1 \exists e < 2 (x_\alpha = \langle \gamma, e \rangle)\}$ .  $H$  contains an uncountable subset  $H'$  orthogonal to  $\mathcal{A}$ . For each  $A \in \mathcal{A}$ , let  $N'(A) = \{A\} \cup ((A \setminus H') \times 2)$ . Let  $U'$  be the neighborhood of  $\Delta$  defined as in the previous paragraph using  $N'(A)$  instead of  $N(A)$ . Then  $U'$  misses the uncountably many points with  $x_\alpha \in H' \times 2$ . That completes the proof.  $\square$

**Theorem 5.2.** *There is an almost disjoint collection  $\mathcal{A}$  of countable subsets of  $\omega_1$  satisfying the conditions of Lemma 5.1, and hence there is a locally compact locally countable  $T_2$ -space with a small diagonal but no  $G_\delta$ -diagonal.*

*Proof.* For each limit ordinal  $\alpha$  in  $\omega_1$ , let  $y_\alpha \in \omega_1^\omega$  be such that  $y_\alpha(n)$ ,  $n \in \omega$ , is an increasing sequence of ordinals with supremum  $\alpha$ . Let  $Y = \{y_\alpha : \alpha < \omega_1, \alpha \text{ is a limit ordinal}\}$ . Viewed as a subset of the metric space  $\omega_1^\omega$ , where  $\omega_1$  is given the discrete topology, the space  $Y$  was considered by A.H. Stone [St] in his non-separable Borel theory. A pertinent fact here is that every separable subspace of  $Y$  is countable, so in particular  $Y$  contains no copy of a Cantor set.

Let  $\mathcal{A} \subset [Y]^\omega$  be a maximal almost disjoint family of sets  $A$  that have a unique limit point  $\omega_1^\omega \setminus Y$ . We will see that this  $\mathcal{A}$  satisfies conditions (a) and (b) of Lemma 5.1.

Let  $H \in [Y]^{\omega_1}$ . In the metric topology,  $H$  is not separable, hence has an uncountable discrete (relative to  $H$ ) subset  $H'$ . Since the set  $K$  of limit points of  $H'$  in  $\omega_1^\omega$  is closed in that metric space,  $K$  is  $G_\delta$  and hence  $H'$  has an uncountable subset  $H''$  which is closed in  $\omega_1^\omega$ . Clearly  $H''$  is orthogonal to  $\mathcal{A}$ . Thus  $\mathcal{A}$  satisfies condition (a).

We check that  $\mathcal{A}$  satisfies condition (b). Let  $B \subset Y$  be orthogonal to  $\mathcal{A}$ . Condition (b) will follow if we show that  $S = \{\alpha : y_\alpha \in B\}$  is non-stationary. Suppose  $S$  is stationary; then by Stone's result [St], the closure of  $B$  in  $\omega_1^\omega$  contains a copy of a Cantor set (in fact a copy of  $\omega_1^\omega$ ).  $Y$  does not contain a Cantor set, so some sequence  $\{b_n : n \in \omega\}$  of points of  $B$  converges to some point of  $\omega_1^\omega \setminus Y$ . But this sequence must meet some member of  $\mathcal{A}$  in an infinite set, contradicting  $B$  orthogonal to  $\mathcal{A}$ .  $\square$

**Remark.** S. Todorčević independently discovered a (different) almost disjoint collection  $\mathcal{A}$  satisfying the conditions of Lemma 5.1.

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