Metrizable spaces and generalizations

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Contents

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1. Introduction

Classes of "generalized" metric or metrizable spaces are those which possess some of the useful structure of metrizable spaces. They have had many applications in the theory of topological groups, in function space theory, dimension theory, and other areas. Even some applications in theoretical computer science are appearing– see, e.g., the article by G.M. Reed in this volume.

In GRUENHAGE[1992], we discussed research activity in generalized metrizable spaces and metrization which occurred primarily over the seven years previous to the 1991 Prague Topological Symposium. Here we discuss activity in the ten years since that time. Of course, there were too many results to include everything, so this article is a quite imperfect selection of them, reflecting to some extent the author's interests, as well as his lack of expertise in certain areas. The article is divided into sections on topics where most of the recent activity has occurred. In the final section, we discuss a variety of open problems.

There are a number of sources for more basic information about the concepts discussed here, e.g., GRUENHAGE[1984] in the Handbook of Set-theoretic Topology, and several articles in the book Topics in Topology (MORITA and NAGATA[1989]).

Unless otherwise stated, all spaces are assumed to be regular and T_1.

2. Metrics, metrizable spaces, and mappings

In our previous article, not much about metrizable spaces or metrics themselves were discussed, but let us mention here a few results in this area that have a set-theoretic topology flavor.

First, an outstanding result in the dimension theory of metrizable spaces was obtained by MROWKA[1997][2000]. Mrowka constructed a metrizable space M (he denoted it by $\nu\mu_0$) with ind $M = 0$, such that if the set-theoretic axiom denoted by $S(\aleph_0)$ is assumed, then any completion of M contains an interval (hence Ind $M \geq 1$) , and further every completion of M^2, which of course also has small inductive dimension 0, contains the square, and hence Ind $M^2 \geq 2$. M^2 is the first known metrizable space in which the gap between the small and large inductive dimensions is at least 2. KULESZA[20∞] extended this to show that every completion of M^n contains an n-cube, and hence the gap between these dimensions can be arbitrarily large.

The only rub with these fascinating examples is that they are far from being ZFC examples. The space M is constructed in ZFC, and M *fails* to be an example under the continuum hypothesis! Furthermore, the axiom $S(\aleph_0)$ under which M is an example is very strong. DOUGHERTY[1997] showed $S(\aleph_0)$ is consistent modulo large cardinals and has large cardinal strength. More specifically, its consistency follows from the existence of the Erdös cardinal $E(\omega_1 + \omega)$ and implies the consistency of $E(\omega)$.

Some interesting mapping theory questions of E. Michael were answered. A continuous map $f : X \to Y$ is *compact covering* (resp., *countable-compact covering*) every compact (resp., compact countable) subset L of Y is the image of some compact subset K of X, and is *inductively perfect* if there is some $X' \subset X$ such that the restricted map $f{\restriction}X'$ is a perfect map of X' onto Y. MICHAEL[1981] asks the following questions, which were repeated (in Problems 392 and 393) in his article in the book Open Problems in Topology:

Suppose X and Y are separable metric spaces, and $f : X \to Y$ continuous.

(a) Suppose f is compact covering. Must f be inductively perfect if either (i) Y is countable, or (ii) each $f^{-1}(y)$ is compact?

(b) If each $f^{-1}(y)$ is compact, and f is countable-compact covering, must f be compact covering?

Debs and Saint Raymond([1996] and [1997]) give counterexamples to (a)(ii) and (b), respectively[2] (contradicting Theorem 2.4 in Just and Wicke[1994] and Theorem 0.2 in Cho and Just[1994]). On the other hand, the answer to (a)(i) is positive (Just and Wicke[1994]), even if the condition on Y is generalized to σ-compact (Ostrovsky[1994]).

Of course, (a) is a special version of the more general question of when compact covering maps between separable metric spaces must be inductively perfect, and it turns out that this can be the case if X is "nice" in a descriptive set-theoretical sense. For example, it is known to be the case if X is Polish Christensen[1973]Saint Raymond[1971-1973]. Under Analytic Determinacy, it holds if X is absolutely Borel (Debs and Saint Raymond([1996]), but under $V = L$, there is a counterexample where X is an F_σ-subset of the irrationals (Debs and Saint Raymond([1999]).

Another question on compact covering maps, due to Michael and Nagami and also appearing in Open Problems in Topology, was answered by H. Chen [1999]. Chen constructed a Hausdorff space Y which is the image of a metrizable space under a quotient map with separable fibers, such that Y is not a compact covering image of any metrizable space. His space Y is not regular; he asks if there can be a regular example.

A space Y is called a *connectification* of a space X if X is dense in Y and Y is connected. It is easy to see that if X has a compact open subset, then X has no Hausdorff connectification. There seem to be no other obvious general conditions which preclude spaces from having "nice" connectifications. Watson and Wilson[1993] gave the first systematic study of when spaces have a Hausdorff connectification. Included in this work, they show that every metrizable nowhere locally compact space has a Hausdorff connectification. Alas, Tkachuk, Tkacenko and Wilson[1996] then showed that every separable metrizable space with no compact open sets has a metrizable connectifaction, and asked if this is true in the non- separable case as well. This question was answered in the negative by Gruenhage, Kulesza and Le Donne[1998], who gave a construction (due primarily to Kulesza) of a metrizable space with no compact open sets which does not have a metrizable, or even perfectly normal, connectification. It is also proven there that nowhere locally compact metrizable spaces do have metrizable connectifications. Whether or not every metrizable space with no compact open sets has a Tychonoff connectification remains an open question.

Now we present a sampling of results about metrics with special properties. *Ultrametric spaces*, also called *non-Archimedean* metric spaces, are metric spaces with a distance d such that $d(x,z) \leq max\{d(x,y), d(y,z)\}$. The metrizable s-

[2] This implies a negative answer to Michael's question on triquotient maps mentioned in Michael[1981]

paces which admit such a metric are exactly those having covering dimension 0. They have a long history and have found many diverse applications. Here we mention recent results on universal (in the sense of isometry) ultrametric spaces. Any universal space for ultrametric spaces of cardinality two must have cardinality continuum. A. Lemin and V. Lemin[2000] constructed for every infinite cardinal number τ, a universal ultrametric space LW_τ with weight τ^ω. The Lemin's asked if their result could be improved for cardinals $\tau > \mathfrak{c}$. This was answered by Vaughan[1999], who showed that there is a subspace LW'_τ of LW_τ which is universal for ultrametric spaces of cardinality τ, and assuming the singular cardinal hypothesis, has weight τ whenever $\tau \geq \mathfrak{c}$ (and in ZFC has weight τ for an unbounded set of cardinals). It is unknown if this can be done in ZFC; if so, apparently a different example would be needed.

Nagata[1983] showed that every metric space has a metric d such that, for each $\epsilon > 0$, the collection $\mathcal{B}_d(\epsilon)$ of all ϵ-balls with respect to d is closure-preserving. His method shows that for separable metric spaces $\mathcal{B}_d(\epsilon)$ can be made finite. In [1999] he asked if any metrizable space admits a metric d such that $\mathcal{B}_d(\epsilon)$ is locally finite. Gruenhage and Balogh[20∞] gave a negative answer by showing that the class of metrizable spaces which admit a metric d such that $\mathcal{B}_d(\epsilon)$ is locally finite for all $\epsilon > 0$ is exactly the class of strongly metrizable spaces, i.e., those spaces which embed in $\kappa \times [0,1]^\omega$ for some cardinal κ, where κ carries the discrete topology.

Nagata has also asked if every metrizable space admits a metric d such that X has a σ-locally finite (σ-discrete) base consisting of open d-balls. Hattori[1986] has shown that the answer to the σ-locally finite question is positive; the σ-discrete question is still unsettled.

"Midset" metric properties have been studied by several authors. The midset between points x and y is the set of all z such that $d(z,x) = d(z,y)$. Hattori and Ohta[1993] showed that a separable metric space X is homeomorphic to a subspace of the real line iff there exists a metric d for X such that the cardinality of each midset is at most one, and for each x there are at most two points the same distance from x. A metrizable space X is said to have the *unique midset property (UMP)* if there is a metric d on X such that each midset has exactly one point. Ito,Ohta and Ono[1999] showed that discrete spaces with the UMP are exactly the ones of cardinality $\leq \mathfrak{c}$ other than 2 or 4. They also showed that the countable power of any discrete space of size $\leq \mathfrak{c}$ has the UMP; hence, the Cantor set and the irrationals have the UMP. But the question of Hattori and Ohta, whether any separable metrizable space having the UMP must be homeomorphic to a subset of the real line, remains open.

3. Networks

Recall that $\mathcal{F}$ is a *network* for a space X if $x \in U$, where U is open, implies $x \in F \subset U$ for some $F \in \mathcal{F}$. A σ-space is a space with a σ-discrete network. Spaces with a countable network are exactly the continuous images of separable metric spaces, and are sometimes called *cosmic spaces.*

Delistathis and Watson[2000] made an important advance in the dimension theory of general spaces by constructing, under CH, a cosmic space X (in fact, X is the union of countably many separable metrizable subspaces) in which $dim\ X \neq Ind\ X$. Of course, all three of the standard dimensions agree for separable metrizable spaces; this shows that they may differ for their continuous images, and answers a question of Arhangel'skii. It was also known that the dimensions agree for paracompact Hausdorff spaces which are μ-spaces, i.e., embeddable in the countable product of paracompact spaces which are F_σ-metrizable (= the union of countably many closed metrizable subspaces). So this is also a consistent example of a cosmic space which is not a μ-space. TAMANO[2001] later obtained a ZFC example of a cosmic non-μ-space; and subsequently TAMANO and TODORCEVIC[20∞] obtained rather natural examples by showing that for separable metric spaces X, $C_p(X)$ is not a μ-space if X is not σ-compact. See Section 5 for the relevance of these examples to the "stratifiable implies M_1" problem.

I am a little embarrassed to mention that the definition of Σ-spaces (a generalization of σ-spaces) given in my article in the Handbook of Set-theoretic Topology is incorrect, as was pointed out by TAMANO[1997]. I had said that X is a Σ-space if there are a cover $\mathcal{C}$ by closed countably compact sets and a σ-discrete collection $\mathcal{F}$ of subsets of X such that, for any $C \in \mathcal{C}$ and open U with $C \subset U$, there is $F \in \mathcal{F}$ with $C \subset F \subset U$. I should have replaced "σ-discrete" with "σ-locally finite", and required members of $\mathcal{F}$ to be closed. For the class of σ-spaces, i.e., where $\mathcal{C}$ can be taken to be the collection of singletons, these differences can be ignored. However, Tamano showed that they can't be ignored here by obtaining an example which satisfies my definition, but is not a Σ-space. It is apparently not known if my definition would have been equivalent to the original had I required the members of $\mathcal{F}$ to be closed (but keeping "σ-discrete" in place of "σ-locally finite").

Cosmic spaces, which are exactly the Lindelöf σ-spaces, are properly contained in the class of Lindelöf Σ-spaces, which can be characterized as the continuous images of perfect pre-images of separable metric spaces. One motivation for studying Lindelöf Σ-spaces comes from Banach space theory. If X is Eberlein compact, then $C_p(X)$ is Lindelöf Σ. Indeed, the class of Gul'ko compacta is precisely the class of compact spaces which have Lindelöf Σ function spaces and is an important generalization of the class of Eberlein compacta.

Several questions of Arhangel'skii concerning Lindelöf Σ-spaces were answered. Let $C_{p,1}(X) = C_p(X)$, and $C_{p,n+1}(X) = C_p(C_{p,n}(X))$. OKUNEV[1993] showed that if X and $C_p(X)$ are Lindelöf Σ, then so is $C_{p,n}(X)$ for all $n > 0$; hence a compact space X is Gul'ko compact iff $C_{p,n}(X)$ is Lindelöf Σ for some $n \in \omega \backslash \{0\}$ iff $C_{p,n}(X)$ is Lindelöf Σ for all $n \in \omega \backslash \{0\}$. TKACHUK[2000] shows further that there are exactly four possibilities for which $C_{p,n}(X)$'s are Lindelöf Σ: either this holds for no n, for all n, or for exactly all even n, or exactly all odd n. He also shows that if ω_1 is a caliber of X (equivalently, $C_p(X)$ has a small diagonal[3]), or [2001] if X has countable spread, then $C_p(X)$ Lindelöf Σ implies X is cosmic. (Arhangel'skii had obtained these results consistently.) OKUNEV and TKACHUK[2001] answered another question of Arhangel'skii by showing that the aforementioned countable spread result fails if this condition is weakened to $p(X) = \omega$, i.e., every point-finite open collection in X is countable. It is not known if $C_p(X)$ Lindelöf Σ and ω_1 a

[3] A space Y has a *small diagonal* if any uncountable subset Z of $Y^2 \backslash \Delta$ contains an uncountable Z' such that $\overline{Z'} \cap \Delta = \emptyset$

caliber for $C_p(X)$ implies X cosmic.

Arhangel'skii (see YASCHENKO[1994]) also asked the following question about network properties in $C_p(X)$, which is still open: Does $C_p(X)$ a σ-space imply that X and $C_p(X)$ are cosmic?

GRUENHAGE[20∞] partially answered another question of Arhangel'skii by showing that under CH a Lindelöf Σ-space with a small diagonal is cosmic. See Problem 10 in the problems section for other results on small diagonals.

A stronger network notion is that of a *k-network* for a space X, i.e., a collection $\mathcal{F}$ of subsets of X such that, whenever K is compact and U is an open set containing K, then $K \subset \cup\mathcal{F}' \subset U$ for some finite $\mathcal{F}' \subset \mathcal{F}$. k-networks have been useful, among other things, in the study of certain kinds of images of metrizable spaces (e.g., see my earlier surveys GRUENHAGE[1984] and [1992]). In the last ten years, many results and examples concerning k-networks that are point-countable, star-countable, compact-countable, and so forth have been obtained. Rather than attempt to summarize them here, we refer the interested reader to the excellent and very complete surveys of Y. TANAKA[1994][2001].

4. Monotone normality

The definition of *monotonically normal*, due to Heath, Lutzer, and Zenor, is probably what you would guess if asked to define "normal in a monotone way". It means that one can assign to each pair (H,K) of disjoint closed sets an open set $U(H,K)$ with $H \subset U(H,K) \subset \overline{U(H,K)} \subset X \backslash K$, so that $H \subset H'$ and $K \supset K'$ implies $U(H,K) \subset U(H',K')$. Every metrizable space and every linearly ordered space is monotonically normal.

Surely the most exciting recent development in this area is the proof of RUDIN [2001] that the compact monotonically normal spaces are precisely the continuous images of compact ordered spaces. This answered a question of J. Nikiel. By an earlier result of Nikiel and (independently) Treybig, it also implies the following non-metric analogue of the Hahn-Mazurkiewicz theorem: X is a continuous image of a connected ordered compact space iff X is compact, connected, locally connected, and monotonically normal.

The earlier work of WILLIAMS and ZHOU[1991][1998] on the structure of compact monotonically normal spaces, which was discussed to some extent in HUŠEK and VAN MILL[1992], has continued to play a role, in particular, the so-called "Williams-Zhou" trees. The idea of these trees is part of the difficult proof of Rudin's result above, and the trees are used by GARTSIDE[1997] in his thorough study of cardinal invariants of monotonically normal spaces.

Another result of RUDIN[1996] answered a question of Purisch; she constructed a locally compact monotonically normal space which has no monotonically normal compactification.

Some interesting results regarding products were obtained. PURISCH and RUDIN[1997] showed that if X and Y are monotonically normal, and Y is countable, then $X \times Y$ is normal. They construct an example demonstrating that the monotone normality assumption on Y is necessary.

NYIKOS[1999] studied monotone normality in trees with the interval topology. He shows that a tree is monotonically normal iff it is the topological sum of convex chains of the tree (hence of ordinal spaces); this generalizes a result proven by K. P. Hart for $\aleph_1$-trees.

A few more results about monotonically normal spaces are mentioned in the next two sections, since they are related to the classes discussed there. Also, we refer the reader to Collins[1996] for an excellent survey of monotone normality up to 1996.

5. Stratifiable and related spaces

Ceder[1961] defined the class of M_1 spaces to be those spaces which have a σ-closure preserving base. He also defined M_2 and M_3-spaces, now known to be equal and usually known as *stratifiable* spaces. A nice characterization of stratifiable spaces is that they are exactly the *monotonically perfectly normal* spaces; i.e., to each closed set H one can assign a sequence $U_n(H)$ of open sets satisfying $H = \bigcap_{n<\omega} U_n = \bigcap_{n<\omega} \overline{U_n}$ so that $H \subset H'$ implies $U_n(H) \subset U_n(H')$. They are also exactly the class of monotonically normal σ-spaces.

The main question about stratifiable spaces remains open: are they the same as M_1-spaces, i.e., do they have a σ-closure-preserving base? There have been a number of interesting advances and related partial results. Mizokami and Shimane[2000a] improved Ito's much earlier result that first-countable stratifiable spaces are M_1 by proving that stratifiable k-spaces (or, equivalently in this context, sequential spaces) are M_1. Their result is a bit more general than that, as follows:

5.1. Theorem. *Suppose X is stratifiable and has the following property:*

($*$) Whenever U is open and $x \in \overline{U} \backslash U$, there exists a closure-preserving collection $\mathcal{F}$ of closed subsets of X that is a net at x and $\overline{F \cap U} = F$ for each $F \in \mathcal{F}$.

Then X is an M_1-space.

It is easy to see that Frechét spaces satisfy ($*$): let $\mathcal{F}$ be the collection of tails of a sequence in U converging to x (along with the limit x itself). Mizokami and Shimane show that sequential stratifiable spaces satisfy ($*$).[4] The quite lengthy proof of their main theorem involves using ($*$) to build closure-preserving collections of various sorts until finally all can be put together to get the sought-after σ-closure-preserving base.

Recently, this result has been improved by Mizokami, Shimane and Kitamura [20∞], who showed that Theorem 5.1 holds with ($*$) restricted to dense open sets U.

We mentioned the class of μ-spaces in the section on networks. The examples of Delistathis and Watson, and Tamano, mentioned there are the first known examples of paracompact σ-spaces which are not μ-spaces. This is relevant here because stratifiable spaces are paracompact σ-spaces, and it is known that all stratifiable μ-spaces are M_1, but it is not known whether all stratifiable spaces are μ-spaces.

Gartside and others have looked, with some success, for "natural" classes of spaces that are stratifiable. First, in a negative direction, Gartside[1998] proved

[4] In fact, one can use an induction on the sequential order of a point x with respect to U to show more generally that regular sequential spaces with points G_δ satisfy ($*$)

that a compact κ-metrizable (in the sense of ŠČEPIN[1980]; one may use Dugundji compacta here instead) space with a dense monotonically normal subspace is metrizable. Since Tychonoff cubes are κ-metrizable, and any linear subspace of a product of lines can be re-embedded densely in a product of lines, and hence densely in a compact κ-metrizable space, it follows that (i) $C_p(X)$ is monotonically normal (or stratifiable, or metrizable) iff X is countable, and (ii) a Banach space in its weak topology is monotonically normal (or stratifiable, or metrizable) iff it is finite dimensional. Results (i) and (ii) answer questions of Arhangel'skii and Wheeler, respectively. In a more positive direction, SHKARIN[1999] shows that the locally convex direct sum of stratifiable locally convex spaces, as well as the strict inductive limit of a sequence of metrizable locally convex spaces, is stratifiable (see ROBERTSON and ROBERTSON[1964] for definitions of the functional analytic terms used here).

Gartside's result that $C_p(X)$ is stratifiable if and only if X is countable was proven independently by both YASCHENKO[1994] and SAKAI. On the other hand, GARTSIDE and REZNICHENKO[2000] show that for $C_k(X)$, the space of continuous real-valued functions on X with the compact-open topology, the situation is quite different:

5.2. THEOREM. *Let X be a Polish space (i.e., complete separable metric). Then $C_k(X)$ is stratifiable.*

This result provides us with a very natural class of stratifiable spaces. It is interesting that the proof gives no clue as to whether or not these function spaces are in general M_1. In particular, it is not known if $C_k(\mathbb{P})$, where $\mathbb{P}$ is the space of irrationals, is M_1. In unpublished results, Gruenhage and Balogh have shown that there is no σ-closure-preserving base consisting of finite unions of standard basic open sets, and K. Tamano showed that standard basic open sets cannot witness another base property known to imply that the space is a μ-space (and hence M_1).

It is also not known if $C_k(X)$ for other separable metric spaces X are stratifiable; deciding whether or not $C_k(\mathbb{Q})$ is stratifiable is probably key here.

Besides function spaces, spaces of subsets with the Vietoris topology were studied by several researchers. It follows easily from classical results that the hyperspace of all (non-empty) closed sets is monotonically normal, stratifiable, or cosmic iff X is compact metric. FISHER, GARTSIDE, MIZOKAMI and SHIMANE[1997] show that the space $\mathcal{F}(X)$ of all finite subsets of X is monotonically normal iff X^2 is monotonically normal iff (by Gartside's result mentioned in the next section) X^n is monotonically normal for all $n < \omega$, and thereby obtain as a corollary the result of MIZOKAMI and KOIWA[1987] that $\mathcal{F}(X)$ is stratifiable iff X is. Furthermore, they show that the space $\mathcal{K}(X)$ of all compact subsets of X is stratifiable if it is monotoncially normal and every non-empty open set in X contains an infinite compact set.

GUO and SAKAI[1993] showed that if X is a connected CW-complex, then the space of compact (resp. compact connected) subsets of X is an absolute retract (AR) for the class of stratifiable spaces. CAUTY, GUO and SAKAI[1995] showed that the space of non-empty finite subsets of X is an absolute neighborhood retract (resp., AR) for stratifiable spaces iff X is stratifiable and 2-hyper-locally connected (resp., and connected). In the negative direction, CAUTY[1998] obtained a result implying that none of the classical characterizations of ANR's for metrizable spaces extend in general to the class of stratifiable spaces.

A long and difficult argument of SIPACHEVA[1993] shows that the free abelian group of a stratifiable space is stratifiable. Arhangel'skii's question whether the same result holds in the non-abelian case remains unsolved. In any case, Sipacheva's result shows that any stratifiable space can be embedded as a closed subspace of a stratifiable abelian group.

KUBIAK[1993] characterized monotonically normal spaces in terms of the insertion of a continuous function between upper and lower semi-continuous functions. Specifically, X is monotonically normal iff X has the *monotone insertion property*, i.e., for every pair (f, g) of real-valued functions on X with f upper semi-continuous and g lower semi-continuous, and $f(x) \leq g(x)$ for all $x \in X$, one can assign a continuous function $\lambda(f, g)$ on X with $f(x) \leq \lambda(x) \leq g(x)$ for all x, and $f \leq f', g \leq g'$ implies $\lambda(f, g)(x) \leq \lambda(f', g')(x)$ for all x.

LANE, NYIKOS and PAN[20∞] showed that also requiring $\lambda(x)$ to be strictly between $f(x)$ and $g(x)$ for all x with $f(x) < g(x)$ characterizes stratifiablity; GOOD and STARES[2000] show that stratifiability is also characterized by requiring this strict betweenness only for those pairs (f, g) with $f(x) < g(x)$ for all $x \in X$. [5]

6. Some higher cardinal generalizations

Some interesting work has been done on higher cardinal generalizations of metrizable and stratifiable spaces, and related classes. A space X is *non-archimedean* if it admits a base which is a tree of open sets under reverse inclusion. If each level of such a tree base covers X, and the height of the tree has uncountable cofinality ω_μ (i.e., $\mu > 0$), then X is *ω_μ-metrizable*. A space X with topology $\mathcal{T}$ is said to be *stratifiable over the cardinal* ω_μ if for each open set U, U can be written as the increasing union of open subsets $U(\alpha)$, $\alpha < \omega_\mu$, whose closures are also contained in U, such that $U \subset U'$ implies $U(\alpha) \subset U'(\alpha)$ for each $\alpha < \omega_\mu$. If X is stratifiable over some cardinal, X is called *linearly stratifiable*, and is *ω_μ-stratifiable* if ω_μ is the least cardinal over which it is stratifiable. If X is ω_μ-stratifiable and each point has a totally ordered local base, X is *ω_μ-Nagata*. ω_μ-metrizability implies ω_μ-Nagata implies ω_μ-stratifiable.

See VAUGHAN[20∞] for a nice discussion of the above classes of spaces, along with other characterizations of them and examples illustrating their differences. He shows there that ω_μ-Nagata spaces are ultraparacompact (for $\mu > 0$ of course); it is apparently not known if the same is true for ω_μ-stratifiable spaces. He also shows that, unlike ω_μ-metrizable spaces, ω_μ-Nagata spaces need not have an orthobase (i.e., a base $\mathcal{B}$ such that for any $\mathcal{B}' \subset \mathcal{B}$, either $\cap\mathcal{B}'$ is open, or $\cap\mathcal{B}'$ is a single point and $\mathcal{B}'$ is a base at that point). Vaughan's original argument [1972] that linearly stratifiable spaces are paracompact turned out to have a gap, which is filled here, though it was earlier fixed in a slightly different way by HARRIS[1991] (who was the one who noticed the gap).

[5] The above insertion results should be compared with classical results of Katetov and Tong, Michael, and Dowker, asserting that the existence of at least *one* continuous function between pairs as above characterizes normal spaces, perfectly normal spaces, and normal and countably paracompact spaces, respectively.

An interesting result of Stares and Vaughan[1996] shows that the space 2^{ω_1} with the countable box topology is an ω_1-metrizable topological group without the Dugundji extension property. This demonstrates that certain results claimed by van Douwen and Borges, namely, that the Dugundji extension property holds for ω_μ-metrizable and linearly stratifiable spaces, respectively, are false.

A non-archimedean space can equivalently be described as a space with a *rank 1 base* $\mathcal{B}$; i.e., $B, B' \in \mathcal{B}$ and $B \cap B' \neq \emptyset$ implies either $B \subset B'$ or $B' \subset B$. X is *proto-metrizable* if X has a *rank 1 pair-base* $\mathcal{B}$, i.e, $\mathcal{B} = \{B = (B_1, B_2) : B \in \mathcal{B}\}$ such that (i) B_1 is open; (ii) $B_1 \subset B_2$; (iii) $x \in U$, U open, implies $x \in B_1 \subset B_2 \subset U$ for some $B \in \mathcal{B}$; and (iv)$B, B' \in \mathcal{B}$ and $B_1 \cap B_1' \neq \emptyset$ implies either $B_1 \subset B_2'$ or $B_1' \subset B_2$.

All of the classes of spaces discussed in this section are hereditarily paracompact and monotonically normal. Gartside[1999] proved that X^2 monotonically normal implies X^n is monotonically normal and hereditarily paracompact for all $n < \omega$, and under some further assumptions is linearly stratifiable. On the other hand, he constructs a non-linearly stratifiable topological group every finite power of which is monotonically normal. Cammaroto[1994] obtains a related result: X is ω_μ-stratifiable iff $X \times Y$ is monotonically normal for every ω_μ-metrizable Y.

Gartside and Moody[1993] characterized proto-metrizable spaces as "monotonically paracompact" spaces, and Junnila and Kunzi[1993] characterized them as those spaces which are both monotonically normal and "monotonically orthocompact". Here, a space X is *monotonically paracompact* if one can assign to each open cover $\mathcal{U}$ a star-refinement $\mu(\mathcal{U})$ such that $\mathcal{V}$ refines $\mathcal{U}$ implies $\mu(\mathcal{V})$ refines $\mu(\mathcal{U})$; monotonic orthocompactness, which we will not define here, has a bit more complicated definition in terms of "transitive neighbornets". We should mention that Gartside and Moody also characterize ω_μ-metrizable topological groups.

A class of spaces related to the above classes is the class of *elastic* spaces, due to H. Tamano and Vaughan[1971]. Gartside and Moody[1992] showed that every elastic space has the "well-ordered F" property (called "well-ordered point-network" in Gruenhage[1992]); recall that well-ordered F spaces are also monotonically normal and hereditarily paracompact. Later [1997a], they showed that every protometrizable space is elastic, and in [1997b] they obtained a counterexample to a long-standing conjecture of Tamano and Vaughan by constructing an example of a perfect image of an elastic space which is not elastic.

7. Moore and developable spaces

Recall that a space X is *developable* if there is a sequence $\mathcal{G}_n$, $n < \omega$, of open covers such that, for every $x \in X$, $\{st(x, \mathcal{G}_n) : n \in \omega\}$ is a base at x. A *Moore space* is a regular developable space.

Shakhmatov, Tall and Watson[1996] showed that it is consistent for there to be a normal Moore space which is not submetrizable (i.e., it has no weaker metrizable topology). Tall[1994] shows that there are also models of set theory in which there is a non-metrizable normal Moore space and a non-submetrizable countably paracompact Moore space, yet every normal Moore space is submetrizable.

There have been other results related to countable paracompactness in Moore spaces. Knight[1993] used a complicated forcing argument to obtain (consistently) a subset Δ of the real line which is a Δ-set but not a Q-set, where $A \subset \mathbb{R}$ is a Δ-*set* if for every decreasing sequence of subsets A_n, $n < \omega$, of A with empty intersection, there are open subset $U_n \supset A_n$ such that $\bigcap_{n \in \omega} U_n = \emptyset$. It follows that it is consistent that there is a separable countably paracompact non-normal Moore space (namely, the tangent disk space over Δ).

A space is δ-*normal* if every pair of disjoint closed sets, one of which is a regular G_δ-set, can be separated by disjoint open sets. Good and Tree[1994], answer a question of Nyikos by constructing a δ-normal Moore space Y which is not countably paracompact (recall that normal Moore spaces must be countably paracompact). It is also the case that $Y \times [0,1]$ fails to be δ-normal.

Laberge[1999] answered a question of Tall by constructing a consistent example of a normal Moore space in which every open cover of cardinality $\aleph_1$ has a locally finite open refinement, but not every open cover of cardinality ω_2 does. (Tall did not include the Moore property in the statement of his question.) The continuum hypothesis holds in the model, and the example has the property that every countable-to-one pre-image of the space is normal.

A π-*base* for a space X is a collection $\mathcal{P}$ of non-empty open sets such that every non-empty open set in X contain a member of $\mathcal{P}$. G. M. Reed had shown long ago that a Moore space has a σ-discrete π-base iff it can be densely embedded in a developable T_2-space with the Baire property, and asked if "developable T_2-space" could be replaced with "Moore space". This question appears as Problem 303 in the book Open Problems in Topology. D. Fearnley[1999] answered the question in the negative by constructing a Moore space with a σ-discrete π-base which cannot be densely embedded in any Moore space with the Baire property.

Gartside, Good, Knight and Mohamad[2001] have answered a couple of questions of P. Nyikos by constructing a quasi-developable (defined like developable, except that the $\mathcal{G}_n$'s need not be covers) manifold with a G_δ-diagonal which is not developable, and a consistent example which is also countably metacompact. Tree and Watson[1993] answer questions of Reed by constructing two non-metrizable Moore manifolds, one of which is pseudonormal, while the other, done under CH, is pseudocompact.

"Property (a)" is a property which had its origins in the theory of countably compact spaces and appears close to normality. X has *property (a)* if for every open cover $\mathcal{U}$ of X and every dense subset D of X, there is a closed discrete subset F of D such that $st(F,\mathcal{U}) = X$. Rudin, Stares and Vaughan[1997] proved that monotonically normal spaces have property (a). Matveev [1997] showed that separable Moore spaces having property (a) are metrizable. However, this does not carry over to the non-separable case, as Just, Matveev and Szeptycki [2000] show that in ZFC there is a non-metric Moore space having property (a).

8. Bases with certain order properties.

There has been some interesting work done on weakly uniform bases and related properties. Recall that a base $\mathcal{B}$ is *weakly uniform* if the intersection of any infinite subcollection of $\mathcal{B}$ is either empty or a singleton. Also, a base $\mathcal{B}$ is *uniform* (resp, *sharp*) if every infinite subfamily is a local base (resp., subbase) at each point of its intersection. Sharp bases were introduced by ALLECHE, ARHANGEL'SKII and CALBRIX[2000]. Note that if a base is sharp, it is also weakly uniform. A weak uniform base could reasonably be (and now is) also called 2-*in-finite*, since any *two* distinct points are in only *finitely* many members of the base. Similarly, a point-countable base might be called "1-in-countable". This leads to the notions of n-in-finite, ω-in-countable, and so forth.

Motivating some of the results here is an old question of HEATH and LINDGREN [1976] : Does every first-countable space with a weakly uniform base have a (possibly different) point-countable base? In other words, does the existence of a 2-in-finite base along with first-countability imply the existence of a 1-in-countable base? Put in those terms, it shouldn't be too surprising that some interesting combinatorics get involved. Old partial results of DAVIS, REED and WAGE[1976] say that there is a counterexample under $MA(\omega_2)$, though the answer is positive in ZFC if there are not more than $\aleph_1$-many isolated points.

Here are some interesting new results. ARHANGEL'SKII, JUST, REZNICHENKO and SZEPTYCKI [2000] show that a space with a sharp base has a point-countable base, generalizing the corresponding known (and easy) result for uniform bases. They also show, under CH, that every first-countable space with a weakly uniform base and no more than $\aleph_\omega$-many isolated points has a point-countable base. BALOGH, DAVIS, JUST, SHELAH and SZEPTYCKI[2000] obtain a stronger (topological) result, which in particular eliminates any condition on the number of isolated points, by introducing the axiom CECA, which is equivalent to GCH plus a bit of $\square_\lambda$ for singular λ and thus follows from $V = L$. They show that the following holds under CECA: X has a point-countable base if it is first-countable and has a base $\mathcal{B}$ such that, for every infinite subset A of X, some finite subset of A is included in only finitely many members of $\mathcal{B}$. Note that the stated base condition is weaker than n-in-finite for any fixed n.

In the paper of ALLECHE ET AL above, an example of a non-developable space with a sharp base is given. In ARHANGEL'SKII ET AL it is asked whether a pseudocompact space with a sharp base must be metrizable (this is known to be the case if "sharp" is strengthened to "uniform"). GOOD, KNIGHT and MOHAMAD[20∞] answer this in the negative; their counterexample has the additional property that its product with the unit interval fails to have a sharp base, which answers a question of ALLECHE ET AL.

BALOGH and GRUENHAGE[2001] generalize the classical result that compact spaces with a point-countable base are metrizable, by showing that compact spaces with an ω-in-countable base are metrizable. They also show that the corresponding statement for countably compact spaces is independent of ZFC. Generalizing results of Peregudov, and Burke and Davis, they show that a locally compact space is metrizable if it has an n-in-countable base, or, provided it has no isolated points, if it has a $\mathfrak{c}$-in-countable base.

BALOGH, BENNETT, BURKE, GRUENHAGE, LUTZER and MASHBURN[2000] study the notion of an *open-in-finite (OIF)* base $\mathcal{B}$, i.e., every non-empty open set is contained in at most finitely many members of $\mathcal{B}$. They show that a base $\mathcal{B}$ is uniform iff the restriction of $\mathcal{B}$ to any subspace Y is an OIF base for Y. They also show, among other things, that every space is an open perfect image as well as a closed subset of a space with an OIF base, and give an example of a space with a point-countable base and an OIF base which is not quasi-developable.

Bennett and Lutzer[1998a] study several of these base properties in ordered spaces. They show that a linearly ordered space has a point-countable base iff it has an ω-in-countable base. They also show that a generalized ordered space has a weak uniform base iff it is quasi-developable with a G_δ-diagonal, and is metrizable iff it has an OIF base iff it has a sharp base.

There have been some results on products involving point-countable bases. Zhu[1993] showed that if X is a metalindelöf Morita P-space (see Section 9 for the definition), and Y has a point-countable base, then $X \times Y$ is metalindelöf. Alster and Gruenhage[1995] showed that the same holds if X is a paracompact monotonically normal space, part of the motivation here being the corollary that there can be no monotonically normal counterexample in ZFC to Michael's question whether the product of a Lindelöf space with the irrationals must be Lindelöf. (A Lindelöf version of the Michael line is a monotonically normal counterexample which exists under CH, or more generally, under $\mathfrak{b} = \omega_1$.) Zhu also showed $X \times Y$ is metalindelöf in case X and Y are metalindelöf and Y is a strong Σ-space.

Balogh([20∞a], [20∞b]) obtained some interesting reflection theorems regarding point-countable bases. He showed that for spaces of density not greater than $\aleph_1$, if every subspace of cardinality ω_1 has a point-countable base, then so does the whole space. A very interesting question is whether or not this can be consistently true, for first-countable spaces, without the density restriction (it is known to be consistently false). Balogh also proved that under the so-called Axiom R of Fleissner, a locally compact space is metrizable if every subspace of cardinality ω_1 has a point-countable base. (Compare with Dow's ZFC theorem [1988]that compact space is metrizable if every subset of cardinality $\aleph_1$ is metrizable.)

9. Normality in products.

A most exciting development here is the solution by Larson and Todorcevic of the following long-standing problem of Katetov [1948]: If X^2 is compact and hereditarily normal, must X be metrizable? Katetov had shown the answer to be positive if X^3 is hereditarily normal. Nyikos discovered a counterexample under $MA(\omega_1)$, and Gruenhage under CH. The question remained whether or not there is a consistent positive answer.

The difficulty in the problem lay in the fact that certain consequences of CH simultaneously with certain consequences of $MA(\omega_1)$ appeared to be needed to solve it. In particular, it is (in effect) shown in Gruenhage and Nyikos[1993] that a model with no Q-sets and no $S-$ or $L-$ subspaces of first countable compacta would be a model in which the answer to Katetov's question is affirmative. Larson and Todorcevic [20∞] succeeded in constructing such a model. To get their model, they take a model M in which there is a "coherent" Souslin tree S such that $MA(\omega_1)$ holds for all posets P such that $P \times S$ is *ccc*. (Such models were already known to exist.) They prove that in the model obtained from M by forcing with S, there are no Q-sets (in fact, there are no Q-sets in any model obtained by forcing with a Souslin tree), and that also (here is the most difficult part of the argument) a certain partition relation holds which implies that the hereditary Lindelöf property and the hereditary separable property are equivalent in subspaces of first countable compacta.

It is not difficult to observe that in a model with no Q-sets and no S- or L- subspaces of first countable compacta, e.g., the Larson-Todorcevic model, any compact space X such that $X^2\backslash\Delta$ is perfectly normal must be metrizable; this answers a question mentioned in GRUENHAGE and NYIKOS[1993]. However, Przymusinski's problem whether there are ZFC examples of non-metrizable compact X and Y such that $X \times Y$ is perfectly normal is still open.

Another old problem involving normality in products was solved by Z. Balogh. Morita long ago characterized the spaces X such that $X \times Y$ is normal for every metric space Y; such X are now called (Morita) P-spaces. Morita also stated the following three problems:

(1) Must X be discrete if $X \times Y$ is normal for any normal space Y?

(2) Must X be metrizable if $X \times Y$ is normal for any normal P-space Y?

(3) Must X be metrizable and σ-locally compact if $X \times Y$ is normal for any countably paracompact normal space Y?

In GRUENHAGE[1992], we reported that M. Atsuji and M.E. Rudin had answered (1) affirmatively in ZFC, and that K. Chiba, T. Przymusinski, and Rudin had answered (2) and (3) affirmatively assuming $V = L$. Now, BALOGH[2001] has used extensions of the idea of his recent ZFC Dowker space construction to give affirmative answers to (2) and (3) in ZFC.

Motivated by normality of product questions, JUNNILA and YAJIMA[1998] introduce some new classes of spaces defined by special networks. One such class is the class of *LF-netted* spaces, which are spaces X having a σ-locally finite network $\mathcal{F}$ such that, for every closed $H \subset X$, the collection $\{F \in \mathcal{F} : F \cap H \neq \emptyset\}$ is locally finite at each point of $X \setminus H$. They show that if X is LF-netted and Y normal and countably paracompact, then $X \times Y$ is normal iff countably paracompact. Since metric spaces are LF-netted, this extends classical results of Morita, and Rudin and Starbird. Since MIZOKAMI and SHIMANE[2000b] show that every *Lašnev space*, i.e., closed image of a metric space, is LF-netted, it also generalizes a result of Hoshina. It is not known if every stratifiable space, or stratifiable μ-space, is LF-netted, though stratifiable F_σ-metrizable spaces are.[6]

Next we mention an interesting result on normality in Σ-products, where X is a *Σ-product* of the the spaces X_α $\alpha < \kappa$, if there are $p_\alpha \in X_\alpha$ such that

$$X = \{\vec{x} \in \Pi_{\alpha<\kappa} X_\alpha : |\{\alpha : x_\alpha \neq p_\alpha\}| \leq \omega\}.$$

Recall that Gul'ko and Rudin independently proved that any Σ-product of metrizable spaces is normal. There is by now a whole theory of normality in Σ-products inspired by this result. One of the more interesting open problems, due to Kodama and mentioned in my earlier survey GRUENHAGE[1992], is whether or not the Gul'ko-Rudin theorem holds for the wider class of Lašnev spaces. There is now a consistent negative example, essentially due to P. Koszmider and appearing in EDA, GRUENHAGE, KOSZMIDER, TAMANO and TODORCEVIC[1995]. Sequential fans are key in this result. The *sequential fan with κ-many spines*, denoted by $S(\kappa)$, is the (Lašnev) space obtained from the topological sum of κ-many convergent sequences by identifying their limit points. Since the space ω_1 of countable

[6] Mizokami recently announced that stratifiable μ-spaces are LF-netted.

ordinals appears as a closed subspace in any uncountable Σ-product of non-trivial T_1-spaces, the product $S(\omega_2)^2 \times \omega_1$ is a closed subspace of a Σ-product of Lašnev spaces. Koszmider constructs a model in which this product is non-normal. On the other hand, he shows that this product *is* normal under $MA(\omega_2)$ and the negation of Chang's Conjecture. It is still open if there are ZFC examples of non-normal Σ-products of Lašnev spaces. In particular, it is not known if $S(2^{\mathfrak{c}})^2 \times \omega_1$ is non-normal in ZFC.

10. Sums of metrizable subspaces

Tkachuk[1994] defines a property P to be *weakly n-additive* if X has P whenever X^n is the union of at most n subspaces having property P. He showed many properties–most of them local properties such as first-countability and local compactness–are weakly n-additive for finite n. He also showed that metrizability is weakly n-additive for finite n in the class of compact or *ccc* spaces. He asked if metrizability is weakly n-additive in general. Balogh, Gruenhage and Tkachuk[1998] show that the answer is positive for pseudocompact spaces, but negative in general. Any "ladder space" X on the countable ordinals is a counterexample; indeed, X^n is the union of two metrizable subspaces for any finite n. Independently, Ohta and Yamada[1998] obtained another example.

Tkachuk's investigation was motivated by an old question of Arhangel'skii. To put it in the above terminology, it asks whether metrizability is weakly ω-additive, i.e., must X be metrizable if X^ω is the union of countably many metrizable subspaces? Tkacenko had shown that the answer is positive for separable or countably compact spaces, and Tkachuk showed it to be the case if X^ω is the union of finitely many metrizable subspaces. However, Gruenhage[1997] answered the question in the negative by showing that any ladder system space on ω_1 in which the set of non-isolated points is G_δ (such spaces can easily be constructed in ZFC) is a counterexample.

In another direction, Ismail and Szymanski[1995],[1996],[2001] have a series of papers in which they investigate the *metrizability number* $m(X)$ of a space X; $m(X)$ is the least cardinal κ such that X is the union of κ-many metrizable subspaces. They obtain nice structure theorems for locally compact X when $m(X)$ is finite; e.g., in this case X must have a dense open metrizable subspace whose complement has smaller metrizability number. From their structure theorems, they easily conclude that, in the finite case, the metrizability number of a locally compact space cannot be raised by a perfect mapping. It is not known if this remains true for infinite metrizability numbers, even if the domain is compact (in particular, the case $m(X) = \omega$ is unsettled).

We take the opportunity to mention here an old unsolved problem on metrizability number due to van Douwen, Lutzer, Pelant and Reed[1980]: Is $m(X) \leq \mathfrak{c}$ whenever X has a point-countable base?

11. Open problems.

In this section we give a selection of several open problems. In some cases, there are recent partial results to be mentioned that didn't conveniently fit into previous categories. That is one purpose; another is to show the rich variety of interesting questions that remain, and thereby, I hope, stimulate further research in generalized metric spaces and metrization. Some of these questions have appeared in my previous article [1992], and in one case earlier in this article. One is completely new. None are due to me originally.

1. Are stratifiable spaces M_1?

This question of Ceder from 1961, the oldest one on my list, was discussed in detail in Section 5; in particular, $C_k(\omega^\omega)$ was suggested as a potential counterexample.

2. Is it consistent that there are no symmetrizable L-spaces?

This is a problem of Arhangel'skii dating back to 1966. Recall that a function $d : X \times X \to [0, \infty)$ is a *symmetric* on X if $d(x, y) = d(y, x)$ and $d(x, y) = 0 \iff x = y$. Then a space X is *symmetrizable* if there is a symmetric d on X such that U is open in X iff for each $x \in U$, there is some $\epsilon > 0$ such that the ϵ-ball $B(x, \epsilon)$ about x is contained in U. (Note: As d need not satisfy the triangle inequality, $B(x, \epsilon)$ itself need not be open.) Shakhmatov [1992] obtained a consistent example of a symmetrizable L-space by forcing, but no ZFC example is known.

Next we list three problems which would take finding a certain Dowker space (or prove such a Dowker space cannot exist) to answer. In each case, without further assumptions on the space, there are no consistent theorems or counterexamples. I.e, the solution, for all we know, could go either way, in ZFC!

3(a). Is there a symmetrizable Dowker space?

(b). Suppose X is normal, and the union of countably many open metrizable subspaces. Must X be metrizable?

(c). Is every normal space with a σ-disjoint base paracompact?

Problem 3(a) is due to S. Davis. A positive answer would imply a negative answer to an old question of E. Michael: Must every point of a symmetrizable space be G_δ? 3(b) is one of Mike Reed's favorite problems. In unpublished work, Reed has shown the answer is positive for spaces of weight less than $\mathfrak{b}$, and has constructed a regular non-developable space which is the increasing union of open metrizable subspaces. And 3(c) is one of Mary Ellen Rudin's favorites. A counterexample to 3(b) is easily seen to be a counterexample to (c) too.

4. (The point-countable base problem.) Does a space X have a point-countable base iff X has a countable open point-network?

This problem is due to Collins, Reed, and Roscoe. The property "countable open point-network", also called "open(G)", means that one can assign to each $x \in X$ a countable open base $\mathcal{B}(x)$ for x such that, whenever a sequence of points x_n converges to x, then $\bigcup_{n\in\omega} \mathcal{B}(x_n)$ contains a base at x. It is easily seen that a space with a point-countable base $\mathcal{B}$ has a countable open point-network (let $\mathcal{B}(x) = \{B \in \mathcal{B} : x \in B\}$). It is known that the answer to Question 4 is "yes" for spaces of density $\aleph_1$, so a positive answer (necessarily consistent) to the reflection problem mentioned in Section 8 would give a consistent positive answer to this one. See Collins, Reed and Roscoe[1990] for more insight and partial results related to this problem.

5. If every $\aleph_1$-sized subspace of a first-countable space X is metrizable, must X be metrizable?

This reflection problem, a version of a problem due to P. Hamburger, was also mentioned in GRUENHAGE[1992], where further information can be found. Except for Balogh's related results on point-countable base reflection (see Section 8), I know of no progress since then.

6. Is Arhangel'skii's class MOBI preserved by perfect mappings?

Recall that MOBI is the smallest class of spaces containing all metrizable spaces, and closed under open compact images. The above problem is the only one still open of those mentioned in ARHANGEL'SKII's classic survey [1966]. However, it is still open at least in part because some other fundamental questions about MOBI remain unsolved, especially whether or not there is some positive integer n such that every space X is MOBI is "n^{th}-generation", i.e., there is a metrizable space M and a sequence $f_1, f_2, ...f_n$ of open compact mappings such that $X = f_n \circ f_{n-1} ... \circ f_1(M)$. Indeed, it is possible, perhaps likely, that such an n, if it exists, can be 2, as this is the case in a very natural way for every known example. See my earlier survey [1992] for more information.

Part of the motivation for MOBI was the search for "nice" classes of spaces which generalized metrizable ones; part of the definition of "nice" included preservation under various standard topological constructions. Now we state a new question of this sort, asked of the author in a recent private communication by E. Michael.

7. Is there a class of spaces (and if so, describe it) which:

(i) contains all metrizable spaces;

(ii) is closed under the taking of closed subspaces, closed images, perfect pre-images, and countable products; and

(iii) is contained in the class of paracompact spaces?

If one only asked for preservation under perfect mappings, then the class of paracompact p-spaces (i.e., the class of all perfect pre-images of metrizable spaces) would satisfy all the conditions. The required class of course must contain all paracompact p-spaces, but also Lašnev spaces (=closed images of metric spaces). Both Lašnev and paracompact p-spaces are subclasses of the class of paracompact Σ-spaces, which satisfies all conditions except preservation under closed maps. The somewhat wider class of paracompact $\Sigma^{\#}$-spaces[7] is closed under closed maps, and it would work if paracompactness were countably productive in the class of $\Sigma^{\#}$-spaces. But that is an unsolved problem! In fact, it is not known if X, Y paracompact $\Sigma^{\#}$ implies $X \times Y$ is paracompact (it is $\Sigma^{\#}$).

Another approach to the question might be to consider the smallest class $\mathcal{M}^{\#}$ containing all metrizable spaces and closed under the operations mentioned in (ii). Then $\mathcal{M}^{\#}$ is contained in the class of $\Sigma^{\#}$ spaces. The question becomes: Is every member of $\mathcal{M}^{\#}$ paracompact? If the answer is affirmative, one would also like an internal characterization of $\mathcal{M}^{\#}$. I don't know if there is a paracompact $\Sigma^{\#}$-space which is not in $\mathcal{M}^{\#}$.

8. Is there in ZFC a non-metrizable perfectly normal non-archimedean space?"

Recall that X is *non-archimedean* if it has a base which is a tree of open sets under reverse inclusion. A Souslin tree implies a consistent counterexample (namely, the "branch space" of a Souslin tree). QIAO and TALL[20∞] proved

[7] $\Sigma^{\#}$-spaces are defined like Σ-spaces were in Section 3, except that the collection $\mathcal{F}$ is assumed to be only σ-closure-preserving instead of σ-locally finite.

that this problem, originally due to Nyikos, is actually equivalent to the following problem of Maurice: "Does every perfect (= closed sets are G_δ) linearly ordered space have a σ-discrete dense set?" See Lutzer's article in this volume for more information.

There is a more general question, which is due to Mike Reed and came out of research from the '60's and '70's on dense metrizable or dense Moore subspaces, which is also unsolved:

9. Is there in ZFC a regular perfect first-countable space with no σ-discrete dense subset?

This question seems to be unsolved even without the "first-countable" assumption. Note that L-spaces do not have σ-discrete dense sets. But it's consistent that there are no first-countable L-spaces, and may be consistent that there are no L-spaces in general, though this is still unsettled.

10. Is there a non-metrizable compact space with a small diagonal?

This is an old problem of Hušek. As was reported in my previous survey, results of Hušek, Dow, and Jushasz and Szentmiklossy show that the answer is "no" under CH or if Cohen reals are added to a model of CH. Gruenhage[20∞] answered questions of Zhou and Shakhmatov by showing that the same question for countably compact spaces is independent of ZFC. A deep result of Eisworth and Nyikos[20∞] about the consistency with CH of countably compact non-compact first-countable spaces containing a copy of the countable ordinals (which does not have a small diagonal) implies the positive result. Pavlov[20∞], answered one of my questions by showing that, under $\diamondsuit^*$, there is a perfect pre-image of ω_1 with a small diagonal; together with my positive result above, this showed independence of the countably compact question with $ZFC+CH$. Pavlov also has shown that the negation of CH implies that there is a Lindelöf space with a small diagonal but no G_δ-diagonal. Gruenhage also showed that there are consistent examples of hereditarily Lindelöf, consistent with CH examples of Lindelöf, and ZFC examples of locally compact spaces having a small diagonal but no G_δ-diagonal.

We should also mention that Arhangel'skii and Bella[1992] showed that CH implies that every Lindel"of p space (i.e., every perfect pre-image of a separable metrizable space) with a small diagonal is metrizable, and Bennett and Lutzer[1998b] answered one of their questions by obtaining a ZFC example of a paracompact p-space (i.e., a perfect pre-image of a metrizable space) with a small diagonal which is not metrizable. See Lutzer's article in this volume for more details.

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